CONJUGACY CLASSES IN $A_n$ ($n \leq 5$) AND APPLICATIONS

As proved in HW#6, conjugacy classes in $S_n$ are very simple to describe, in fact we proved the following Theorem:

**Theorem 1** Two permutations are conjugate in $S_n$ if and only if they have the same cycle structure.

**Proof.** See Hw#6 Extra credit. You can use this Theorem without further proof if you need it. ■

We wish things were this clear for the alternating groups, but they aren’t. (Take a moment to understand why conjugacy classes in $A_n$ can be different from those in $S_n$). The goal of this project is to understand the conjugacy classes in $A_n$ ($n \leq 5$) and obtain some applications.

**Part I**

**Preliminary Results**

Let $G$ be a group. Define on $G$ the conjugation relation by $a \sim b$ iff there exists $g \in G$ such that $b = gag^{-1}$.

1. Show that $\sim$ is an equivalence relation. We will denote the class of $a \in G$ by $[a]_G$ and call it the conjugacy class of $a$ in $G$.

2. What are the conjugacy classes of an Abelian group?

3. If $H$ is a subgroup of $G$ and $a \in H$.
   (i) Show that $[a]_H \subseteq [a]_G$.
   (ii) Give an example of a group $G$, a subgroup $H$ and $a \in H$ so that $[a]_H \nsubseteq [a]_G$.

4. Show that a subgroup $H$ of $G$ is normal in $G$ if and only if $H = \bigcup_{a \in H} [a]_G$.

5. For $a \in G$, $C_G(a)$ denotes the centralizer of $a$ in $G$. (see HW#7 for definition and properties).
   (i) Suppose $|G| < \infty$. Show that for every $a \in G$,
   \[ |[a]_G| = |G : C_G(a)| \]  
   (ii) Show that if $H$ is a subgroup of $G$ and $a \in H$, then $C_H(a) = C_G(a) \cap H$.

**Part II**

**Conjugacy classes in $A_n$ ($n \leq 5$).**

1. Write down explicitly the conjugacy classes in $A_1, A_2, A_3$. 

2. List the conjugacy classes in $S_4$.
3. List the sizes of conjugacy classes in $S_5$. (note that you don’t have to list the permutations)

I want to illustrate through the following example the technique that you will use to describe conjugacy classes in $A_4$ and $A_5$.

I wish to understand the conjugacy class of 3-cycles in $A_5$ that is $[\alpha]_{A_5}$ with $\alpha$ a 3-cycle. If $\alpha$ is a 3-cycle, we know from Theorem 1 and a counting argument that $|[\alpha]|_{S_5} = 20$. As observed in Part 1.3.2i, it may happen that $[\alpha]_{A_5} \subseteq [\alpha]_{S_5}$, which will imply that 3-cycles splits into more than one conjugacy class in $A_5$.

Now let’s see what actually happens. If $\alpha$ is a 3-cycle, we know by equation (*) that $|C_{S_5}(\alpha)|$ = $\frac{|S_5|}{|\alpha|_{S_5}}$ = $\frac{120}{5}$ = 6. Let $t$ be the unique transposition whose support is disjoint from $\alpha$. Observe that $\alpha \in C_{S_5}(\alpha)$, and $t \in C_{S_5}(\alpha)$, hence since $C_{S_5}(\alpha)$ is a subgroup, we get $\varepsilon, \alpha, \alpha^{-1}, t, \alpha t, \alpha^{-1} t$ are all distinct in $C_{S_5}(\alpha)$. But we know that $|C_{S_5}(\alpha)| = 6$, thus $C_{S_5}(\alpha) = \{\varepsilon, \alpha, \alpha^{-1}, t, \alpha t, \alpha^{-1} t\}$. Hence by Part 1.5.2i, $C_{A_5}(\alpha) = \{\varepsilon, \alpha, \alpha^{-1}, t, \alpha t, \alpha^{-1} t\} \cap A_5 = \{\varepsilon, \alpha, \alpha^{-1}\}$. Thus $|C_{A_5}(\alpha)| = 3$.

Using equation (*), we obtain $|[\alpha]|_{A_5}$ = $\frac{|A_5|}{|C_{A_5}(\alpha)|}$ = $\frac{60}{3}$ = 20. Since $[\alpha]_{A_5} \subseteq [\alpha]_{S_5}$ and $|[\alpha]|_{A_5}$ = $|[\alpha]|_{S_5}$ = 20, then $[\alpha]_{A_5} = [\alpha]_{S_5}$. Thus in $A_5$ the 3-cycle are still all in the same conjugacy class. This is nice, but it is not only this way. Put it like this, if you finish the project without realizing this, you must have missed something.

4. List the conjugacy classes in $A_4$.
5. List the sizes of conjugacy classes in $A_5$.

Part III

Applications to Normal subgroups

For this part, you will rely heavily of Part 1.4 and the Lagrange Theorem.

1. Show that $V_4 \supseteq S_4$. Does this imply that $V_4 \supseteq A_4$?
2. Show that $\{\varepsilon\}, V_4, A_4, S_4$ are the only normal subgroups of $S_4$.
3. Show that $\{\varepsilon\}, A_5, S_5$ are the only normal subgroups of $S_5$.
4. Show that $A_5$ have no proper non trivial normal subgroups. (such a group is called a simple group)

Remark 2 One can prove that $A_n$ is simple for every $n \geq 5$. 
Part IV
Applications to Normal series

A subnormal series of a group $G$ is a finite sequence of subgroups $\{H_i\}$ satisfying:
$\{e\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ such that $H_i \unlhd H_{i+1}$ for all $i$. A group $G$ is called solvable if it has a subnormal series $\{H_i\}$ with $H_{i+1}/H_i$ Abelian.

1. Write down all the normal series of $S_4$.
2. Write down all the normal series of $S_5$.
3. Is $S_4$ solvable? what about $S_5$?