Math 251 Solutions to the Review Session Worksheet for Midterm 1

1. (a) (5 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

(b) (14 points) If $f(x) = 3x - x^2$, compute the derivative $f'(5)$ *directly from the definition.* (You should check your answer using the differentiation formula, but no credit will be given for just using the formula.)

Solution: We find the limit of the difference quotient, using the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \to 0} \frac{3(5+h) - (5+h)^2 - (3 \cdot 5 - 5^2)}{h} = \lim_{h \to 0} \frac{15 + 3h - (25 + 10h + h^2) - (1}{h}$$

$$= \lim_{h \to 0} \frac{15 + 3h - 25 - 10h - h^2 - 15 + 25}{h} = \lim_{h \to 0} \frac{-7h - h^2}{h} = \lim_{h \to 0}(-7 - h) = -7.$$

We can check using the differentiation formulas: $f'(x) = 3 - 2x$, so $f'(5) = 3 - 2 \cdot 5 = -7$. (However, you get no credit if this is the only thing you do.)

2. (4 points/part) Lake Ixhakh is located in a desert, surrounded by mountains. It is a salt lake: there is no outlet, and the only way water is removed from it is by evaporation. In the spring, snow on the mountains melts and the water flows into the lake, but it is still cool and there isn't much evaporation. By the middle of the summer, most of the snow is gone, and the streams running into the lake mostly dry up. But it is hot and a lot of water evaporates.

Let $h(t)$ be the water level in Lake Ixhakh, measured in feet above sea level, at time t, with t being measured in months after 1 January 2000.

(a) Explain the practical significance of the statement $T'(1) = -2$.

Solution: On 1 February 2000, the water level in Lake Ixhakh was decreasing at 2 feet per month. If the water level continued to decrease at the same rate for an entire month, then one month later the water level would be two feet lower.

(b) Do you expect $T'(8)$ to be positive or negative? Why?

Solution: $T'(8)$ should be negative. Since lots of water is evaporating from Lake Ixhakh but little water is flowing in, the water level in Lake Ixhakh should be decreasing.

3. (8 points.) Differentiate the function $f(x) = \dfrac{a}{\sqrt{x}} - \dfrac{b}{x} - \dfrac{c}{x^2} - \dfrac{7}{71}$. ($a$, b, and c are constants.)

Solution: We have

$$f(x) = ax^{-1/2} - bx^{-1} - cx^{-2} - \sqrt{7}.$$

Therefore

$$f'(x) = -\tfrac{1}{2}ax^{-1/2-1} - b(-1)x^{-2} - c(-2)x^{-3} = -\tfrac{1}{2}ax^{-3/2} + bx^{-2} + 2cx^{-3}.$$

Note that $\frac{7}{71}$ is a *constant*, so its derivative is zero. (I have seen people waste lots of time using the quotient rule to differentiate expressions like $\frac{7}{71}$.)

4. (8 points.) Let b be a function such that $b'(t) = e^{-t^2} - e^t$. Find the derivative of the function $f(t) = t - b(2t)$. (Your answer might involve the function b.)

Solution: Use the chain rule:

$$f'(t) = 1 - b'(2t) \cdot \tfrac{d}{dt}(2t) = 1 - 2\big(e^{-(2t)^2} - e^{2t}\big) = 1 - 2e^{-4t^2} + 2e^{2t}.$$

5. (8 points/part) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give reasons in all cases.

(a) $\displaystyle\lim_{x\to 6} \frac{x^2 - 36}{x^2 - 5x - 6}$.

Solution: This has the indeterminate form "$\frac{0}{0}$", so work is needed. We factor the numerator and denominator, and then cancel common factors in the fraction:

$$\lim_{x\to 6} \frac{x^2 - 36}{x^2 - 5x - 6} = \lim_{x\to 6} \frac{(x-6)(x+6)}{(x-6)(x+1)} = \lim_{x\to 6} \frac{x+6}{x+1} = \frac{6+6}{6+1} = \frac{12}{7}.$$

(b) $\displaystyle\lim_{t\to\infty} \frac{t^3 - 2t^2 - 3t}{7t^3 + \cos(t)}$

Solution: The limit has the indeterminate form "$\frac{\infty}{\infty}$". Therefore more work is needed. We factor out t^3 from both the numerator and denominator, and then use the limit laws:

$$\lim_{t\to\infty} \frac{t^3 - 2t^2 - 3t}{7t^3 + \cos(t)} = \lim_{t\to\infty} \frac{1 - \frac{2}{t} - \frac{3}{t^2}}{7 + \frac{\cos(t)}{t^3}} = \frac{1 - \lim_{t\to\infty} \frac{2}{t} - \lim_{t\to\infty} \frac{3}{t^2}}{7 + \lim_{t\to\infty} \frac{\cos(t)}{t^3}}.$$

Certainly $\lim_{t\to\infty} \frac{2}{t}$ and $\lim_{t\to\infty} \frac{3}{t^2}$ are zero. Since $-1 \leq \cos(t) \leq 1$ for all t, and $t^3 \to \infty$ as $t \to \infty$, we get $\lim_{t\to\infty} \frac{\cos(t)}{t^3} = 0$. Therefore

$$\lim_{t\to\infty} \frac{t^3 - 2t^2 - 3t}{7t^3 + \cos(t)} = \frac{1}{7}.$$

Here is a different way to arrange essentially the same calculation:

$$\lim_{t\to\infty}\frac{t^3-2t^2-3t}{7t^3+\cos(t)} = \lim_{t\to\infty}\frac{\left(\frac{1}{t^3}\right)(t^3-2t^2-3t)}{\left(\frac{1}{t^3}\right)(7t^3+\cos(t))} = \lim_{t\to\infty}\frac{1-\frac{2}{t}-\frac{3}{t^2}}{7+\frac{\cos(t)}{t^3}}$$

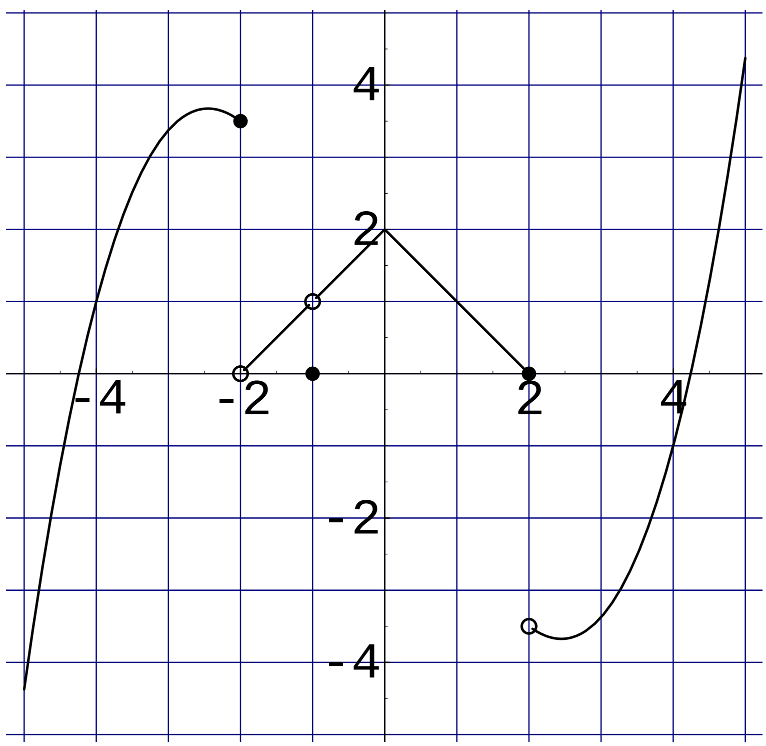
$$= \frac{1-\lim_{t\to\infty}\frac{2}{t}-\lim_{t\to\infty}\frac{3}{t^2}}{7+\lim_{t\to\infty}\frac{\cos(t)}{t^3}} = \frac{1+0+0}{7+0} = \frac{1}{7}.$$

(c) $\lim_{x\to 3}\dfrac{\sqrt{x}-\sqrt{3}}{x-3}$.

Solution: This has the indeterminate form "$\frac{0}{0}$", so work is needed. We rationalize the numerator, and then cancel common factors:

$$\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3} = \lim_{x\to 3}\frac{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}{(x-3)(\sqrt{x}+\sqrt{3})} = \lim_{x\to 3}\frac{x-3}{(x-3)(\sqrt{x}+\sqrt{3})} = \lim_{x\to 3}\frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

5. (4 points/part.) For the function $y = f(x)$ graphed below, answer the following questions:



(a) Give the largest interval containing 0 on which f is continuous.

Solution: The largest interval containing 0 on which f is continuous is $(-1, 2)$. (Note that f is continuous at neither 2 nor at -1, since $\lim_{x\to 2} f(x)$ does not exist, while $\lim_{x\to -1} f(x) = 1 \neq f(0)$.)

(b) Which of the following best describes $f'(3)$? Why?

(1) $f'(3)$ does not exist.

(2) $f'(3)$ is close to 0.

(3) $f'(3)$ is positive and not close to 0.
(4) $f'(3)$ is negative and not close to 0.

Solution: $f'(3)$ is the slope of the tangent line to the graph of $y = f(x)$ at $x = 3$. You can tell from inspection that this slope is positive and not close to 0 (choice (3) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere near 1.

6. (10 points) Find the equation of tangent line to the graph of $f(x) = \dfrac{x}{e^x}$ at $x = -2$. You need not calculate the derivative directly from the definition. (Note that e might appear several places in the answer.)

We need the slope, which is $f'(-2)$, and a point on the line, such as

$$(-2,\, f(-2)) = \left(-2,\, \frac{-2}{e^{-2}}\right) = (-2,\, -2e^2).$$

Now the quotient rule gives

$$f'(x) = \frac{1 \cdot e^x - xe^x}{(e^x)^2} = \frac{e^x(1-x)}{(e^x)^2} = \frac{1-x}{e^x},$$

so

$$f'(-2) = \frac{1-(-2)}{e^{-2}} = 3e^2.$$

Therefore the equation is $y + 2e^2 = 3e^2(x - (-2))$, which can be rearranged to give $y = -2e^2 + 3e^2x + 6e^2$, or $y = 3e^2x + 4e^2$. (If you like, you can rewrite this as $y = e^2(3x + 4)$, but this is not necessary.)

Note that we want the slope at the *particular* value $x = -2$. Therefore we must substitute $x = -2$ in the formula for the derivative $f'(x)$ *before* using it as the slope of a line. The equation

$$y + 2e^2 = \left(\frac{1-x}{e^x}\right)(x - (-2))$$

is wrong—it is not even the equation of a line.

Alternate solution (partial): We can do the differentiation differently. Rewrite the function as $f(x) = xe^{-x}$. Now use the product and chain rules to get $f'(x) = e^{-x} + xe^{-x}(-1) = (1-x)e^{-x}$.

7. (4 points/part) A calculus book is thrown upwards on the planet Yuggxth at 18 meters/second. Its height t seconds after it is thrown is $18t - 3t^2$ meters, until it hits the ground again.

(a) Is the calculus book falling or rising 4 seconds after being thrown? How fast?

Solution: Let $y(t)$ be the height, in meters, of the calculus book t seconds after it is thrown. Then $y(t) = 18t - 3t^2$ for t at most the time at which it hits the ground. Since $y(4) = (18)(4) - (3)(4^2) = 72 - 48 > 0$, after 4 seconds it hasn't hit the ground yet. Therefore the vertical velocity at time t is $y'(t) = 18 - 6t$, and the vertical velocity at time 4 is $y'(4) = 18 - (6)(4) = -6$. Therefore the calculus book is falling at 6 meters/second.

Note: You *must* include the units in this kind of problem.

(b) How high does the calculus book get, and when does it get reach that height?

Solution: Let $y(t)$ be the height, in meters, of the calculus book t seconds after it is thrown. Then $y(t) = 18t - 3t^2$ for t at most the time at which it hits the ground. The calculus book reaches its greatest height when it stops rising and starts falling, that is, when the derivative $y'(t)$ changes from positive to negative. We have $y'(t) = 18 - 6t$, which changes from positive to negative when it is zero, that is, at $t = 3$. So it reaches its greatest height 3 seconds after being thrown, and the height is $y(3) = (18)(3) - (3)(3^2) = 27$ meters above the ground.

Note: You *must* include the units in this kind of problem.

8. (11 points) If $xy = \cos(x + y) + \sin(6)$, find $\frac{dy}{dx}$ by implicit differentiation. (You must solve for $\frac{dy}{dx}$.)

Solution: Differentiate both sides with respect to x, using the product rule on the left and the chain rule on the right:

$$1 \cdot y + x\frac{dy}{dx} = -\sin(x+y)\frac{d}{dx}(x+y) = -\sin(x+y)\left(1 + \frac{dy}{dx}\right).$$

(The derivative of $\sin(6)$ is immediately seen to be zero because $\sin(6)$ is a constant.) Now solve for $\frac{dy}{dx}$:

$$\begin{aligned} y + x\frac{dy}{dx} &= -\sin(x+y) - \sin(x+y)\frac{dy}{dx} \\ [x + \sin(x+y)]\frac{dy}{dx} &= -\sin(x+y) - y \\ \frac{dy}{dx} = \frac{-\sin(x+y) - y}{x + \sin(x+y)} &= -\frac{y + \sin(x+y)}{x + \sin(x+y)}. \end{aligned}$$

This fraction can't be further simplified.

For those who prefer the other notation, here it is written with y as an explicit function $y(x)$ of x. Start with

$$xy(x) = \cos(x + y(x)) + \sin(6).$$

Then differentiate with respect to x, just as before:

$$1 \cdot y(x) + xy'(x) = -\sin(x + y(x))\frac{d}{dx}(x + y(x)) = -\sin(x + y(x))\left(1 + y'(x)\right).$$

Now solve for $y'(x)$:

$$\begin{aligned} y(x) + xy'(x) &= -\sin(x + y(x)) - \sin(x + y(x))y'(x) \\ [x + \sin(x + y(x))]y'(x) &= -\sin(x + y(x)) - y(x) \\ y'(x) = \frac{-\sin(x + y(x)) - y(x)}{x + \sin(x + y(x))} &= -\frac{y(x) + \sin(x + y(x))}{x + \sin(x + y(x))}. \end{aligned}$$

As before, this fraction can't be further simplified.