Hodge theory for combinatorial geometries

June Huh

with Karim Adiprasito and Eric Katz

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the tropical linear space (*Ardila-Klivans*).

The idea of Richard Stanley that the Hodge structure on the cohomology of projective

toric varieties produces fundamental combinatorial inequalities.

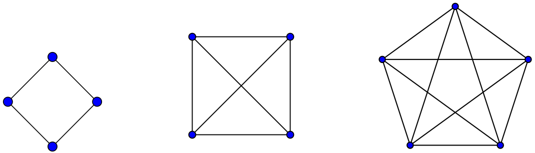
The idea of Peter McMullen that the g-conjecture for polytopes can be proved

using the 'flip connectivity' of simplicial polytopes of given dimension.

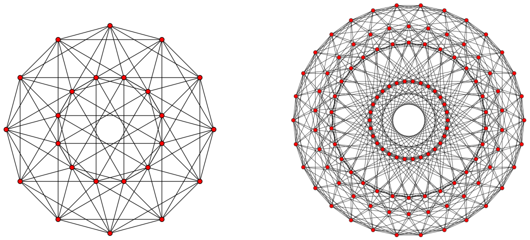
We will not apply any algebraic geometry.

I will present a purely combinatorial solution to a purely combinatorial problem.

A *graph* is a 1-dimensional space, with vertices and edges.



Graphs are the simplest combinatorial structures.



Hassler Whitney (1932): The *chromatic polynomial* of a graph G is the function

$$\chi_G(q) = \text{(the number of proper colorings of } G \text{ with } q \text{ colors).}$$

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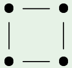
Example



$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q, \qquad \chi_G(2) = 2,\ \chi_G(3) = 18,\ \ \ldots$$

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$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q, \qquad \chi_G(2) = 2,\ \chi_G(3) = 18,\ \ \ldots$$

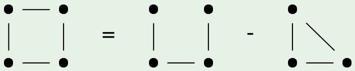
Read's conjecture (1968)

The coefficients of the chromatic polynomial $\chi_G(q)$ form a log-concave sequence for any graph G, that is,

$$a_i^2 \geq a_{i-1}a_{i+1} \ \ \textit{for all } i.$$

Example

How do we compute the chromatic polynomial? We write



and use

$$\begin{aligned}\chi_{G\backslash e}(q) &= q(q-1)^3 \\ \chi_{G/e}(q) &= q(q-1)(q-2).\end{aligned}$$

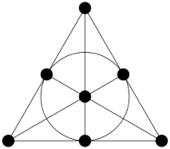
Therefore

$$\chi_G(q) = \chi_{G\backslash e}(q) - \chi_{G/e}(q) = 1q^4 - 4q^3 + 6q^2 - 3q.$$

This algorithmic description of $\chi_G(q)$ makes the prediction of the conjecture interesting.

For any finite set of vectors A in a vector space over a field, define

$$f_i(A) = (\text{number of independent subsets of } A \text{ with size } i).$$



Example

If A is the set of all nonzero vectors in $\mathbb{F}_2^3$, then

$$f_0 = 1, \quad f_1 = 7, \quad f_2 = 21, \quad f_3 = 28.$$

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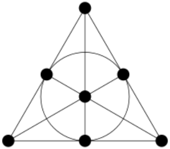

Example

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How do we compute $f_i(A)$? We use

$$f_i(A) = f_i(A \setminus v) + f_{i-1}(A \,/\, v).$$

Welsh's conjecture (1969)

The sequence f_i form a log-concave sequence for any finite set of vectors A

in any vector space over any field, that is,

$$f_i^2 \geq f_{i-1}\, f_{i+1} \text{ for all } i.$$

Hassler Whitney (1935).

A *matroid* M on a finite set E is a collection of subsets of E, called *independent sets*, which satisfy axioms modeled on the relation of linear independence of vectors:

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A *matroid* M on a finite set E is a collection of subsets of E, called *independent sets*, which satisfy axioms modeled on the relation of linear independence of vectors:

1. Every subset of an independent set is an independent set.
2. If an independent set A has more elements than independent set B, then there is an element in A which, when added to B, gives a larger independent set.

We write $n+1$ for the *size* of M, the cardinality of the ground set E.

We write $r+1$ for the *rank* of M, the cardinality of any maximal independent set of M.

In all interesting cases, $r < n$.

1. Let G be a finite graph, and E the set of edges.

   Call a subset of E independent if it does not contain a circuit.

   This defines a *graphic matroid* M.

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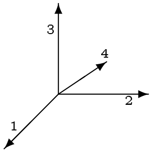
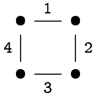
2. Let V be a vector space over a field k, and A a finite set of vectors.

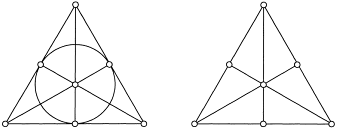
   Call a subset of A independent if it is linearly independent.

   This defines a matroid M *realizable* over k.

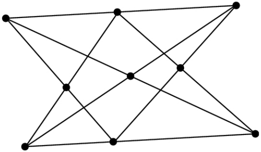




Fano matroid is realizable iff char$(k) = 2$.



Non-Fano matroid is realizable iff char$(k) \neq 2$.



Non-Pappus matroid is not realizable over any field.

How many matroids are realizable over a field?

0% of matroids are realizable.

In other words, almost all matroids are (conjecturally) not realizable over any field.

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Testing the realizability of a matroid over a given field is not easy.

When $k = \mathbb{Q}$, this is equivalent to Hilbert's tenth problem over $\mathbb{Q}$ (*Sturmfels*):

"Is there an algorithm to decide whether a given polynomial equation

with $\mathbb{Q}$ coefficients has a solution over $\mathbb{Q}$?"

One can define the *characteristic polynomial* of a matroid by the recursion

$$\chi_M(q) = \chi_{M \setminus e}(q) - \chi_{M/e}(q).$$

Rota's conjecture (1970)

The coefficients of the characteristic polynomial $\chi_M(q)$ form a log-concave sequence for any matroid M, that is,

$$\mu_i^2 \geq \mu_{i-1}\mu_{i+1} \quad \textit{for all } i.$$

This implies the conjecture on G and the conjecture on A (*Brylawski, Lenz*).

How to show that a given sequence is log-concave?

(*after Teissier and Dimca-Papadima*)

- h : a nonconstant homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_r]$.
- J_h : the jacobian ideal $\big(\partial h/\partial z_0, \ldots, \partial h/\partial z_r\big)$.
- Define the numbers $\mu^i(h)$ by saying that the function

$$\dim_{\mathbb{C}} \mathfrak{m}^u J_h^v / \mathfrak{m}^{u+1} J_h^v$$

  agrees with the polynomial for large enough u and v

$$\frac{\mu^0(h)}{r!} u^r + \cdots + \frac{\mu^i(h)}{(r-i)!i!} u^{r-i} v^i + \cdots + \frac{\mu^r(h)}{r!} v^r + \text{(lower degree terms)}.$$

- The sequence $\mu^i(h)$ is log-concave for any h.

Geometrically, the Milnor numbers are given by the graph Γ of the Gauss map of h:

$$
\begin{tikzcd}
 & \Gamma \subset \mathbb{P}^r \times \mathbb{P}^r \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^r \arrow[rr, dashed, "\left(\frac{\partial h}{\partial z_0} : \frac{\partial h}{\partial z_1} : \cdots : \frac{\partial h}{\partial z_r}\right)"'] & & \mathbb{P}^r
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Γ defines an element in the homology of $\mathbb{P}^r \times \mathbb{P}^r$, which we write as

$$[\Gamma] = \sum_{i=0}^{r} \mu^i(h)[\mathbb{P}^{r-i} \times \mathbb{P}^i] \in H_{2r}(\mathbb{P}^r \times \mathbb{P}^r; \mathbb{Z}).$$

Theorem (-, 2012)

Any nonconstant homogeneous polynomial $h \in \mathbb{C}[z_0, \ldots, z_r]$ *defines a sequence of 'Milnor numbers'* $\mu^0(h), \ldots, \mu^r(h)$ *with the following properties:*

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1. *$\mu^i(h)$ is the number of i-dimensional cells in a CW-model of the complement*

$$D(h) := \{x \in \mathbb{P}^r \mid h(x) \neq 0\}.$$

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When h defines a hyperplane arrangement $\mathscr{A}$, using *Orlik-Solomon* we get

$$\mu^i(h) = \mu_i(\mathscr{A}) := (\text{the } i\text{-th coefficient of the characteristic polynomial of } \mathscr{A}),$$

justifying the log-concavity for matroids realizable over a field of characteristic zero.

Example

- Consider the case when
$$h = z_0^2 z_1^3 \in \mathbb{C}[z_0, z_1].$$

- We have
$$J_h = (2z_0 z_1^3, 3z_0^2 z_1^2) = z_0 z_1^2 (2z_1, 3z_0) = z_0 z_1^2 \mathfrak{m}.$$

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- Therefore
$$\dim_{\mathbb{C}} \mathfrak{m}^u J_h^v / \mathfrak{m}^{u+1} J_h^v = \dim_{\mathbb{C}} \mathfrak{m}^{u+v} / \mathfrak{m}^{u+v+1} = u + v + 1,$$
$$\mu^0(h) = 1 \quad \text{and} \quad \mu^1(h) = 1.$$
- These are the Betti numbers of $D(h) \simeq S^1$.

Example

- Consider the case when
$$h = z_0^d + z_1^d + z_2^d \in \mathbb{C}[z_0, z_1, z_2].$$

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- Therefore

$$\dim_{\mathbb{C}} \mathfrak{m}^u J_h^v / \mathfrak{m}^{u+1} J_h^v = \dim_{\mathbb{C}} \mathfrak{m}^{u+(d-1)v} / \mathfrak{m}^{u+(d-1)v+1},$$

$$\mu^0(h) = 1, \quad \mu^1(h) = d-1, \quad \mu^2(h) = (d-1)^2.$$

Example

This gives

$$\begin{aligned}\chi\big(D(h)\big) &= 1-(d-1)+(d-1)^2,\\ \chi\big(V(h)\big) &= 3-\big(1-(d-1)+(d-1)^2\big)\end{aligned}$$

hence

$$g = 1-\frac{1}{2}\chi\big(V(h)\big) = \frac{1}{2}(d-1)(d-2).$$

The homology group of an algebraic variety is a finitely generated abelian group with several extra structures.

It contains

(i) the set of *prime* classes, the classes of subvarieties,

(ii) the set of *effective* classes, the nonnegative linear combinations of prime classes.

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(ii) the set of *effective* classes, the nonnegative linear combinations of prime classes.

The second set is a cone, and in the case of toric varieties, this cone is generated by the classes of torus orbit closures.

Example

If X is a product of projective spaces, then

$$H_{2k}(X;\mathbb{Z}) = \Big\{\xi = \sum_i d_i[\mathbb{P}^{k-i} \times \mathbb{P}^i], \;\; d_i \in \mathbb{Z}\Big\}.$$

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In this case, some positive multiple of ξ is prime if and only if $\{d_i\}$ form a *log-concave* sequence of *nonnegative* integers with *no internal zeros*.

Example

This structure on the distribution of primes is *not* visible if we do not work up to a multiple.

For example, there is no subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ with the homology class

$$1[\mathbb{P}^5 \times \mathbb{P}^0] + 2[\mathbb{P}^4 \times \mathbb{P}^1] + 3[\mathbb{P}^3 \times \mathbb{P}^2] + 4[\mathbb{P}^2 \times \mathbb{P}^3] + 2[\mathbb{P}^1 \times \mathbb{P}^4] + 1[\mathbb{P}^0 \times \mathbb{P}^5],$$

although $(1, 2, 3, 4, 2, 1)$ is a log-concave sequence with no internal zeros.

I believe that no one will ever be able to characterize prime homology classes in $\mathbb{P}^m \times \mathbb{P}^n$. However, the *asymptotic* distribution of primes has a simple structure (at least in this case).

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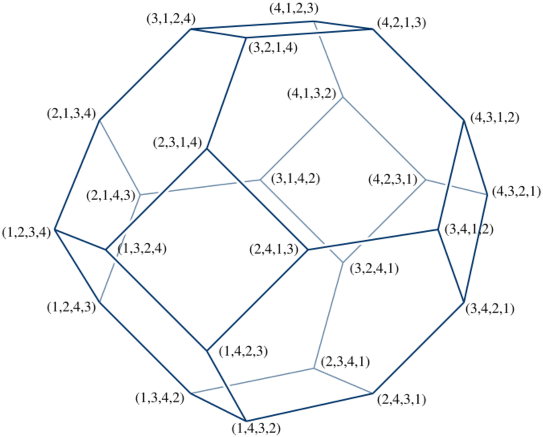
How about for other homogeneous spaces, for example, Grassmannians?

Matroids on $[n] = \{0, 1, \ldots, n\}$ are related to the geometry of the n-dimensional *permutohedron*, the convex hull of an orbit of the symmetric group S_{n+1}.



The above is, in fact, the picture of flags in the Boolean lattice of $[n]$.

- The rays of the dual fan Δ_{A_n} correspond to nonempty proper subsets of $[n]$.
- More generally, k-dimensional cones of Δ_{A_n} correspond to

  flags of nonempty proper subsets of $[n]$:

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k.$$

- The "extra symmetry" of the permutohedron maps a flag

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k.$$

  to the flag of complements

$$[n] \setminus S_1 \supsetneq [n] \setminus S_2 \supsetneq \cdots \supsetneq [n] \setminus S_k.$$

- A matroid M of rank $r+1$ on $[n]$ can be viewed as an r-dimensional subfan

$$\Delta_M \subseteq \Delta_{A_n}$$

which consists of cones corresponding to flags of *flats* of M:

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- The fan Δ_M is the *Bergman fan* of M,

or the *tropical linear space* associated to M.

- A homology class of X_{A_n} can be viewed as a balanced weighted subcomplex of P_n by assigning intersection numbers to the faces of complementary dimension:

  A homology class of a toric variety can be viewed as a *tropical variety*.

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- A matroid M on $[n]$ can be viewed as the balanced weighted subcomplex Δ_M, which has weight 1 on faces corresponding to complete flags of flats of M:

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- A matroid M on $[n]$ can be viewed as the balanced weighted subcomplex Δ_M, which has weight 1 on faces corresponding to complete flags of flats of M:

$$F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r.$$

- The homology class Δ_M determines M, and we know exactly which homology classes in X_{A_n} are of this form.

Theorem

Let k be a field. For any matroid M on $[n]$,

(i) *the homology class Δ_M is effective in X_{A_n} over k, and*

(ii) *the homology class Δ_M is prime in X_{A_n} over k iff M is realizable over k.*

Theorem

Under the "anticanonical" map

$$\pi : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n,$$

the matroid homology class pushforwards

$$\Delta_M \longmapsto (\textit{the coefficients of the chromatic polynomial of } M).$$

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Since prime classes map to prime classes, this implies the log-concavity conjecture for all matroids which are realizable over some field (but it suggests more).

In a joint work with Farhad Babaee, we showed that every cohomology class Δ in a smooth complete toric variety has a canonical representative $\mathscr{T}_\Delta$ in the space of closed currents, the "tropical current" of Δ.

This representative faithfully reflects piecewise linear geometry of the tropical variety Δ.

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This representative faithfully reflects piecewise linear geometry of the tropical variety Δ.

Question

Is the matroid current $\mathscr{T}_{\Delta_M}$ a limit of effective algebraic cycles for any matroid M?

In general, given a smooth projective variety, we wish to understand the limits of

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There is an obvious necessary condition for the approximability of a (p, p)-dimensional closed current $\mathscr{T}$:

$$(*) \quad [\mathscr{T}] \in \mathbb{R} \otimes_{\mathbb{Z}} \left(H^{2q}(X, \mathbb{Z})/\mathrm{tors} \cap H^{q,q}(X) \right), \quad q = n - p.$$

Is this condition sufficient for the approximability?

HC If $\mathscr{T}$ is a (p,p)-dimensional real closed current on X satisfying $(*)$, then $\mathscr{T}$ is a weak limit of the form

$$\mathscr{T} = \lim_{i \to \infty} \mathscr{T}_i, \quad \mathscr{T}_i = \sum_j \lambda_{ij} Z_{ij}, \quad \lambda_{ij} \in \mathbb{R}.$$

HC^+ If $\mathscr{T}$ is a (p,p)-dimensional positive closed current on X satisfying $(*)$, then $\mathscr{T}$ is a weak limit of the form

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Demailly proved that

1. HC^+ holds for any X when $p = n-1$,
2. HC^+ for X implies HC for X, and
3. HC for X is equivalent to the Hodge conjecture for X.

Part of the claim that “tropical currents faithfully reflect . . . ” is the following.

Theorem (*Babaee-H.*)

For any matroid M on $[n]$, the tropical current $\mathscr{T}_{\Delta_M}$ is extremal in the cone of positive closed currents on X_{A_n}.

Part of the claim that “tropical currents faithfully reflect ...” is the following.

Theorem (*Babaee-H.*)

For any matroid M on $[n]$, the tropical current $\mathscr{T}_{\Delta_M}$ is extremal in the cone of positive closed currents on X_{A_n}.

A Krein-Milman type convex analysis shows that

Corollary (*Babaee-H.*)

Assuming HC^+ for X_{A_n}, the main question has an affirmative answer for all M on $[n]$.

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What we show is that the tropical variety Δ_M has a "cohomology ring" which *has the structure of the cohomology ring of a smooth projective variety.*

There is a young Italian, Bombieri, who is working on zeta functions. He noticed all by himself that it was necessary to prove in all characteristics that the intersection form on "primitive" algebraic cycles of half dimension is definite; furthermore, he also apparently spotted the conjecture according to which the factors of an algebraic cycle in a "Künneth" decomposition are algebraic. By the way, what are you up to in these directions?

From Jean-Pierre Serre to Alexander Grothendieck, 1964.

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(2) Hodge-Riemann: Any hyperplane class ℓ defines a definite form of sign $(-1)^k$

$$PC^k(X) \times PC^k(X) \longrightarrow C^r(X) \simeq \mathbb{Q}, \qquad (h_1, h_2) \longmapsto \ell^{r-2k} \cdot h_1 \cdot h_2,$$

where $PC^k(X) \subseteq C^k(X)$ is the kernel of the multiplication by ℓ^{r-2k+1}.

A motivating observation is that the toric variety of Δ_M is, in the realizable case, 'Chow equivalent' to a smooth projective variety (*Feichtner-Yuzvinsky*):

There is a map from a smooth projective variety

$$V \longrightarrow X_{\Delta_M}$$

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It is tempting to think this as a 'Chow homotopy'.

(When the base field is $\mathbb{C}$, it is important not to think this as the usual homotopy.)

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We show that, even in the non-realizable case, $A^*(M) := A^*(X_{\Delta_M})$ has the structure of the cohomology ring of a smooth projective variety (of dimension r).

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and each flip preserves the validity of the ‘Kähler package’ in their cohomology rings.

The intermediate objects are tropical varieties with good cohomology rings,

but not in general associated to a matroid (unlike in McMullen’s case of polytopes).

The cohomology ring $A^*(M)$ can be described explicitly by generators and relations, which can be taken as a definition.

Definition

The cohomology ring of M is the quotient of the polynomial ring

$$A^*(M) := \mathbb{Z}[x_F]/(I_1 + I_2),$$

where the variables are indexed by nonempty proper flats of M, and

$$\begin{aligned} I_1 \quad &:= \quad \text{ideal}\left(\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F \mid i_1 \text{ and } i_2 \text{ are distinct elements of } [n] \right), \\ I_2 \quad &:= \quad \text{ideal}\left(x_{F_1} x_{F_2} \mid F_1 \text{ and } F_2 \text{ are incomparable flats of } M \right). \end{aligned}$$

Theorem

The Chow ring $A^*(M)$ is a Poincaré duality algebra of dimension r:

(1) *Degree map: There is an isomorphism*

$$\deg_M : A^r(M) \longrightarrow \mathbb{Z}, \qquad \prod_{i=1}^{r} x_{F_i} \longmapsto 1,$$

for any complete flag of nonempty proper flats $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r$ of M.

(2) *Poincaré duality: For any nonnegative integer $k \leq r$, the multiplication defines the perfect pairing*

$$A^k(M) \times A^{r-k}(M) \longrightarrow A^r(M) \simeq \mathbb{Z},$$

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Note that the underlying simplicial complex of Δ_M, the *order complex* of M, is not Gorenstein in general.

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Because we do not understand Kähler classes in their ’cohomology’ ring.

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The case of non-realizable matroids contrasts this in an interesting way.

Let $\mathscr{K}_{[n]}$ be the convex cone of linear forms with real coefficients

$$\sum_S c_S x_S$$

that satisfy, for any two incomparable nonempty proper subsets S_1, S_2 of $[n]$,

$$c_{S_1} + c_{S_2} > c_{S_1 \cap S_2} + c_{S_1 \cup S_2} \qquad (c_\emptyset = c_{[n]} = 0).$$

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Definition

The *ample cone* of M, denoted $\mathscr{K}_M$, is defined to be the image

$$\mathscr{K}_{[n]} \longrightarrow \mathscr{K}_M \subseteq A^1(M)_\mathbb{R},$$

where all the non-flats of M are mapped to zero.

Main Theorem

Let ℓ be an element of $\mathscr{K}_M$ and let k be a nonnegative integer $\leq r/2$.

(1) *Hard Lefschetz: The multiplication by ℓ defines an isormophism*

$$A^k(M)_{\mathbb{R}} \longrightarrow A^{r-k}(M)_{\mathbb{R}}, \qquad h \longmapsto \ell^{r-2k} \cdot h.$$

(2) *Hodge-Riemann: The multiplication by ℓ defines a definite form of sign $(-1)^k$*

$$PA^k(M)_{\mathbb{R}} \times PA^k(M)_{\mathbb{R}} \longrightarrow A^r(M)_{\mathbb{R}} \simeq \mathbb{R}, \qquad (h_1, h_2) \longmapsto \ell^{r-2k} \cdot h_1 \cdot h_2,$$

where $PA^k(M)_{\mathbb{R}} \subseteq A^k(M)_{\mathbb{R}}$ is the kernel of the multiplication by ℓ^{r-2k+1}.

Why does this imply the log-concavity conjecture?

Let i be an element of $[n]$, and consider the linear forms

$$\alpha(i) := \sum_{i \in S} x_S,$$
$$\beta(i) := \sum_{i \notin S} x_S.$$

Note that these linear forms are 'almost' ample:

$$c_{S_1} + c_{S_2} \geq c_{S_1 \cap S_2} + c_{S_1 \cup S_2} \qquad (c_\emptyset = c_{[n]} = 0).$$

Their images in the cohomology ring $A^*(M)$ does not depend on i;

they will be denoted by α and β respectively.

Proposition

Under the isomorphism deg : $A^r(M) \longrightarrow \mathbb{Z}$*, we have*

$$\alpha^{r-k}\beta^k \longmapsto (k\text{-th coefficient of the reduced characteristic polynomial of } M).$$

While neither α nor β are in the ample cone $\mathscr{K}_M$, we may take the limit

$$\ell_1 \longrightarrow \alpha, \qquad \ell_2 \longrightarrow \beta, \qquad \ell_1, \ell_2 \in \mathscr{K}_M.$$

This may be one reason why direct combinatorial reasoning for log-concavity was not easy.