

Kazhdan-Lusztig polynomials of matroids

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AMS Special Session on Arrangements of Hypersurfaces

Arrangements and Flats

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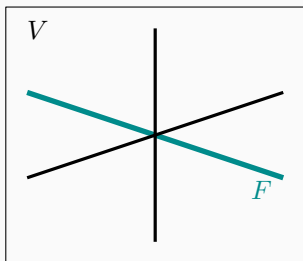
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Definition

The **contraction of \mathcal{A} at F** is the arrangement

$$\mathcal{A}^F := \{H \cap F \mid F \not\subset H \in \mathcal{A}\}$$

in the vector space F .

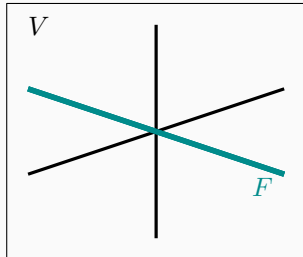
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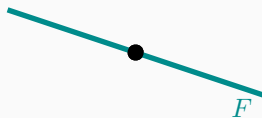
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Definition

The **localization of \mathcal{A} at F** is the arrangement

$$\mathcal{A}_F := \{H/F \mid F \subset H \in \mathcal{A}\}$$

in the vector space V/F .

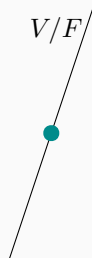
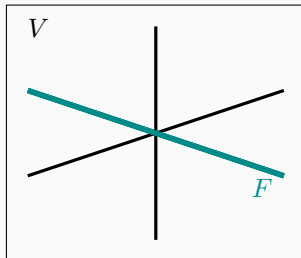
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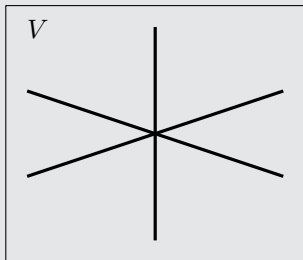
If V is a vector space over \mathbb{F}_q , $\chi_{\mathcal{A}}(q) = |V \setminus \bigcup_{H \in \mathcal{A}} H|$.

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Example



$$\chi_{\mathcal{A}}(t) = t^2 - 3t + 2$$

Kazhdan-Lusztig Polynomial

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There exists a unique way to assign to each arrangement \mathcal{A} a polynomial $P_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ subject to the following conditions:

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Remark

The theory of Kazhdan-Lusztig-Stanley polynomials provides a common generalization of these polynomials and classical Kazhdan-Lusztig polynomials.

Geometric Interpretation

$$V \hookrightarrow \bigoplus_{H \in \mathcal{A}} V/H \cong \bigoplus_{H \in \mathcal{A}} \mathbb{A}^1 \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1$$

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Have $H^*(Y_{\mathcal{A}}) \cong IH^*(Y_{\mathcal{A}})$, both concentrated in even degree.

Theorem (Huh-Wang, P-Xu-Young, Elias-P-Wakefield)

- $$\sum t^i \dim H^{2i}(Y_{\mathcal{A}}) = \sum_F t^{\text{codim } F}$$

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Remark

The definition of $P_{\mathcal{A}}(t)$ makes sense for matroids, but when the matroid is not realizable, non-negativity is still a conjecture. Work in progress by Braden-Huh-Matherne-P-Wang.

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$$\begin{aligned} t^{n-1} P_{\mathcal{A}_n}(t^{-1}) &= \sum_{k=0}^{n-2} \binom{n}{k} (t-1)^k P_{\mathcal{A}_{n-k}}(t) + \chi_{\mathcal{A}_n}(t) \\ &= \sum_{k=0}^{n-2} \binom{n}{k} (t-1)^k P_{\mathcal{A}_{n-k}}(t) + \frac{(t-1)^n + (-1)^n (t-1)}{t} \end{aligned}$$

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Put it in a generating function:

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Then our recursion becomes

$$\begin{aligned} \Phi(t^{-1}, tu) &= \sum_{n=2}^{\infty} t^{n-1} P_{\mathcal{A}_n}(t^{-1}) u^{n-1} \\ &= \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n-2} \binom{n}{k} (t-1)^k P_{\mathcal{A}_{n-k}}(t) + \chi_{\mathcal{A}_n}(t) \right) u^{n-1} \\ &= \dots \\ &= \frac{(1+u) \Phi\left(t, \frac{u}{1-u(t-1)}\right) + u(t-1)(1-u(t-1))}{(1+u)(1-u(t-1))^2} \end{aligned}$$

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Theorem (P-Wakefield-Young)

- $$\Phi(t, u) = \frac{2}{u} \cdot \frac{(2t + 1)u - 1 + \sqrt{1 - 2(2tu + 1)u + u^2}}{1 - (2tu + 1)^2}$$

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Remark

The polynomial $P_{\mathcal{A}}(t)$ is conjecturally real-rooted for any \mathcal{A} , but this is the only non-trivial family of examples for which we can prove it!

Thanks!