



1. (1 point) Are you awake?

2. (16 points) Helium is leaking from a spherical balloon. At a certain time, helium is being lost at the rate of 2 cubic centimeters per minute, and the radius of the balloon is 10 centimeters. At what rate is the radius decreasing at that time? (Be sure to include the correct units in your answer.)

3. (8 points) Let  $q(x) = \pi^4 - \sin(x^2 \ln(x))$ . Find  $q'(x)$ .

4. (25 points) A xenobiologist wants to wall off a rectangular enclosure for a small herd of fire breathing monsters from the planet Yuggxth. One side of the enclosure will be a long straight river which flows from east to west; no wall is needed here. A second side will be along an already existing wall which is perpendicular to the river, and which can be used as it is. His research grant contains enough money to build 6 kilometers of wall for the remaining two sides. What is the largest area he can enclose?

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

(Show a full mathematical solution. A correct guess with no valid work will receive no credit, and a correct number supported only by heuristic reasoning will receive very little credit.)

5. (5 points) The derivative of the function  $f(x) = (x - 3)e^{-x}$  is given by  $f'(x) = -(x - 4)e^{-x}$ , and the second derivative is given by  $f''(x) = (x - 5)e^{-x}$ . Determine whether  $f$  has a local minimum, a local maximum, or neither at  $x = 4$ .

6. A pendulum swings back and forth on the surface of the earth. The relation between the period  $T$  and the length  $l$  of the pendulum can be modelled by the equation  $T = \sqrt{k l}$ , where  $T$  is measured in seconds,  $l$  is measured in meters, and  $k = 5 \text{ s}^2/\text{m}$ . Answer the following questions, being careful to include units when appropriate.

(a) (1 point) When  $l$  is 20 meters, compute the period of the pendulum.

(b) (7 points) Using part (a) as your base value, give a linear approximation for the period of the pendulum when the length  $l$  is changed to 18 meters.

7. (24 points) Graph the function  $f(x) = x^3 - 3x^2$  using the methods of calculus. In particular, determine exactly the  $x$ -intercept(s),  $y$ -intercept(s), asymptotes, intervals of increase and decrease, critical numbers, local maximums, local minimums, intervals of concavity, and inflection points. Be sure to label your axes, and include the scales. **Be sure to organize your work so that it is easy to follow, and explain what you are doing. A list of equations, with no explanation of how they relate to each other or to the problem, will receive little credit.** Fill in the table (writing “none” if appropriate), and show full work and the graph elsewhere on the page.

Domain.	
$x$ -intercepts.	
$y$ -intercepts.	
Horizontal asymptotes.	
Vertical asymptotes.	
Critical numbers.	
Local minimums.	
Local maximums.	
Intervals of strict increase.	
Intervals of strict decrease.	
Intervals of upwards concavity.	
Intervals of downwards concavity.	
Inflection points.	

8. (13 points) Use the methods of calculus to find the exact values of  $x$  at which the function  $f(x) = \frac{x}{x^2 + 1}$  takes its absolute minimum and maximum on the interval  $[0, 7]$ .

(No credit will be given for correct guesses without supporting work that is valid for general functions of the sort considered in this course.)

Extra credit. (Do not attempt these problems until you have done and checked your answer to all the ordinary problems on this exam. They will only be counted if you get 75 or more points on the main part of this exam.)

EC. Use the following steps to prove that if  $f$  is a function defined for all real numbers, and there is a constant  $k$  such that  $f'(x) = kf(x)$  for all real  $x$ , then there is a constant  $C$  such that  $f(x) = Ce^{kx}$  for all real  $x$ . (That is, prove that the exponential functions are the *only* functions for which the growth rate is proportional to the function value.)

(a) (5 extra credit points) Suppose  $f'(x) = kf(x)$  for all real  $x$ . Define

$$g(x) = \frac{f(x)}{e^{kx}}.$$

Show that  $g'(x) = 0$  for all  $x$ .

(b) (10 extra credit points) Use the Mean Value Theorem to conclude that  $g$  is a constant function. (Show that for any two real numbers  $a < b$ , we have  $g(a) = g(b)$ .)

(c) (5 extra credit points) Explain why parts (a) and (b) show that  $f(x)$  has the required form.