Math 251 Midterm 1 retake NAME: SOLUTIONS

F 27 Oct. 2017 Student ID: $\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$

Reminder: Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned. If, as planned, I return the exam Monday, this means complaints must be received by Tuesday 27 Oct.

1. (1 point) True of false: I hate limits which contain square roots.

Solution: If you think those are bad, try limits which contain cube roots.

2. (a) (6 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

(b) (14 points) If $f(x) = 5x - x^2$, compute the derivative $f'(4)$ *directly from the definition*. (You should check your answer using the differentiation formula, but no credit will be given for just using the formula.)

Solution: We find the limit of the difference quotient, using the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$\begin{aligned} f'(4) &= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{5(4+h) - (4+h)^2 - (5 \cdot 4 - 4^2)}{h} \\ &= \lim_{h \to 0} \frac{20 + 5h - (16 + 8h + h^2) - (20 - 16)}{h} \\ &= \lim_{h \to 0} \frac{20 + 5h - 16 - 8h - h^2 - 20 + 16}{h} = \lim_{h \to 0} \frac{-3h - h^2}{h} = \lim_{h \to 0} (-3 - h) = -3. \end{aligned}$$

We can check using the differentiation formulas: $f'(x) = 5 - 2x$, so $f'(4) = 5 - 2 \cdot 4 = -3$. (However, you get no credit if this is the only thing you do.)

3. (9 points) Let c be a constant. Find the equation of the tangent line to the graph of $p(x) = 16/x + c$ at $x = 2$. You need not calculate the derivative directly from the definition.

Solution: We need the slope, which is $p'(2)$, and a point on the line, such as $(2, p(2)) = (2, 8 + c)$. Now $p(x) = 16x^{-1} + c$, so $p'(x) = -16x^{-2}$ (because c is a constant), and $p'(1) = -4$. So the equation is $y - (8 + c) = -4(x - 2)$, which can be rearranged to give $y = -4x + 16 + c$.

Note that we want the slope at the *particular* value $x = 2$. Therefore we must substitute $x = 2$ in the formula for the derivative $p'(x)$ *before* using it as the slope of a line. The equation

$$\cancel{y - (8 + c) = -16x^{-2}(x - 2)}$$

is wrong—it is not even the equation of a line.

4. (9 points/part) Differentiate the following functions:

(a) $w(t) = \dfrac{\cos(t)}{t} - \dfrac{8}{\sqrt{t}} + e^2$.

Solution: We rewrite the second term of the function to make it easy to differentiate:

$$w(t) = \frac{\cos(t)}{t} - 8t^{-1/2} + e^2.$$

Use the quotient rule on the first term, use the power rule on the second term, and note that e^2 is a constant so that its derivative is zero. This gives

$$g'(t) = \frac{-\sin(t)\cdot t - \cos(t)\cdot 1}{t^2} - 8\cdot\left(-\tfrac{1}{2}\right)t^{-1/2-1} = -\frac{t\sin(t)+\cos(t)}{t^2} + 4t^{-3/2}.$$

Alternate (easier) solution: Rewrite both the first and second terms to make them easier to differentiate:

$$w(t) = t^{-1}\cos(t) - 8t^{-1/2} + e^2.$$

Use the product rule on the first term, use the power rule on the second term, and note that e^2 is a constant so that its derivative is zero. This gives

$$g'(t) = (-1)t^{-2}\cos(t) + t^{-1}(-\sin(t)) - 8\cdot\left(-\tfrac{1}{2}\right)t^{-1/2-1} = -t^{-2}\cos(t) - t^{-1}\sin(t) + 4t^{-3/2}.$$

(One can check that this answer is the same as in the first solution.)

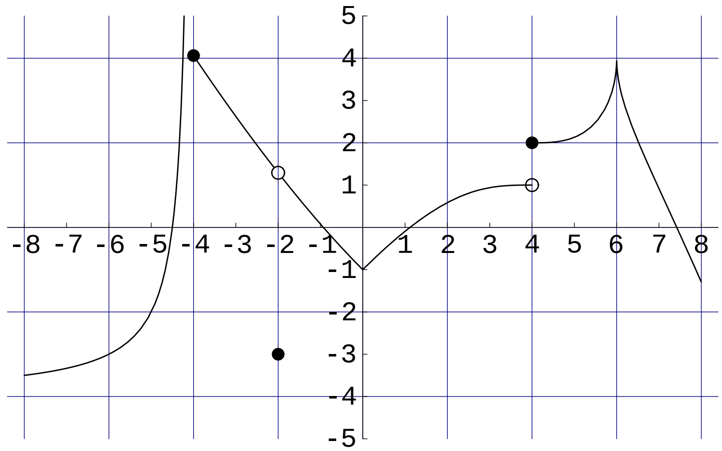
(b) $s(x) = e^{7x-3x^4}$.

Solution: Use the chain rule:

$$s'(x) = e^{7x-3x^4}\cdot\frac{d}{dx}(7x-3x^4) = e^{7x-3x^4}\cdot(7-3\cdot 4x^3) = e^{7x-3x^4}\cdot(7-12x^3) = (7-12x^3)e^{7x-3x^4}.$$

The last rewriting is conventional but not necessary.

5. For the function $y = r(x)$ graphed below, answer the following questions:



(a) (4 points.) Find $\lim_{x\to 4^-} r(x)$.

Solution: $\lim_{x\to 4^-} r(x) = 1$. (Note that it is different from $r(4) = 2$ and $\lim_{x\to 4^+} r(x) = 2$.)

(b) (4 points.) Which of the following best describes $r'(6)$?

(1) $r'(6)$ does not exist.
(2) $r'(6)$ is close to 0.
(3) $r'(6)$ is positive and not close to 0.
(4) $r'(6)$ is negative and not close to 0.
(5) None of the above.

Solution: $r'(6)$ is the slope of the tangent line to the graph of $y = k(x)$ at $x = 6$. The function has a cusp at $x = 6$, so (depending on how you think of it) there is no tangent line, or the tangent is vertical. Whichever way you think of it, the slope doesn't exist. So choice (1) above is correct.

6. (9 points/part) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give reasons in all cases.

(a) $\displaystyle\lim_{x\to-\infty}\frac{7x^3+817x+42}{19x^3-9x^2+42}$.

Solution: This has the indeterminate form "$\frac{\infty}{\infty}$", so work is needed. We factor out x^3 from both the numerator and denominator, and then use the limit laws:

$$\begin{aligned}\lim_{x\to-\infty}\frac{7x^3+817x+42}{19x^3-9x^2+42} &= \lim_{x\to-\infty}\frac{x^3\left(7+\frac{817}{x^2}+\frac{42}{x^3}\right)}{x^3\left(19-\frac{9}{x}+\frac{42}{x^3}\right)} = \lim_{x\to-\infty}\frac{7+\frac{817}{x^2}+\frac{42}{x^3}}{19-\frac{9}{x}+\frac{42}{x^3}}\\ &= \frac{7+\lim_{x\to-\infty}\frac{817}{x^2}+\lim_{x\to-\infty}\frac{42}{x^3}}{19-\lim_{x\to-\infty}\frac{9}{x}+\lim_{x\to-\infty}\frac{42}{x^3}} = \frac{7+0+0}{19-0+0} = \frac{7}{19}.\end{aligned}$$

Here is a different way to arrange essentially the same calculation:

$$\begin{aligned}\lim_{x\to-\infty}\frac{7x^3+817x+42}{19x^3-9x^2+42} &= \lim_{x\to-\infty}\frac{\left(\frac{1}{x^3}\right)\left(7x^3+817x+42\right)}{\left(\frac{1}{x^3}\right)\left(19x^3-9x^2+42\right)} = \lim_{x\to-\infty}\frac{7+\frac{817}{x^2}+\frac{42}{x^3}}{19-\frac{9}{x}+\frac{42}{x^3}}\\ &= \frac{7+\lim_{x\to-\infty}\frac{817}{x^2}+\lim_{x\to-\infty}\frac{42}{x^3}}{19-\lim_{x\to-\infty}\frac{9}{x}+\lim_{x\to-\infty}\frac{42}{x^3}} = \frac{7+0+0}{19-0+0} = \frac{7}{19}.\end{aligned}$$

(b) $\displaystyle\lim_{x\to3^-}\frac{x^2-4}{x-3}$.

Solution: At 3, the denominator is zero but the numerator is not. Moreover, both the denominator and the numerator are continuous at 3. Therefore the function $f(x)=\frac{x^2-4}{x-3}$ has a vertical asymptote at $x=3$.

For $x<3$ but very close to 3, the denominator is negative and very close to zero. The numerator is very close to $3^2-4=5$, so is positive and not close to zero. Therefore the quotient is negative and very far from zero. Thus

$$\lim_{x\to3^-}\frac{x^2-4}{x-3}=-\infty.$$

(c) $\displaystyle\lim_{x\to2}\frac{\sqrt{2x}-2}{3(x-2)}$.

Solution: This has the indeterminate form "$\frac{0}{0}$", so work is needed. We rationalize the numerator and then cancel common factors:

$$\begin{aligned}\lim_{x\to2}\frac{\sqrt{2x}-2}{3(x-2)} &= \lim_{x\to2}\frac{\left(\sqrt{2x}-2\right)\left(\sqrt{2x}+2\right)}{3\,(x-2)\left(\sqrt{2x}+2\right)} = \lim_{x\to2}\frac{2x-4}{3\,(x-2)\left(\sqrt{2x}+2\right)}\\ &= \lim_{x\to2}\frac{2(x-2)}{3\,(x-2)\left(\sqrt{2x}+2\right)} = \lim_{x\to2}\frac{2}{3\left(\sqrt{2x}+2\right)} = \frac{2}{3\left(\sqrt{4}+2\right)} = \frac{1}{6}.\end{aligned}$$

7. (4 points/part) A mosquito takes off from a log and buzzes above a pool of stagnant water. Its height $h(t)$ above the surface of the water, measured in millimeters, at time t, measured in seconds after it takes off, is modelled by $h(t)=200+10t^2-t^3$, until it hits the water (sometime between 11 and 12 seconds after it takes off).

(a) Find a time after the mosquito takes off but before it hits the water when it is neither rising nor falling.

Solution: We want a time t with $0<t<12$ such that $h'(t)=0$. Now $h'(t)=20t-3t^2=t(20-3t)$, which is zero when $t=0$ and when $t=\frac{20}{3}$. Since we want $t>0$, we take $t=\frac{20}{3}$. This number is in fact less than 11, so is definitely a legitimate solution. So the answer is $\frac{20}{3}$ seconds.

Note: You *must* include the units in this kind of problem.

(b) What is the average upwards velocity of the mosquito in the period between 1 and 3 seconds after it takes off?

Solution: The average velocity is the how far it went divided by how long it took to go that far, which here is

$$\frac{h(3)-h(1)}{3-1} = \frac{200+10\cdot 3^2-3^3-(200+10\cdot 1^2-1^3)}{2}$$
$$= \frac{200+90-27-(200+10-1)}{2} = 27.$$

So the average upwards velocity is 27 millimeters/second.

Note: You *must* include the units in this kind of problem.

It is not correct to average the velocities at the endpoints of the interval. That is, do not use

$$\frac{h'(3)+h'(1)}{2} = 25.$$

8. (9 points) If $x^2y = (3x+y)^5 + \cos(11)$, find $\frac{dy}{dx}$ by implicit differentiation. (You must solve for $\frac{dy}{dx}$.)

Solution: Differentiate both sides with respect to x, using the product rule on the left and the chain rule on the right:

$$2x\cdot y + x^2\frac{dy}{dx} = 5(3x+y)^4\frac{d}{dx}(3x+y) = 5(3x+y)^4\left(3+\frac{dy}{dx}\right).$$

(The derivative of $\cos(11)$ is immediately seen to be zero because $\cos(11)$ is a constant.) Now solve for $\frac{dy}{dx}$:

$$2xy + x^2\frac{dy}{dx} = 15(3x+y)^4 + 5(3x+y)^4\frac{dy}{dx}$$
$$[x^2 - 5(3x+y)^4]\frac{dy}{dx} = 15(3x+y)^4 - 2xy$$
$$\frac{dy}{dx} = \frac{15(3x+y)^4 - 2xy}{x^2-5(3x+y)^4}.$$

This fraction can't be simplified.

For those who prefer the other notation, here it is written with y as an explicit function $y(x)$ of x. Start with

$$x^2y(x) = [3x+y(x)]^5 + \cos(11).$$

Then differentiate with respect to x, just as before:

$$2x\cdot y(x) + x^2y'(x) = 5[3x+y(x)]^4\frac{d}{dx}(3x+y(x)) = 5[3x+y(x)]^4(3+y'(x)).$$

Now solve for $y'(x)$:

$$2xy(x) + x^2y'(x) = 15[3x+y(x)]^4 + 5[3x+y(x)]^4y'(x)$$
$$\left[x^2 - 5[3x+y(x)]^4\right]y'(x) = 15[3x+y(x)]^4 - 2xy(x)$$
$$y'(x) = \frac{15[3x+y(x)]^4 - 2xy(x)}{x^2-5[3x+y(x)]^4}.$$

As before, this fraction can't be simplified.