Recall (from the formula for the regular representation when $G$ is discrete):

**Corollary**

Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$. Let $\pi_0 : A \to L(H_0)$ be a representation, and let $\sigma : C^*_r(G, A, \alpha) \to L(H) = L(L^2(G, H_0))$ be the associated regular representation. Let $a = \sum_{g \in G} a_g u_g \in C^*_r(G, A, \alpha)$, with $a_g = 0$ for all but finitely many $g$. 

For $g \in G$, let $s_g \in L(H_0)$ be the isometry which sends $\eta \in H_0$ to the function $\xi \in L^2(G, H_0)$ given by $$\xi(h) = \begin{cases} \eta & h = g \\ 0 & h \neq g. \end{cases}$$

Then $s_g^* h \sigma(a) s_k = \pi_0(\alpha^{-1} h(a_{hk^{-1}}))$ for all $h, k \in G$. 

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Coefficients in reduced crossed products

Proposition

Let $\alpha: G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$.

Then for each $g \in G$, there is a linear map $E_g: C^*_r(G, A, \alpha) \to A$ with $\|E_g\| \leq 1$ such that if $a = \sum_{g \in G} a_g u_g \in C_c(G, A, \alpha)$, then $E_g(a) = a_g$.

Moreover, with $s_g$ as above, we have $s_g^*h \sigma(a)s_k = \pi_0(\alpha^{-1}(h)(E_{hk}a))$ for all $h, k \in G$.

Proof.

The first part is immediate from the inequality $\|a\|_\infty \leq \|a\|_r$ from the last lecture.

The last statement follows by continuity from "picking off coordinates" in the regular representation.

Proposition

Let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$. Then for each $g \in G$, there is a linear map $E_g : C^*_r(G, A, \alpha) \rightarrow A$ with $\|E_g\| \leq 1$ such that if $a = \sum g \in G a_g u_g \in C_c(G, A, \alpha)$, then $E_g(a) = a_g$. Moreover, with $s_g$ as above, we have $s_g^* h \sigma (a) s_k = \pi_0 (\alpha - 1 h (E_{hk} - 1(a)))$ for all $h, k \in G$. 

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Moreover, with \( s_g \) as above, we have \( s_h^* \sigma(a) s_k = \pi_0(\alpha_h^{-1}(E_{hk^{-1}}(a))) \) for all \( h, k \in G \).
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Coefficients in reduced crossed products: Properties

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Proposition

Let $\alpha: G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$. Then:

1. If $a \in C^*_r(G, A, \alpha)$ and $E_g(a) = 0$ for all $g \in G$, then $a = 0$.
2. If $\pi_0: A \to L(H_0)$ is a nondegenerate representation such that $\bigoplus_{g \in G} \pi_0 \circ \alpha_g$ is injective, then the regular representation $\sigma$ of $C^*_r(G, A, \alpha)$ associated to $\pi_0$ is injective.
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**Proposition**

Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a $\text{C}^*$-algebra $A$. Let the maps $E_g : C^*_r(G, A, \alpha) \to A$ be as in the previous proposition. Then:

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Proof of the properties of coefficients

(1): Let $\pi_0: A \to L(H_0)$ be a representation, and let the notation be as above.

Since $\pi_0$ is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{r}(G, A, \alpha)$ and $\sigma(a) = 0$.

Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1}g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$.

So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C^c(G, A, \alpha)$.

Then $a^* a = \sum_{g, h \in G} u^*_g a^*_g a_h u_h$, so $E_1(a^* a) = \sum_{g \in G} \alpha^{-1}_g(E_g(a)^* E_g(a))$.

In particular, for each fixed $g$, we have $E_1(a^* a) \geq \alpha^{-1}_g(E_g(a)^* E_g(a))$.

By continuity, this inequality holds for all $a \in C^*_{r}(G, A, \alpha)$.

Thus, if $E_1(a^* a) = 0$, then $E_g(a)^* E_g(a) = 0$ for all $g$, so $a = 0$ by Part (1).

This completes the proof.
Proof of the properties of coefficients

(1): Let \( \pi_0 : A \to L(H_0) \) be a representation, and let the notation be as above. If \( a \in C^*_r(G, A, \alpha) \) satisfies \( E_g(a) = 0 \) for all \( g \in G \), then \( s_h^* \sigma(a) s_k = 0 \) for all \( h, k \in G \),
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$$E_1(a^*a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1}(E_g(a)^* E_g(a)).$$

In particular, for each fixed $g$, we have $E_1(a^*a) \geq \alpha_g^{-1}(E_g(a)^* E_g(a))$. By continuity, this inequality holds for all $a \in C^*_r(G, A, \alpha)$. 
Proof of the properties of coefficients

(1): Let $\pi_0 : A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C_r^*(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since $\pi_0$ is arbitrary, it follows that $a = 0$.

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Injective representations of $A$ always give injective regular representations of the reduced crossed product

It is true for general locally compact groups, not just discrete groups, that the regular representation of $C^*_r(G, A, \alpha)$ associated to an injective representation of $A$ is injective. See Theorem 7.7.5 of Pedersen’s book.
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Exercise

Let $\alpha: G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$. 

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1. $E(E(b)) = E(b)$ for all $b \in C^*_r(G, A, \alpha)$. 

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2. If $b \geq 0$ then $E(b) \geq 0$.
3. $\|E(b)\| \leq \|b\|$ for all $b \in C^*_r(G, A, \alpha)$.
4. If $a \in A$ and $b \in C^*_r(G, A, \alpha)$, then $E(ab) = aE(b)$ and $E(ba) = E(b)a$. 
The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_r(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_r(G, A, \alpha)$.

In fact, the situation is intractable even for the case of the trivial action of $\mathbb{Z}$ on $C^*(\mathbb{C})$. In this case, $l_1(\mathbb{Z}) = l_1(\mathbb{C})$. The crossed product is the group C*-algebra $C^*(\mathbb{Z})$, which can be identified with $C(S^1)$. The map $l_1(\mathbb{Z}) \to C(S^1)$ is given by Fourier series: the sequence $a = (a_n)_{n \in \mathbb{Z}}$ goes to the function $\zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$. (This looks more familiar when expressed in terms of $2\pi$-periodic functions on $\mathbb{R}$: it is $t \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$.)

Failure of convergence of $\sum_{n \in \mathbb{Z}} a_n u_n$ corresponds to the fact that the Fourier series of a continuous function need not converge uniformly. Identifying the coefficient sequences which correspond to elements of the crossed product corresponds to giving a criterion for exactly when a sequence $(a_n)_{n \in \mathbb{Z}}$ of complex numbers is the sequence of Fourier coefficients of some continuous function on $S^1$, a problem for which I know of no satisfactory solution.
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Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda: C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$.

Let $\delta_n \in l^2(\mathbb{Z})$ be the function $\delta_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$. Then the Fourier coefficient $a_n$ is recovered as $a_n = \langle \lambda(a) \delta_0, \delta_n \rangle$. That is, $\lambda(a) \delta_0 \in l^2(\mathbb{Z})$ is given by $\lambda(a) \delta_0 = \sum_{n \in \mathbb{Z}} a_n \delta_n$.

Thus, the sequence of Fourier coefficients of a continuous function is always in $l^2(\mathbb{Z})$. (Of course, we already know this, but the calculation here can be applied to more general crossed products.) Unfortunately, this fact is essentially useless for the study of the group C*-algebra. Not only is the Fourier series of a continuous function always in $l^2(\mathbb{Z})$, but the Fourier series of a function in $L^\infty(S_1)$, which is the group von Neumann algebra of $\mathbb{Z}$, is also always in $l^2(\mathbb{Z})$.

One will get essentially no useful information from a criterion which can't even exclude any elements of $L^\infty(S_1)$. 
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As far as I know, this problem is also in general intractable. There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by $\mathbb{Z}$ of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of Davidson's book.

The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder. Computing the K-theory of a crossed product by $\mathbb{Z}/2\mathbb{Z}$ is harder than computing the K-theory of a crossed product by any of $\mathbb{Z}$, $\mathbb{R}$, or even a (nonabelian) free group!
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The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder.
The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_r(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_r(G) \otimes_{\text{min}} A$. As far as I know, this problem is also in general intractable.

There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by $\mathbb{Z}$ of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of Davidson’s book.

The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder. Computing the K-theory of a crossed product by $\mathbb{Z}/2\mathbb{Z}$ is harder than computing the K-theory of a crossed product by any of $\mathbb{Z}$, $\mathbb{R}$, or even a (nonabelian) free group!
We will shortly do some explicit computations of examples. First, though, we give some useful preliminaries:

- Equivariant maps and functoriality.
- Crossed products of exact sequences.
- Crossed products and direct limits.
- Notation for matrix units.

In many of the cases we consider, the ideal structure of the crossed product can be derived from the Gootman-Rosenberg theorem. (See below for a little more about this theorem.) In some cases, one can then use this information to determine the entire structure of the crossed product.
Preliminaries for computing crossed products

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Equivariant homomorphisms

Let $G$ be a locally compact group. A C*-algebra $A$ equipped with an action $G \to \text{Aut}(A)$ will be called a $G$-algebra. We sometimes refer to $(G, A, \alpha)$ as a $G$-algebra.
Equivariant homomorphisms

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**Definition**

If $(G, A, \alpha)$ and $(G, B, \beta)$ are $G$-algebras, then a homomorphism $\varphi : A \to B$ is said to be **equivariant** (or $G$-equivariant if the group must be specified) if for every $g \in G$, we have $\varphi \circ \alpha_g = \beta_g \circ \varphi$. 
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For a fixed locally compact group $G$, the $G$-algebras and equivariant homomorphisms form a category.
The crossed product construction is functorial for equivariant homomorphisms

**Theorem**

Let $G$ be a locally compact group. If $(G, A, \alpha)$ and $(G, B, \beta)$ are $G$-algebras and $\varphi: A \rightarrow B$ is an equivariant homomorphism,
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Let $G$ be a locally compact group. If $(G, A, \alpha)$ and $(G, B, \beta)$ are $G$-algebras and $\varphi: A \to B$ is an equivariant homomorphism, then there is a homomorphism $\psi: C_c(G, A, \alpha) \to C_c(G, B, \beta)$ given by the formula

$$\psi(a)(g) = \varphi(a(g))$$

for $a \in C_c(G, A, \alpha)$ and $g \in G$, and this homomorphism extends by continuity to a homomorphism $L^1(G, A, \alpha) \to L^1(G, B, \beta)$, and then to homomorphisms $C^*(G, A, \alpha) \to C^*(G, B, \beta)$ and $C^*_r(G, A, \alpha) \to C^*_r(G, B, \beta)$.

This construction makes the crossed product and reduced crossed product constructions functors from the category of $G$-algebras to the category of $C^*$-algebras. This is straightforward. See the notes for details.
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This construction makes the crossed product and reduced crossed product constructions functors from the category of $G$-algebras to the category of $\text{C}^\ast$-algebras.

This is straightforward. See the notes for details.
Full crossed products preserve exact sequences

**Theorem**

Let $0 \to J \to A \to B \to 0$ be an exact sequence of $G$-algebras, with actions $\gamma$ on $J$, $\alpha$ on $A$, and $\beta$ on $B$. Then the sequence

$$0 \to C^*(G, J, \gamma) \to C^*(G, A, \alpha) \to C^*(G, B, \beta) \to 0$$

is exact.

Proofs can be found in the three places listed in the notes.

The analog for reduced crossed products is in general false.
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The analog for reduced crossed products is in general false.
Full crossed products preserve direct limits

Theorem

Let \( ( (G, A_i, \alpha^{(i)})_{i \in I}, (\varphi_{j,i})_{i \leq j}) \) be a direct system of \( G \)-algebras.
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Let \( \left( \left( G, A_i, \alpha^{(i)} \right)_{i \in I}, \left( \varphi_{j,i} \right)_{i \leq j} \right) \) be a direct system of \( G \)-algebras. Let \( A = \lim_{\rightarrow} A_i \), with action \( \alpha : G \to \text{Aut}(A) \) given by \( \alpha_g = \lim_{\rightarrow} \alpha_g^{(i)} \).
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Let \(((G, A_i, \alpha^{(i)}))_{i \in I}, (\varphi_{j,i})_{i \leq j}\) be a direct system of \(G\)-algebras. Let \(A = \varprojlim A_i\), with action \(\alpha: G \to \text{Aut}(A)\) given by \(\alpha_g = \varprojlim \alpha^{(i)}_g\). Let

\[\psi_{j,i}: C^*(G, A_i, \alpha^{(i)}) \to C^*(G, A_j, \alpha^{(j)})\]

be the map obtained from \(\varphi_{j,i}\). Using these maps in the direct system of crossed products, there is a natural isomorphism
Full crossed products preserve direct limits

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\[
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\[
C^* \left( G, A, \alpha \right) \cong \lim \rightarrow C^* \left( G, A_i, \alpha^{(i)} \right).
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\]

The proof is done by combining the universal properties of direct limits and crossed products. See the notes.
Notation for matrix units

For any index set $S$, let $\delta_s \in l^2(S)$ be the standard basis vector, determined by

$$\delta_s(t) = \begin{cases} 
1 & t = s \\
0 & t \neq s.
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$$e_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad e_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
Example: The trivial action

Example

If \( G \) acts trivially on the C*-algebra \( A \), then

\[
C^*\left( G, A \right) \cong C^*\left( G \right) \otimes_{\text{max}} A
\]

and

\[
C^r\left( G, A \right) \cong C^r\left( G \right) \otimes_{\text{min}} A
\]

We describe how to see this when \( G \) is discrete and \( A \) is unital. Then

\[
C^*\left( G, A \right)
\]

is the universal unital C*-algebra generated by a unital copy of \( A \) and a commuting unitary representation of \( G \) in the algebra. Since

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C^*\left( G \right)
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If $G$ acts trivially on the $C^*$-algebra $A$, then

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We describe how to see this when \( G \) is discrete and \( A \) is unital. Then \( C^*(G, A) \) is the universal unital C*-algebra generated by a unital copy of \( A \).
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Example: The trivial action (continued)

For the reduced crossed product, the point is that a regular covariant representation of $(G, A)$ has the form $(\lambda \otimes 1_{H^0}, 1_{L^2(G)} \otimes \pi_0)$ for $\pi_0: A \to L(H^0)$ an arbitrary nondegenerate representation and with $\lambda: G \to U(L^2(G))$ being the left regular representation. As we saw above, it suffices to take $\pi_0$ to be a single injective representation. Now we are looking at $C^*_r(G)$ on one Hilbert space and $A$ on another, and taking the tensor product of the Hilbert spaces. This is exactly how one gets the minimal tensor product of two C*-algebras. Note how full and reduced crossed products parallel maximal and minimal tensor products.
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Note how full and reduced crossed products parallel maximal and minimal tensor products.
Example: Inner actions

Example

Let $\alpha: G \to \text{Aut}(A)$ be an inner action of a discrete group $G$ on a unital C*-algebra $A$. 

Thus, there is a homomorphism $g \mapsto \gamma_g$ from $G$ to $\text{U}(A)$ such that $\alpha_g(a) = \gamma_g a \gamma_g^*$ for all $g \in G$ and $a \in A$.

Then $C^*(G, A, \alpha) \cong C^*(G) \otimes_{\text{max}} A$. (This is true even if $G$ is not discrete.)

One shows that the crossed product is the same as for the trivial action.

Let $\iota: G \to \text{Aut}(A)$ be the trivial action of $G$ on $A$.

As usual, for $g \in G$ let $u_g \in C_c(G, A, \alpha)$ be the standard unitary, but let $v_g \in C_c(G, A, \iota)$ be the standard unitary in the crossed product by the trivial action.

Define $\phi_0: C_c(G, A, \alpha) \to C_c(G, A, \iota)$ by $\phi_0(au_g) = az_g v_g$ for $a \in A$ and $g \in G$, and extend linearly.

This map is obviously bijective (the inverse sends $av_g$ to $az_g^* u_g$) and isometric for $\| \cdot \|_1$. 

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SNU crossed products course: Lecture 2 
14 December 2009 20 / 52
Example: Inner actions

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Let \( \alpha: G \to \text{Aut}(A) \) be an inner action of a discrete group \( G \) on a unital C*-algebra \( A \). Thus, there is a homomorphism \( g \mapsto z_g \) from \( G \) to \( U(A) \) such that \( \alpha_g(a) = z_g a z_g^* \) for all \( g \in G \) and \( a \in A \).
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Example: Inner actions (continued)

For multiplicativity, it suffices to check the following, for $a, b \in A$ and $g, h \in H$, 

$$\phi_0(au^g) \phi_0(bu^h) = az^g v^g bz^h v^h = az^g z^h v^g v^h = \phi_0(\alpha g^{-1} (a^* u^g) v^g v^{-1} (b^* h^g)) = \phi_0((au^g) (bu^h)).$$

For preservation of adjoints:

$$\phi_0(au^g) = (az^g v^g)^* = v^g z^* g^* a^* = (z^* g^* a^* z^g v^g)^* = \phi_0(\alpha g^{-1} (a^* u^g)).$$

So $\phi_0$ is an isometric isomorphism of $*$-algebras, and therefore extends to an isomorphism of the universal C*-algebras.
Example: Inner actions (continued)

For multiplicativity, it suffices to check the following, for $a, b \in A$ and $g, h \in H$, using the fact that $v_g$ commutes with all elements of $A$:
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\varphi_0(u_g)\varphi_0(u_h) = az_g v_g b z_h v_h = az_g b z^*_g z_g h v_g v_h
$$

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= a \alpha_g(b) z_g h v_g h = \varphi_0(\alpha_g(b) u_g h) = \varphi_0((u_g)(u_h)).
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For multiplicativity, it suffices to check the following, for $a, b \in A$ and $g, h \in H$, using the fact that $v_g$ commutes with all elements of $A$:

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$$

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$$
\varphi_0(a u_g)^* = (az_g v_g b z_h v_h)^* = v_h^* g z^* g a^* z^* g v_g^* g v_h = \alpha_g^{-1}(a^*) z^* g v_g^{-1} z^* g v_h = \varphi_0(\alpha_g^{-1}(a^*) u_{gh}).
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Example: $G$ acting on itself by translation

Example

If $G$ is discrete and acts on itself by translation, then the crossed product is $K(l^2(G))$. 

Thus $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$.

Also, span($\{\delta_g : g \in G\}$) is dense in $C^*(G, C^0(G), \alpha)$.

For $g, h \in G$, we have $v_{g, h} = \delta_g u_{gh}^{-1} \in C^*(G, C^0(G), \alpha)$.

Moreover, $v_{g_1, h_1} v_{g_2, h_2} = \delta_{g_1} u_{g_1 h_1}^{-1} \delta_{g_2} u_{g_2 h_2}^{-1} = \delta_{g_1} \alpha_{g_1 h_1}^{-1}(\delta_{g_2}) u_{g_1 h_1}^{-1} u_{g_2 h_2}^{-1}$.

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1, h_2}$.

Similarly, $v^*_{g, h} = v_{h, g}$.

That is, the elements $v_{g, h}$ satisfy the relations for a system of matrix units indexed by $G$. 

Also, span($\{v_{g, h} : g, h \in G\}$) is dense in $l^1(G, C^0(G), \alpha)$, and hence in $C^*(G, C^0(G), \alpha)$. 
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If $G$ is discrete and acts on itself by translation, then the crossed product is $K(l^2(G))$. (This is actually true for general $G$.)

Let $\alpha: G \to \text{Aut}(C_0(G))$ denote the action. For $g \in G$, let $u_g$ be the standard unitary, and let $\delta_g \in C_0(G)$ be the function $\chi_{\{g\}}$. 

Thus $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. Also, $\text{span}\{\delta_g: g \in G\}$ is dense in $C_0(G)$. 

For $g, h \in G$, we have $v_g, h = \delta_g u_{gh} - 1 \in C^*(G, C_0(G), \alpha)$. Moreover, $v_g, 1v_g, h = \delta_g u_g h - 1 \delta_g u_g h - 1 = \delta_g \alpha_g h - 1(\delta_g) u_g h - 1 u_g h - 1 = \delta_g \delta_{gh}$. 

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_g, h$. Similarly, $v^* g, h = v h, g$. That is, the elements $v_g, h$ satisfy the relations for a system of matrix units indexed by $G$. 

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\[ v_g, h = \delta_g u_g h^{-1} \in C^*(G, C_0(G), \alpha) \] 

Moreover, 

\[ v_{g_1}, h_1 v_{g_2}, h_2 = \delta_{g_1} u_{g_1} h_1^{-1} \delta_{g_2} u_{g_2} h_2^{-1} = \delta_{g_1} \alpha_{g_1}(h_1^{-1}) (\delta_{g_2}) u_{g_1} h_1^{-1} u_{g_2} h_2^{-1} = \delta_{g_1} \delta_{g_1} h_1^{-1} g_2 u_{g_1} h_1^{-1} u_{g_2} h_2^{-1} . \]

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1}, h_2$.

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Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is $v_{g_1,h_2}$. Similarly, $v_{g,h}^* = v_{h,g}$. That is, the elements $v_{g,h}$ satisfy the relations for a system of matrix units indexed by $G$. 
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sending the matrix unit $e_{g,h} \in L(l^2(F))$ to $v_{g,h}$. If $G$ is finite, we have a surjective homomorphism $L(l^2(G)) \to C^*(G, C_0(G), \alpha)$. 

Since the full crossed product is simple, the map to the reduced crossed product is an isomorphism. If $G$ acts on $G \times X$ by translation in the first factor and trivially in the second factor, we get the crossed product $C(X, K(l^2(G)))$. (In fact, this is true for an arbitrary action of $G$ on $X$. See the notes.)
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sending the matrix unit \( e_{g,h} \in L(l^2(F)) \) to \( \nu_{g,h} \). If \( G \) is finite, we have a surjective homomorphism \( L(l^2(G)) \to C^*(G, C_0(G), \alpha) \), necessarily injective since \( L(l^2(G)) \) is simple.
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In general, one assembles the maps $\psi_F$ to get an isomorphism $K(l^2(G)) \to C^*(G, C_0(G), \alpha)$. For details, see the notes.
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In general, one assembles the maps $\psi_F$ to get an isomorphism $K(l^2(G)) \to C^*(G, C_0(G), \alpha)$. For details, see the notes.

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For any finite set $F \subset G$, we thus get a homomorphism

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$$

sending the matrix unit $e_{g,h} \in L(l^2(F))$ to $v_{g,h}$. If $G$ is finite, we have a surjective homomorphism $L(l^2(G)) \to C^*(G, C_0(G), \alpha)$, necessarily injective since $L(l^2(G))$ is simple.

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If $G$ acts on $G \times X$ by translation in the first factor and trivially in the second factor, we get the crossed product $C(X, K(l^2(G)))$. (In fact, this is true for an arbitrary action of $G$ on $X$. See the notes.)
Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$

Fix $n \in \mathbb{Z}_{>0}$, and let $G = \mathbb{Z}/n\mathbb{Z}$ act on $S^1$ via the rotation by $2\pi/n$, that is, with generator the homeomorphism $h(\zeta) = e^{2\pi i/n} \zeta$ for $\zeta \in S^1$. 
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Example: \( n = 8 \)

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The heavy sections are the 8 translates of the closed arc \( Y \) from 1 to \( \exp(2\pi i/9) \). They give a quotient of the crossed product isomorphic to \( C(Y, M_8) \).
Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$

(continued)

Here are the details. Let $\alpha \in \text{Aut}(C(S^1))$ be the order $n$ automorphism $\alpha(f) = f \circ h^{-1}$. 

Thus, $\alpha(f)(\zeta) = f(e^{-2\pi i/n}\zeta)$ for $\zeta \in S^1$. 

Let $s \in M_n$ be the shift unitary $s = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \end{pmatrix}$.

The key computation is $s\operatorname{diag}(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n)s^* = \operatorname{diag}(\lambda_n,\lambda_1,\lambda_2,\ldots,\lambda_n-1)$. 

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\[
s = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\
1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
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\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
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Example: \( \mathbb{Z}/n\mathbb{Z} \) acting on \( S^1 \) by rotation by \( e^{2\pi i/n} \)
(continued)

Set

\[
B = \{ f \in C([0,1], M_n) : f(0) = sf(1)s^* \}.
\]
Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$ (continued)

Set

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Define $\varphi_0 : C(S^1) \to B$ by sending $f \in C(S^1)$ to the continuously varying diagonal matrix

$$\varphi_0(f)(t) = \text{diag}(f(e^{2\pi i t/n}), f(e^{2\pi i(t+1)/n}), \ldots, f(e^{2\pi i(t+n-1)/n})).$$

(For fixed $t$, the diagonal entries are obtained by evaluating $f$ at the points in the orbit of $e^{2\pi it/n}$.)
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$$\varphi_0(\alpha(f))(t) = \text{diag}(f(e^{2\pi i(t-1)/n}), f(e^{2\pi it/n}), \ldots, f(e^{2\pi i(t+n-2)/n}))$$

$$= s\varphi_0(f)(t)s^*.$$
Example: \( \mathbb{Z}/n\mathbb{Z} \) acting on \( S^1 \) by rotation by \( e^{2\pi i/n} \)

(continued)

Now let \( \nu \in C([0,1], M_n) \) be the constant function with value \( s \).
Example: \( \mathbb{Z}/n\mathbb{Z} \) acting on \( S^1 \) by rotation by \( e^{2\pi i/n} \) (continued)

Now let \( \nu \in C([0, 1], M_n) \) be the constant function with value \( s \). Note that \( \nu \in B \).
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(continued)

Now let \( v \in C([0, 1], M_n) \) be the constant function with value \( s \). Note that \( v \in B \). The calculation just done implies that

\[
\varphi_0(\alpha^k(f)) = v^k \varphi_0(f) v^{-k}
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for \( 0 \leq k \leq n - 1 \).
Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$ (continued)

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Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$ (continued)

We prove directly that $\varphi$ is bijective.
Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$ (continued)

We prove directly that $\varphi$ is bijective. We can rewrite $\varphi$ as the map $C(\mathbb{Z}/n\mathbb{Z} \times S^1) \to B$ given by

$$\varphi(f) = \sum_{k=0}^{n-1} \varphi_0(f(k, -))v^k.$$
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Injectivity now reduces to the easily verified fact that if $a_0, a_1, \ldots, a_{n-1} \in M_n$ are diagonal matrices, and $\sum_{k=0}^{n-1} a_k v^k = 0$, then $a_0 = a_1 = \cdots = a_{n-1} = 0$. 

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(continued)

For surjectivity, let $a \in B$, and write

$$a(t) = \begin{pmatrix}
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with $a_{j,k} \in C([0,1])$ for $1 \leq j, k \leq n$. 
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\vdots & \vdots & \ddots & \vdots \\
a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t)
\end{pmatrix}$$

with $a_{j,k} \in C([0, 1])$ for $1 \leq j, k \leq n$. The condition $a \in B$ implies that, taking the indices mod $n$ in $\{1, 2, \ldots, n\}$, we have $a_{j,k}(1) = a_{j+1,k+1}(0)$ for all $j$ and $k$. So we get a well defined element of $C(\mathbb{Z}/n\mathbb{Z} \times S^1)$ via

$$f(l, e^{2\pi i(t+j)/n}) = a_{l+j,j}(t)$$

for $t \in [0, 1]$, $1 \leq j \leq n$, and $0 \leq l \leq n - 1$, with $l + j$ taken mod $n$ in $\{1, 2, \ldots, n\}$. 

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Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$

(continued)

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Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$ (continued)

It remains to prove that $B \cong C(S^1, M_n)$.
Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$ (continued)

It remains to prove that $B \cong C(S^1, M_n)$. Since $U(M_n)$ is connected, there is a unitary path $t \mapsto s_t$, for $t \in [0, 1]$, such that $s_0 = 1$ and $s_1 = s$. 
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For $f \in C(S^1, M_n)$, we have $\psi(f)(1) = s_1^* f(1)s_1 = s_0^* f(0)s_0 = \psi(f)(0)s_0$, so $\psi(f)$ really is in $B$. It is easily checked that $\psi$ is bijective.
Example: $\mathbb{Z}/n\mathbb{Z}$ acting on $S^1$ by rotation by $e^{2\pi i/n}$ (continued)

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$$\psi(f)(1) = s^* f(1) s = s^* f(0) s = s^* \psi(f)(0) s.$$
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Example: \( x \mapsto -x \) on \( S^n \)

Example

Let \( X = S^n = \{ x \in \mathbb{R}^{n+1} : \|x\|_2 = 1 \} \), and let \( \mathbb{Z}/2\mathbb{Z} \) act by sending the nontrivial group element to the order 2 homeomorphism \( x \mapsto -x \).
Example: $x \mapsto -x$ on $S^n$

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Example: Complex conjugation on $S^1$

**Example**

Take $X = S^1 = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \}$, and let $\mathbb{Z}/2\mathbb{Z}$ act by sending the nontrivial group element to the order 2 homeomorphism $\zeta \mapsto \overline{\zeta}$. Let $\alpha \in \text{Aut}(C(S^1))$ be the corresponding automorphism.
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$$B = \{f \in C([-1, 1], M_2): f(1) \text{ and } f(-1) \text{ are diagonal matrices}\}.$$
Example: Complex conjugation on $S^1$ (continued)

Here are the details. First, let $C_0 \subset M_2$ be the subalgebra consisting of all matrices of the form $\begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix}$ with $\lambda, \mu \in \mathbb{C}$. 
Example: Complex conjugation on $S^1$ (continued)

Here are the details. First, let $C_0 \subset M_2$ be the subalgebra consisting of all matrices of the form $\begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix}$ with $\lambda, \mu \in \mathbb{C}$. Then define

$$C = \{ f : [-1, 1] \to M_2 : f \text{ is continuous and } f(1), f(-1) \in C_0 \}. $$
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$$\varphi_0(f)(t) = \begin{pmatrix} f(t + i\sqrt{1 - t^2}) & 0 \\ 0 & f(t - i\sqrt{1 - t^2}) \end{pmatrix}$$

for $f \in C(S^1)$ and $t \in [-1, 1]$. One checks that the conditions at $\pm 1$ for membership in $C$ are satisfied. Moreover, $v^2 = 1$ and $v \varphi_0(f) v^* = \varphi_0(\alpha(f))$ for $f \in C(S^1)$. 
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Example: Complex conjugation on $S^1$ (continued)
Picture for the definition of $\varphi_0(f)$

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Example: Complex conjugation on $S^1$ (continued)

Therefore there is a homomorphism $\varphi: C^*(\mathbb{Z}/2\mathbb{Z}, X) \to C$ such that $\varphi|_{C(S^1)} = \varphi_0$ and $\varphi$ sends the standard unitary $u$ in $C^*(\mathbb{Z}/2\mathbb{Z}, X)$ to $v$. It is given by the formula

$$\varphi(f_0 + f_1 u)(t) = (f_0(t + i\sqrt{1-t^2}) f_1(t + i\sqrt{1-t^2}))$$

for $f_1, f_2 \in C(S^1)$ and $t \in [-1, 1]$.

We claim that $\varphi$ is an isomorphism. Since $C^*(\mathbb{Z}/2\mathbb{Z}, X) = \{f_0 + f_1 u : f_1, f_2 \in C(S^1)\}$, it is easy to check injectivity.
Example: Complex conjugation on $S^1$ (continued)

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Example: Complex conjugation on \( S^1 \) (continued)

Therefore there is a homomorphism \( \varphi: C^*(\mathbb{Z}/2\mathbb{Z}, X) \rightarrow \mathbb{C} \) such that \( \varphi|_{C(S^1)} = \varphi_0 \) and \( \varphi \) sends the standard unitary \( u \) in \( C^*(\mathbb{Z}/2\mathbb{Z}, X) \) to \( v \). It is given by the formula

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We claim that \( \varphi \) is an isomorphism. Since

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C^*(\mathbb{Z}/2\mathbb{Z}, X) = \{ f_0 + f_1 u : f_1, f_2 \in C(S^1) \},
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it is easy to check injectivity.
Example: Complex conjugation on $S^1$ (continued)

For surjectivity, let

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}$$

define an element $a \in C$. 

Example: Complex conjugation on $S^1$ (continued)

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define an element $a \in C$. Then

$$a_{1,1}(-1) = a_{2,2}(-1) \quad \text{and} \quad a_{2,1}(-1) = a_{1,2}(-1), \quad (1)$$
Example: Complex conjugation on $S^1$ (continued)

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Now set

$$f_0(\zeta) = \begin{cases} a_{1,1}(\text{Re}(\zeta)) & \text{Im}(\zeta) \geq 0 \\ a_{2,2}(\text{Re}(\zeta)) & \text{Im}(\zeta) \leq 0 \end{cases}$$
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for $\zeta \in S^1$. 

Example: Complex conjugation on $S^1$ (continued)

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for $\zeta \in S^1$. The relations (1) and (2) ensure that $f_0$ and $f_1$ are well defined at $\pm 1$, and are continuous.
Example: Complex conjugation on $S^1$ (continued)

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for $\zeta \in S^1$. The relations (1) and (2) ensure that $f_0$ and $f_1$ are well defined at $\pm 1$, and are continuous. One easily checks that $\varphi(f_0 + f_1 u) = a$. This proves surjectivity.
Example: Complex conjugation on $S^1$ (continued)

The algebra $C$ is not quite what was promised. Set

$$w = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
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\end{pmatrix},$$

which is a unitary in $M_2$.

(Check this!)
Example: Complex conjugation on $S^1$ (continued)

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In this example, one choice of matrix units in $M_2$ was convenient for the free orbits, while another choice was convenient for the fixed points.
Example: Complex conjugation on $S^1$ (continued)

The algebra $C$ is not quite what was promised. Set

$$w = \begin{pmatrix} 1 \sqrt{2} & 1 \sqrt{2} \\ -1 \sqrt{2} & 1 \sqrt{2} \end{pmatrix},$$

which is a unitary in $M_2$. Then the isomorphism $\psi: C^*(\mathbb{Z}/2\mathbb{Z}, X) \to B$ is given by $\psi(a)(t) = w\varphi(a)(t)w^*$. (Check this!)

In this example, one choice of matrix units in $M_2$ was convenient for the free orbits, while another choice was convenient for the fixed points. It seemed better to compute everything in terms of the choice convenient for the free orbits, and convert afterwards.
Example: \( x \mapsto -x \) on \([-1, 1]\)

Exercise

Let \( \mathbb{Z}/2\mathbb{Z} \) act on \([-1, 1]\) via \( x \mapsto -x \). Compute the crossed product.
Example: \((x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (x_1, x_2, \ldots, x_n, -x_{n+1})\) on \(S^n\)

Exercise

Let \(\mathbb{Z}/2\mathbb{Z}\) act on

\[ S^n = \{(x_1, x_2, \ldots, x_{n+1}): x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\} \]

via \((x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (x_1, x_2, \ldots, x_n, -x_{n+1})\). Compute the crossed product.
Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation

Example

Let $X = \mathbb{Z}/n\mathbb{Z}$, and let $\mathbb{Z}$ act on $X$ by translation. We show that $C^*(\mathbb{Z}, X) \cong M_n \otimes C(S^1)$. 

Note that there is no twisting. We will be sketchy. See the notes for details.
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This is a special case of $G$ acting on $G/H$ by translation. In the general case, it turns out that $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no twisting.
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Identify $\mathbb{Z}/n\mathbb{Z}$ with $\{1, 2, \ldots, n\}$. (We start at 1 instead of 0 to be consistent with common matrix unit notation.)
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Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

In $C(S^1)$ let $z$ be the function $z(\zeta) = \zeta$ for all $\zeta$. 
Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

In $C(S^1)$ let $z$ be the function $z(\zeta) = \zeta$ for all $\zeta$. In $M_n(C(S^1)) \cong M_n \otimes C(S^1)$, abbreviate $e_{j,k} \otimes 1$ to $e_{j,k}$,
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\[
\nu = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 & 0 & z \\
1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\
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\end{pmatrix}.
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0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 1 & 0 
\end{pmatrix}.
$$

(This unitary differs from the unitary $s$ used before only in that here the upper right corner entry is $z$ instead of 1.)
Define $\varphi_0 : C(\mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ by $\varphi_0(\chi_{\{k\}}) = e_{k,k}$. 
Define $\varphi_0 : C(\mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ by $\varphi_0(\chi_{\{k\}}) = e_{k,k}$. Then one checks that $\nu \varphi_0(f)\nu^* = \varphi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$. 

Therefore there is a homomorphism $\varphi : C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ such that $\varphi|_{C(\mathbb{Z}/n\mathbb{Z})} = \varphi_0$ and $\varphi(u) = v$. We claim that $\varphi$ is an isomorphism.

We use the following description of $M_n \otimes C(S^1)$: it is the universal unital $C^*$-algebra generated by a system $(e_{j,k})_{1 \leq j,k \leq n}$ of matrix units such that $\sum_{j=1}^n e_{j,j} = 1$ and a central unitary $y$.

The $e_{j,k}$ are the matrix units we have already used, and the central unitary is $1 \otimes z$.

(Proof: Exercise.)
Define $\varphi_0 : C(\mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ by $\varphi_0(\chi_{\{k\}}) = e_{k,k}$. Then one checks that $v\varphi_0(f)v^* = \varphi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$. Therefore there is a homomorphism $\varphi : C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ such that $\varphi|_{C(\mathbb{Z}/n\mathbb{Z})} = \varphi_0$ and $\varphi(u) = v$. We claim that $\varphi$ is an isomorphism. We use the following description of $M_n \otimes C(S^1)$: it is the universal unital $C^*$-algebra generated by a system $(e_{j,k})_{1 \leq j, k \leq n}$ of matrix units such that $\sum_{j=1}^n e_{j,j} = 1$ and a central unitary $y$. The $e_{j,k}$ are the matrix units we have already used, and the central unitary is $1 \otimes z$. (Proof: Exercise.)
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Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

Define $\varphi_0 : C(\mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ by $\varphi_0(\chi_{\{k\}}) = e_{k,k}$. Then one checks that $\nu \varphi_0(f) \nu^* = \varphi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$. Therefore there is a homomorphism $\varphi : C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ such that $\varphi|_{C(\mathbb{Z}/n\mathbb{Z})} = \varphi_0$ and $\varphi(u) = \nu$. We claim that $\varphi$ is an isomorphism.

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To prove that $\varphi$ is surjective, it suffices to prove that its image contains the generators above.
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To prove injectivity, it suffices to prove that whenever $A$ is a unital C*-algebra, $\psi_0: C(\mathbb{Z}/n\mathbb{Z}) \rightarrow A$ is a unital homomorphism, and $w \in A$ is a unitary such that $w\psi_0(f)w^* = \psi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$, then there is a homomorphism $\gamma: \mathcal{M}_n \otimes C(S^1) \rightarrow A$ such that $\gamma \circ \varphi_0 = \psi_0$ and $\gamma(v) = w$. That is, we are showing that $\mathcal{M}_n \otimes C(S^1)$ satisfies the universal property of the crossed product.

If $\varphi$ were not injective, taking $\psi_0$ and $w$ to come from $\text{id}_{C^*}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ would yield a contradiction. It suffices to define $\gamma$ on the generators above. The rest of the details are omitted; see the notes. The main point is to use the description of $\mathcal{M}_n \otimes C(S^1)$ as the universal C*-algebra on the generators and relations above.
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The rest of the details are omitted; see the notes. The main point is to use the description of $M_n \otimes C(S^1)$ as the universal C*-algebra on the generators and relations above.
Where ideals in crossed products come from

We have implicitly seen two sources of ideals in a reduced crossed product $C^*_r(G, A, \alpha)$:

invariant ideals in $A$, and group elements which act trivially on $A$.

There is a theorem due to Gootman and Rosenberg which gives a description of the primitive ideals of any crossed product $C^*_r(G, A)$ with $G$ amenable, and which, very roughly, says that they all come from some combination of these two sources.

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Example: A product type action

We consider the action of $\mathbb{Z}/2\mathbb{Z}$ on the $2^\infty$ UHF algebra $A$ generated by $igotimes_{n=1}^{\infty} \text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$.
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Write $A = \lim_{\rightarrow} M_{2^n}$, with maps $\varphi_n: M_{2^n} \to M_{2^{n+1}}$ given by $a \mapsto \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right)$. 
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Define unitaries $z_n \in M_{2^n}$ inductively by $z_0 = 1$ and $z_{n+1} = \left( \begin{array}{cc} z_n & 0 \\ 0 & -z_n \end{array} \right)$.
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Write $A = \lim \longrightarrow M_{2^n}$, with maps $\varphi_n: M_{2^n} \rightarrow M_{2^{n+1}}$ given by $a \mapsto \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right)$. Define unitaries $z_n \in M_{2^n}$ inductively by $z_0 = 1$ and $z_{n+1} = \left( \begin{array}{cc} z_n & 0 \\ 0 & -z_n \end{array} \right)$. In tensor product notation, and with an appropriate choice of isomorphism $M_{2^n} \otimes M_2 \rightarrow M_{2^{n+1}}$, these are
Example: A product type action

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Let

$$\overline{\varphi}_n : C^*(\mathbb{Z}/2\mathbb{Z}, M_{2^n}, \text{Ad}(z_n)) \to C^*(\mathbb{Z}/2\mathbb{Z}, M_{2^{n+1}}, \text{Ad}(z_{n+1}))$$

be the corresponding map on the crossed products.
Example: A product type action (continued)

From what we did with inner actions, we get isomorphisms

$$\sigma_n: C^*(\mathbb{Z}/2\mathbb{Z}, M_{2^n}, \text{Ad}(z_n)) \to M_{2^n} \oplus M_{2^n}$$

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$$\psi_n : M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$$

which makes the following diagram commute:
Example: A product type action (continued)

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which makes the following diagram commute:

\[
\begin{array}{ccc}
C^*(\mathbb{Z}/2\mathbb{Z}, M_{2n}, \text{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2n} \oplus M_{2n} \\
\downarrow{\bar{\varphi}_n} & & \downarrow{\psi_n} \\
C^*(\mathbb{Z}/2\mathbb{Z}, M_{2n+1}, \text{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2n+1} \oplus M_{2n+1}
\end{array}
\]
Example: A product type action (continued)

We need a map

\[
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which makes the diagram on the previous slide commute. That is, \( \psi_n \) sends

\[
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We have enough information to write an explicit formula for this expression (see the notes),
Example: A product type action (continued)

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which makes the diagram on the previous slide commute. That is, \( \psi_n \) sends

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to

\[ \sigma_{n+1}(\varphi_n(a) + \varphi_n(b)u_{n+1}) \].

We have enough information to write an explicit formula for this expression (see the notes), and it turns out that we can take

\[ \psi_n(b, c) = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \].
Example: A product type action (continued)

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Remarks on the product type example

The fact that we got the same algebra back is somewhat special, but the general principle of the computation is much more generally applicable.
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The theorem of Gootman and Rosenberg described above gives no information here.

Exercises 4.23 and 4.24 in the notes combine direct limit methods with computations of the sort done above.
Example: The irrational rotation algebras

Let $\theta \in \mathbb{R}$. Recall that the rotation algebra $A_\theta$ is the universal C*-algebra generated by unitaries $u$ and $v$ satisfying $vu = e^{2\pi i \theta} uv$. 
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Let $h_\theta : S^1 \to S^1$ be the homeomorphism $h_\theta(\zeta) = e^{2\pi i \theta} \zeta$. We claim that there is an isomorphism $\phi : A_\theta \to C^*(\mathbb{Z}, S^1, h_\theta)$ which sends $u$ to the standard unitary $u_1$ in the crossed product,
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Let $h_\theta : S^1 \to S^1$ be the homeomorphism $h_\theta(\zeta) = e^{2\pi i \theta} \zeta$. We claim that there is an isomorphism $\varphi : A_\theta \to C^*(\mathbb{Z}, S^1, h_\theta)$ which sends $u$ to the standard unitary $u_1$ in the crossed product, and sends $v$ to the function $z \in C(S^1)$ defined by $z(\zeta) = \zeta$ for all $\zeta \in S^1$. 
The proof of the claim is by comparison of universal properties.

First, one checks that
\[ z u_1 = e^{2 \pi i \theta} u_1 z, \]
so at least there is a homomorphism \( \phi \) as claimed.

Next, define \( \psi_0 : C(S^1) \to A_\theta \) by
\[ \psi_0(f) = f(v) \] (continuous functional calculus) for \( f \in C(S^1) \).

For \( n \in \mathbb{Z} \), we have
\[ u \psi_0(z_n u^*) = (uvu^*)^n = e^{2 \pi i \theta} v^n = \psi_0(e^{2 \pi i \theta} z_n) = \psi_0(z_n \circ h^{-1} \theta) . \]

Since the functions \( z_n \) span a dense subspace of \( C(S^1) \), it follows that
\[ u \psi_0(f) u^* = \psi_0(f \circ h^{-1} \theta) \]
for all \( f \in C(S^1) \).

By the universal property of the crossed product, there is a homomorphism \( \psi : C^*(\mathbb{Z}, S^1, h \theta) \to A_\theta \) such that \( \psi |_{C(S^1)} = \psi_0 \) and \( \psi(u_1) = u_1 \).

We have \((\psi \circ \phi)(u) = u \) and \((\psi \circ \phi)(v) = v \).

Since \( u \) and \( v \) generate \( A_\theta \), we conclude that \( \psi \circ \phi = \text{id}_{A_\theta} \).

Similarly, one proves \( \phi \circ \psi = \text{id}_{C^*(\mathbb{Z}, S^1, h \theta)} \).

We will see below that for \( \theta \in \mathbb{R} \setminus \mathbb{Q} \), the algebra \( C^*(\mathbb{Z}, S^1, h \theta) \) is simple.
Example: The irrational rotation algebras (continued)

The proof of the claim is by comparison of universal properties. First, one checks that $zu_1 = e^{2\pi i \theta} u_1 z$, so at least there is a homomorphism $\varphi$ as claimed.

Next, define $\psi_0: C(S^1) \to A_\theta$ by $\psi_0(f) = f(v)$ (continuous functional calculus) for $f \in C(S^1)$. For $n \in \mathbb{Z}$, we have $u \psi_0(z^n) u^* = (uvu^*)^n = e^{2\pi i \theta n} v^n = \psi_0(e^{2\pi i \theta} z^n) = \psi_0(z^n \circ h^{-1} \theta)$.

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By the universal property of the crossed product, there is a homomorphism $\psi: C^*(\mathbb{Z}, S^1, h_\theta) \to A_\theta$ such that $\psi|_{C(S^1)} = \psi_0$ and $\psi(u_1) = u$.

We have $(\psi \circ \varphi)(u) = u$ and $(\psi \circ \varphi)(v) = v$. Since $u$ and $v$ generate $A_\theta$, we conclude that $\psi \circ \varphi = \text{id}_{A_\theta}$.

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