Kazhdan-Lusztig polynomials of matroids

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AMS Special Session on Arrangements of Hypersurfaces

Arrangements and Flats

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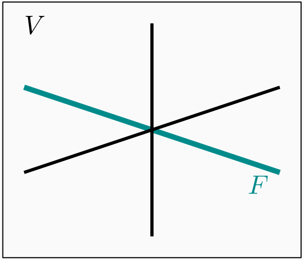

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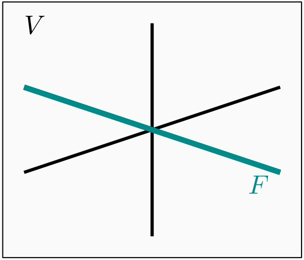
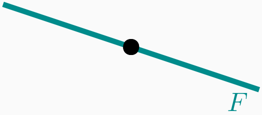


$\mathcal{A}$



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The **localization of $\mathcal{A}$ at F** is the arrangement

$$\mathcal{A}_F := \{H/F \mid F \subset H \in \mathcal{A}\}$$

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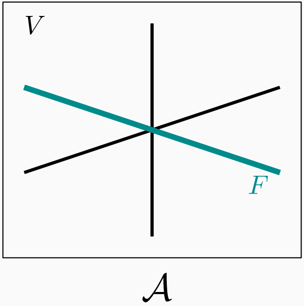

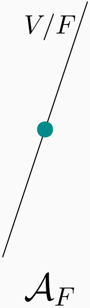

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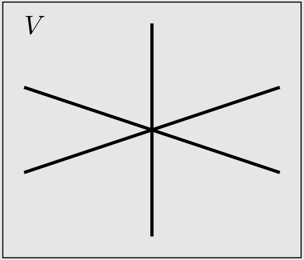
Example



$\chi_{\mathcal{A}}(t) = t^2 - 3t + 2$

Kazhdan-Lusztig Polynomial

Theorem

There exists a unique way to assign to each arrangement $\mathcal{A}$ a polynomial $P_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ subject to the following conditions:

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Remark

The theory of Kazhdan-Lusztig-Stanley polynomials provides a common generalization of these polynomials and classical Kazhdan-Lusztig polynomials.

Geometric Interpretation

$$V \hookrightarrow \bigoplus_{H \in \mathcal{A}} V/H \cong \bigoplus_{H \in \mathcal{A}} \mathbb{A}^1 \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1$$

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Have $H^*(Y_{\mathcal{A}}) \curvearrowright IH^*(Y_{\mathcal{A}})$, both concentrated in even degree.

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Theorem (Huh-Wang, P-Xu-Young, Elias-P-Wakefield)

- $\sum t^i \dim H^{2i}(Y_{\mathcal{A}}) \; = \; \sum_F t^{\operatorname{codim} F}$

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Corollary

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Remark

The definition of $P_{\mathcal{A}}(t)$ makes sense for matroids, but when the matroid is not realizable, non-negativity is still a conjecture. Work in progress by Braden-Huh-Matherne-P-Wang.

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$$\begin{aligned} t^{n-1} P_{\mathcal{A}_n}(t^{-1}) &= \sum_{k=0}^{n-2} \binom{n}{k} (t-1)^k P_{\mathcal{A}_{n-k}}(t) \ \ + \ \ \chi_{\mathcal{A}_n(t)} \\ &= \sum_{k=0}^{n-2} \binom{n}{k} (t-1)^k P_{\mathcal{A}_{n-k}}(t) \ \ + \ \ \frac{(t-1)^n + (-1)^n (t-1)}{t} \end{aligned}$$

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Then our recursion becomes

$$\begin{aligned}
\Phi(t^{-1},tu) &= \sum_{n=2}^{\infty} t^{n-1} P_{\mathcal{A}_n}(t^{-1})u^{n-1} \\
&= \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n-2} \binom{n}{k} (t-1)^k P_{\mathcal{A}_{n-k}}(t) \;\;+\;\; \chi_{\mathcal{A}_n}(t) \right) u^{n-1} \\
&= \cdots \\
&= \frac{(1+u)\;\Phi\Big(t, \frac{u}{1-u(t-1)}\Big) + u(t-1)\,(1-u(t-1))}{(1+u)\,(1-u(t-1))^2}
\end{aligned}$$

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Theorem (P-Wakefield-Young)

- $\Phi(t,u) = \frac{2}{u} \cdot \frac{(2t+1)u - 1 + \sqrt{1 - 2(2tu+1)u + u^2}}{1 - (2tu+1)^2}$

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Remark

The polynomial $P_{\mathcal{A}}(t)$ is conjecturally real-rooted for any $\mathcal{A}$, but this is the only non-trivial family of examples for which we can prove it!

Thanks!