Operator Algebras and Conformal Field Theory

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1 Introduction

We report on a programme to understand unitary conformal field theory (CFT) from the point of view of operator algebras. The earlier stages of this research were carried out with Jones, following his suggestion that there might be a deeper "subfactor" explanation of the coincidence between certain braid group representations that had turned up in subfactors, statistical mechanics, and conformal field theory. (Most of our joint work appears in Section 10.) The classical *additive* theory of operator algebras, due to Murray and von Neumann, provides a framework for studying unitary Lie group representations, although in specific examples almost all the hard work involves a quite separate analysis of intertwining operators and differential equations. Analogously, the more recent *multiplicative* theory provides a powerful tool for studying the unitary representations of certain infinite-dimensional groups, such as loop groups or Diff S^1. It must again be complemented by a detailed analysis of certain intertwining operators, the primary fields, and their associated differential equations. The multiplicative theory of von Neumann algebras has appeared in three separate but related guises: first in the algebraic approach to quantum field theory (QFT) of Doplicher, Haag and Roberts; then in Connes' theory of bimodules or correspondences and their tensor products; and last (but not least) in Jones' theory of subfactors. Our results so far include:

(1) Several new constructions of subfactors.
(2) Nontrivial algebraic QFT's in $1+1$ dimensions with finitely many sectors and noninteger statistical (or quantum) dimension ("algebraic CFT").
(3) A definition of quantum invariant theory without using quantum groups at roots of unity.
(4) A computable and manifestly unitary definition of fusion for positive energy representations ("Connes fusion") making them into a tensor category.
(5) Analytic properties of primary fields ("constructive CFT").

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2 Classical Invariant Theory

The irreducible unitary (finite-dimensional) representations of $G = SU(N)$ can be studied in two distinct approaches. These provide a simple but important prototype for developing the theory of positive energy loop group representations and primary fields.

Borel-Weil Approach. This constructs all irreducible representations uniformly in a Lie algebraic way via highest weight theory. The representations are described as quotients of Verma modules, that is in terms of lowering and raising operators. This approach gives an important *uniqueness* result — such a representation is uniquely determined by its highest weight — but has the disadvantage that it is not manifestly unitary.

Hermann Weyl Approach. This starts from a special representation, $V = \mathbb{C}^N$ or ΛV, and realizes all others in the tensor powers $V^{\otimes \ell}$ or $(\Lambda V)^{\otimes \ell}$. The key to understanding the decomposition of $V^{\otimes \ell}$ is *Schur-Weyl duality*: $\mathrm{End}_G V^{\otimes \ell}$ is the image of $\mathbb{C}S_\ell$, where the symmetric group S_ℓ acts by permuting the tensor factors in $V^{\otimes \ell}$. This sets up a one-one correspondence between the irreducible representations of G and the symmetric groups and gives a manifestly unitary construction of the irreducible representations of G (on multiplicity spaces of S_ℓ). The irreducible unitary representation V_f with character χ_f and signature $f : f_1 \geq \cdots \geq f_N (= 0)$ is generated by the vector $e_f = e_1^{\otimes(f_1 - f_2)} \otimes (e_1 \wedge e_2)^{\otimes(f_2 - f_3)} \otimes \cdots$ in $(\Lambda V)^{\otimes \ell}$. The signature can be written in the usual way as a Young diagram and we then have the tensor product rule $V_f \otimes V_\Box = \oplus V_g$, where g runs over all diagrams that can be obtained by adding one box to f.

Thus, the Borel-Weil Lie algebraic approach leads to *uniqueness* results, whereas the Hermann Weyl approach leads to *existence* results and an explicit *construction*, giving analytic unitary properties.

3 Fermions and Quantization

Let H be a complex Hilbert space. Bounded operators $a(f)$ for $f \in H$ are said to satisfy the *canonical anticommutation relations* (CAR) if $[a(f), a(g)]_+ = 0$, $[a(f), a(g)^*]_+ = (f, g) \cdot I$, where $f \mapsto a(f)$ is $\mathbb{C}$-linear and $[x, y]_+ = xy + yx$. The *complex wave representation* π of the CAR on fermionic Fock space $\mathcal{F} = \Lambda H$ is given by $a(f)\omega = f \wedge \omega$. It is irreducible. Now the equations $c(f) = a(f) + a(f)^*$, $a(f) = \frac{1}{2}(c(f) - ic(if))$ give a correspondence with *real* linear maps $f \mapsto c(f)$ such that $c(f) = c(f)^*$ and $[c(f), c(g)]_+ = 2\mathrm{Re}\,(f, g) \cdot I$. Any projection P in H defines a new complex structure on H, by taking multiplication by i as i on PH and $-i$ on $P^\perp H$. So through c, this gives a new irreducible representation π_P of the CAR on Fock space $\mathcal{F}_P$. By considering approximations by finite-dimensional systems, Segal showed that $\pi_P \cong \pi_Q$ iff $P - Q$ is Hilbert-Schmidt. This leads to the following *quantization criterion*. Any $u \in U(H)$ gives a Bogoliubov automorphism of the CAR, $\alpha_u : a(f) \mapsto a(uf)$. The automorphism α_u is said to be implemented in $\mathcal{F}_P$ if $a(uf) = Ua(f)U^*$ for some unitary $U \in U(\mathcal{F}_P)$, unique up to a phase. The quantization criterion states that α_u is implemented in $\mathcal{F}_P$ iff $[u, P]$ is Hilbert-Schmidt. Thus, we get a homomorphism from the subgroup of implementable unitaries into $\mathcal{PU}(\mathcal{F}_P)$, the *basic* projective representation. As a special case, there are *canonical quantizations*: any unitary u with $uPu^* = P$ is canonically implemented in Fock space; and if $uPu^* = I - P$, then u is canonically implemented by a conjugate-linear isometry in Fock space.

4 Positive Energy Representations

Let $G = SU(N)$ and define the loop group $LG = C^\infty(S^1, G)$, the smooth maps of the circle into G. The diffeomorphism group of the circle $\mathrm{Diff}\, S^1$ is naturally a subgroup of $\mathrm{Aut}\, LG$ with the action given by reparametrization. In particular the group of rotations $\mathrm{Rot}\, S^1 \cong U(1)$ acts on LG. We look for projective representations $\pi : LG \to PU(H)$ that are both *irreducible* and have *positive energy*. This means that π should extend to $LG \rtimes \mathrm{Rot}\, S^1$ so that $H = \oplus_{n \geq 0} H(n)$, where the $H(n)$'s are eigenspaces for the action of $\mathrm{Rot}\, S^1$, i.e. $r_\theta \xi = e^{in\theta}\xi$ for $\xi \in H(n)$, and $\dim H(n) < \infty$ with $H(0) \neq 0$. Because the constant loops G commute with $\mathrm{Rot}\, S^1$, the $H(n)$'s are automatically G-modules.

Uniqueness. An irreducible positive energy representation π on H is uniquely determined by its *level* $\ell \geq 1$, a positive integer specifying the central extension or 2-cocycle of LG, and its *lowest energy space* $H(0)$, an irreducible representation of G. Only finitely many irreducible representations of G occur at level ℓ: their signatures must satisfy the quantization condition $f_1 - f_N \leq \ell$ and form a set $\mathbf{Y}_\ell$.

Existence/Analytic Properties. Let $H = L^2(S^1) \otimes V$ and let P be the projection onto the Hardy space $H^2(S^1) \otimes V$ of functions with vanishing negative Fourier coefficients (or equivalently boundary values of functions holomorphic in the unit disc). The semidirect product $LG \rtimes \mathrm{Diff}^+ S^1$ acts unitarily on H and satisfies the quantization criterion for P, so gives a projective representation of $LG \rtimes \mathrm{Diff}^+ S^1$ in $\mathcal{F}_P$. The irreducible summands of $\mathcal{F}_P^{\otimes \ell}$ give all the level ℓ representations of LG and this construction shows that any positive energy representation extends to $LG \rtimes \mathrm{Diff}^+ S^1$ ("invariance under reparametrization").

If H is a positive energy representation of level ℓ, the C^∞ vectors H^∞ for $\mathrm{Rot}\, S^1$ are acted on continuously by $LG \rtimes \mathrm{Rot}\, S^1$ (or more generally $LG \rtimes \mathrm{Diff}\, S^1$) and its Lie algebra. This can be seen in a variety of ways, using representations of the Heisenberg group or the infinitesimal version of the fermionic construction. If $\mathfrak{g} = \mathrm{Lie}(G)$, then $\mathrm{Lie}(LG) = L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$. Its complexification is spanned by the functions $e^{in\theta}x$ with $x \in \mathfrak{g}$. Let $x(n)$ be the corresponding unbounded operators on H^∞ (or H^0, the subspace of finite energy vectors) and let d be the self-adjoint generator for $\mathrm{Rot}\, S^1$ (so that $r_\theta = e^{i\theta d}$). Then $[x(n), y(m)] = [x, y](n + m) + \ell n\, \delta_{n+m,0} \operatorname{tr}(xy) \cdot I$ and $[d, x(n)] = -nx(n)$.

5 Formal Conformal Field Theory

Purely as motivation, we sketch the standard approach to CFT based on formal quantum fields.

State-Field Correspondence. For fixed level ℓ, there should be a one-one correspondence between states $v \in H = \oplus_{f \in \mathbf{Y}_\ell} H_f^0$ and field operators $\phi(v, z) = \sum_n \phi(v, n) z^{-n-h_{ij}}$, where $\phi_{ij}(v, n) : H_j^0 \to H_i^0$ and z is a formal parameter. The field "creates the state from the vacuum", i.e. $\phi(v, 0)\Omega = v$. Fields are first defined for vectors $x \in H_0(1) \cong \mathfrak{g}$ by $x(z) = \sum \pi(x(n))z^{-n-1}$. For $a \in H_0$, fields $\phi(a,z)$ are uniquely determined by $\phi(a,0)\Omega = a$, rotation invariance $[d,\phi(v,n)] = -n\phi(v,n)$ and the gauge condition $x(z)\phi(a,w) \sim \phi(a,w)x(z)$. (This notation means that, on taking matrix coefficients, one side is the analytic continuation of the other, with the domains of definition given by decreasing moduli of arguments.) The fields $\phi(a,z)$ for $a \in H_0^0$ form a *vertex algebra*, Borcherds' analogue of a commutative ring. Commutativity and associativity are replaced by $\phi(a,z)\phi(b,w) \sim \phi(b,w)\phi(a,z)$ and $\phi(a,z)\phi(b,w) \sim \phi(\phi(a,z-w)b,w)$. The operator product expansion (OPE) is obtained by expanding the right-hand side of this last equation as a power series in $(z-w)$: the resulting coefficients are the fields arising from the fusion of $\phi(a,z)$ and $\phi(b,w)$. In this sense the $x(z)$'s generate the vertex algebra. The H_i's become modules over the vertex algebra and the gauge condition defines general fields as intertwiners. The fields corresponding to vectors in $H_i(0)$ are called *primary fields*. Other secondary fields are obtained by successive fusion with $x(z)$'s.

Braiding-Fusion Duality. If neither a nor b lies in H_0, the commutativity and associativity relations must be replaced by braiding and fusion relations:

$$\phi_{ij}^p(a,z)\phi_{jk}^q(b,w) \sim \textstyle\sum_h \alpha_h \phi_{ih}^q(b,w)\phi_{hk}^p(a,z) \text{ (where } a \in H_p, b \in H_q).$$
$$\phi_{ij}^p(a,z)\phi_{jk}^q(b,w) \sim \textstyle\sum_h \beta_h \phi_{ik}^h(\phi_{hq}^p(a,z-w)b,w).$$

These are first proved as identities between lowest energy matrix coefficients of primary fields and follow in general by fusion. The matrix coefficients give a vector-valued function $f(\zeta)$ of one variable $\zeta = z/w$. It satisfies the Knizhnik-Zamolodchikov ODE $f'(z) = z^{-1}Pf(z) + (1-z)^{-1}Qf(z)$, with P, Q constant matrices. α_h and β_h are entries in the matrices connecting the solutions at 0 with the solutions at ∞ and 1 respectively. The evident algebraic relations between these two matrices constitute "braiding-fusion duality".

6 Construction of Primary Fields

Let H_i, H_j be positive energy representations of level ℓ and let W be an irreducible representation of G. A *primary field* of charge W is a continuous linear map $\phi : H_i^\infty \otimes C^\infty(S^1, W) \to H_j^\infty$ that commutes with the action of $LG \rtimes \mathrm{Rot}\, S^1$. This makes sense because H_i and H_j are projective representations with the same cocycle, whereas $C^\infty(S^1,W)$ is an ordinary representation, with LG acting by pointwise multiplication and $\mathrm{Rot}\, S^1$ by rotation. Any $f \in C^\infty(S^1,W)$ determines a "smeared field" $\phi(f) : H_i^\infty \to H_j^\infty$, which must satisfy the covariance relation $\phi(g \cdot f) = \pi_j(g)\phi(f)\pi_i(g)^*$ for $g \in LG \rtimes \mathrm{Rot}\, S^1$.

Uniqueness. A primary field ϕ is uniquely determined by its initial term $H_i(0) \otimes W \to H_j(0)$, which commutes with G. The charge W must have signature f satisfying $f_1 - f_N \leq \ell$. Moreover the initial term must satisfy an algebraic quantization condition with respect to $SU(2) \subset SU(N)$: $(*)$ when cut down to irreducible summands of $SU(2)$, the resulting intertwiners $V_p \otimes V_q \to V_r$ can only be non-zero if $p + q + r \leq \ell$ where the spins p, q, r are half integers $\leq \ell/2$.

Existence/Construction. Primary fields for the vector representation of G come from compressing fermions $P_j(a(f) \otimes I \otimes \cdots \otimes I)P_i$, where P_i, P_j are projections onto H_i, H_j summands of $\mathcal{F}^{\otimes \ell}$. More generally, primary fields arise from (antisymmetric) external tensor products of fermions, parallelling the explicit construction of highest weight vectors in $(\Lambda V)^{\otimes \ell}$. For $v \in V$, define $v_m(\theta) = e^{im\theta} v$ in $C^\infty(S^1, V)$ and $a(v,m) = a(v_m)$. Introduce the formal Laurent series $a(v,z) = \sum_m a(v,m) z^{-m}$. At level one, the primary field for $\Lambda^k V$ corresponds to compressions of the formal Laurent series $\phi(e_1 \wedge \cdots \wedge e_k, z) = a(e_1,z)a(e_2,z)\cdots a(e_k,z)$ (essentially an external tensor product as the e_i's are orthogonal). At level ℓ, the primary fields of signature f arise as formal Laurent series $\phi(w,z)$, uniquely specified by $\phi(e_f, z) = P_j(\phi(e_1,z)^{\otimes(f_1-f_2)} \otimes \phi(e_1 \wedge e_2, z)^{\otimes(f_2-f_3)} \otimes \cdots)P_i$ and G-covariance. All possible primary fields arise in this way because an intertwiner satisfies $(*)$ iff it appears as a component of the map $\Lambda \otimes \Lambda \to \Lambda$, $\alpha \otimes \beta \mapsto \alpha \wedge \beta$, where Λ is the exterior algebra $(\Lambda V)^{\otimes \ell}$.

This fermionic construction of the primary fields makes manifest their continuity properties on H_i^∞. In particular the primary fields for the vector representation or its dual must satisfy the same kind of L^2 bounds as fermions, $\|\phi(f)\| \le A\|f\|_2$, underlining Haag's philosophy that QFT can and should be understood in terms of (algebras of) *bounded* operators. Here there is no choice.

7 The K-Z ODE and Braiding of Primary Fields

When $f, g \in C^\infty(S^1) \otimes V$ have disjoint support, the corresponding smeared fermi fields satisfy the anticommutative exchange rule $a(f)a(g)^* = -a(g)^* a(f)$. Similarly, if a and b are test functions supported in the upper and lower semicircle of S^1, there are *braiding relations*

$$\phi^f_{f0}(a)\phi^{\Box}_{0\Box}(b) = \sum \lambda_g \, \phi^{\Box}_{fg}(e^{\mu_g} \cdot b)\phi^f_{g\Box}(e^{\nu_g} \cdot a), \tag{1}$$

where the constants λ_g, μ_g, ν_g are to be determined and $e^\tau(\theta) = e^{i\tau\theta}$. The λ_g's arise as the entries of the matrix connecting the solutions at 0 and ∞ of a matrix-valued ODE as follows. Fusion of the $x(z)$'s shows that, up to an additive constant, d is given by $L_0 \equiv (N+\ell)^{-1}[\sum_i \frac{1}{2} x_i(0)^* x_i(0) + \sum_{n>0,i} x_i(n)^* x_i(n)]$ (the *Segal-Sugawara* formula), where (x_i) is an orthonormal basis of $\mathfrak{g}$. Let $f(z) = \sum \langle \phi(v_2, n)\phi(v_3, -n)v_4, v_1^* \rangle z^n$, the reduced 4-point function with values in $(V_f \otimes V_f^* \otimes V_\Box \otimes V_\Box^*)^G$. The two expressions, when d and L_0 are inserted, between the two field operators can be simplified using the commutation relations with primary fields. After the change of variable $z \mapsto (1-z)^{-1}$, they lead to the Knizhnik-Zamolodchikov ODE $f'(z) = z^{-1}Pf(z) + (1-z)^{-1}(P-Q)f(z)$, where P and Q are self-adjoint $(N \times N)$-matrices with P having distinct eigenvalues, Q proportional to a rank one projection, and P, Q in general position. There is then an essentially unique choice of (non-orthogonal) basis so that

$$P = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & 1 & 0 \\ & & & \\ & & & 1 \\ a_1 & & & a_N \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & & 0 \\ & & & \\ & & & \\ & & & \\ b_1 & & & b_N \end{pmatrix}.$$

This is the matrix-valued ODE for the generalized hypergeometric equation. Entries of the transport matrices relating solutions at 0 and 1 are calculated by an extension of the classical method of Gauss and two tricks: the unitary of the transport matrix when the constant $N + \ell$ is made imaginary; and Karamata's Tauberian theorem.

The inner product of both sides of (1) with lowest energy vectors can be expressed through integrals involving a, b, and the branches of $f(z)$ at 0 and ∞ (viewed as vector-valued distributions on S^1). The transport matrix between the branches therefore gives the braiding coefficients (and phase corrections) for the inner products with lowest energy vectors. Using lowering and raising operators, they also work for inner products with arbitrary finite energy vectors and hence, by continuity, with all smooth vectors.

8 Von Neumann Algebras

It is perhaps most natural to define von Neumann algebras as "the symmetry algebras of unitary groups". Thus, if H is a complex Hilbert space, von Neumann algebras $M \subseteq B(H)$ are of the form $M = G'$, where G is a subgroup of the unitary group $U(H)$ and the *commutant* or symmetry algebra of $\mathcal{S} \subseteq B(H)$ is $\mathcal{S}' = \{T : Tx = xT \text{ for all } x \in \mathcal{S}\}$. If $\mathcal{S}^* = \mathcal{S}$, then $\mathcal{S}''$ coincides with the von Neumann algebra generated by $\mathcal{S}$ (i.e. the smallest von Neumann algebra containing $\mathcal{S}$). It is also the strong or weak operator closure of the unital *-algebra generated by $\mathcal{S}$.

If M is a von Neumann algebra, its *center* $Z(M) = M \cap M'$ is an Abelian von Neumann algebra, so of the form $L^\infty(X, \mu)$ for some measure space (X, μ). If X is atomic, then M is canonically a direct sum of *factors*, von Neumann algebras with trivial center, with one factor for each point of X. In general M has an essentially unique direct integral decomposition $M = \int_X^\oplus M_x \, d\mu(x)$, where each M_x is a factor, so the study of von Neumann algebras reduces to that of factors.

Any von Neumann algebra is generated by its projections. Because $M = M'' = (M')'$, these projections correspond to invariant subspaces or submodules for the von Neumann algebra M'. Unitary equivalence of M'-modules translates into a notion of equivalence of projections ("Murray-von Neumann equivalence"). If in addition M is a factor, then simple set-theoretic type arguments show that M falls into one of three types: (I) M has minimal projections; (II) M has projections not equivalent to any proper subprojection; (III) every nonzero projection in M is equivalent to a proper subprojection (so that they are all equivalent).

The type I factors have the form $B(K)$ for some Hilbert space K. In the type II case, Murray and von Neumann defined a countably additive dimension function on projections with range $[0, 1]$ or $[0, \infty]$, with two projections equivalent iff they have the same dimension. This leads to the notion of "continuous dimension" for any M-module. The two possibilities for the range give a further subdivision into type II_1 and type II_∞ factors. Any type II_∞ factor is the von Neumann algebra of infinite matrices with values in some type II_1 factor. For type II_1 factors, Murray and von Neumann proved that the dimension function can be linearized to give a *trace* tr on M, i.e. a state with $\mathrm{tr}(ab) = \mathrm{tr}(ba)$. Conversely, any factor admitting such a tracial state must be a type II_1 factor.

9 Modular Theory

Modular theory has its roots implicitly in QFT (Haag-Araki duality for bosons) and explicitly in statistical physics (the lattice models of Haag-Hugenholtz-Winninck). Independently Tomita proposed a general theory for any von Neumann algebra, developed in detail by Takesaki. For *hyperfinite* von Neumann algebras (those approximable by an increasing sequence of finite-dimensional algebras), Hugenholtz and Wierenga gave a more elementary approach based on the lattice model proof.

Tomita-Takesaki Theory. Let $M \subset B(H)$ be a von Neumann algebra and let $\Omega \in H$ (the "vacuum vector") be a unit vector such that $M\Omega$ and $M'\Omega$ are dense in H. It is then possible to define an operator $S = S_M : M\Omega \to M\Omega$, $a\Omega \mapsto a^*\Omega$. S is conjugate-linear, densely defined, and closeable with closure $\overline{S} = S_{M'}^*$. Let $\overline{S} = J\Delta^{1/2}$ be the polar decomposition of $\overline{S}$, so that J is a conjugate-linear isometry with $J^2 = I$ and Δ is a positive unbounded operator not having 0 as an eigenvalue. Then $JMJ = M'$ and $\Delta^{it} M \Delta^{-it} = M$. Thus, $x \mapsto Jx^*J$ gives an isomorphism between M^{op} (M with multiplication reversed) and M' and $\sigma_t(x) = \Delta^{it} x \Delta^{-it}$ gives a one-parameter group of automorphisms of M and M'.

Connes' (2×2)-Matrix Trick. Connes' fundamental observation was that the image of σ_t in the outer automorphism group of M is independent of the choice of the state Ω, and thus can be used to provide further intrinsic invariants of M.

"Trivial" Example (von Neumann). Let A be a unital *-algebra and tr a tracial state on A. Let $L^2(A, \mathrm{tr})$ be the Hilbert space completion of A for the inner product $\mathrm{tr}(b^*a)$. If λ and ρ denote the actions of A on $L^2(A,\mathrm{tr})$ by right and left multiplication, then $\lambda(A)'' = \rho(A)'$ and $\Delta = I$. In particular, if $A = \mathbb{C}[\Gamma]$ where Γ is a discrete countable group and tr is the Plancherel trace $\mathrm{tr}(\gamma) = \delta_{\gamma,1}$, then $L^2(A,\mathrm{tr}) = \ell^2(\Gamma)$ and λ and ρ become the usual left and right regular representations. If Γ has infinite (non-identity) conjugacy classes, e.g. if $\Gamma = S_\infty$, then $\lambda(\Gamma)''$ is a factor with a trace, so a type II_1 factor.

Easy Consequences. Connes' (2×2)-matrix trick shows that if M is a type I or II factor, then the modular group σ_t must be inner. Hence, if the fixed point algebra M^σ equals $\mathbb{C}$, i.e. σ_t is *ergodic*, then M is a type III factor (in fact, III_1).

Classification of Type III Factors. Connes' "essential spectrum" is defined as $S(M) = \bigcap \mathrm{Sp}(\Delta_\Omega)$, where Ω ranges over vectors cyclic for M and M'. Then $\Gamma = S(M) \cap \mathbb{R}_+^*$ is a closed subgroup of $\mathbb{R}_+^*$, so the type III factors can be subdivided further into: type III_0 when $\Gamma = \{1\}$; type III_λ when $\Gamma = \lambda^{\mathbb{Z}}$ with $\lambda \in (0,1)$; and III_1 when $\Gamma = \mathbb{R}_+^*$. In the type III_0 case, a further invariant is the "flow of weights", an ergodic flow on a Lebesgue space (the action $\hat{\sigma}_\tau = \mathrm{id} \otimes \mathrm{Ad}\, m(e^{i\tau\cdot})$ of $\mathbb{R}$ on the center of $(M \otimes B(L^2(\mathbb{R})))^{\sigma\otimes \mathrm{Ad}\lambda}$). Thanks to the work of von Neumann, Connes, and Haagerup (completed in 1985), any hyperfinite factor is uniquely determined by its type (and flow of weights). In particular, the hyperfinite type II_1 and III_1 factors are unique.

Takesaki Devissage. If $N \subset M$ is a von Neumann subalgebra, normalized by Δ^{it}, then Δ and J restrict to the corresponding operators for N on the closure of $N\Omega$. This result allows one to pass from the modular operators for a theory to those of a subtheory. Thus, if M is a hyperfinite type III_1 factor with σ_t ergodic, so too is N.

10 Haag Duality and Local Loop Groups

Geometric Modular Theory for Fermions on S^1. Let I be an open interval of S^1 and let I^c be the complementary open interval. Let Cliff(I) be the *-algebra generated by $a(f)$ with $f \in L^2(I) \otimes V$. Then Haag-Araki duality holds: Cliff$(I)'' =$ Cliff$(I^c)'$ (graded commutant). This follows directly from the more important fact that the modular operators are *geometric.* Taking I and I^c to be the upper and lower semicircles, this means that J is the canonical quantization of the flip $z \mapsto \overline{z}$, sending $f(z)$ to $\overline{z}f(\overline{z})$. Δ^{it} is the canonical quantization of the Möbius flow fixing the endpoints of I. This is proved directly by "reduction to one-particle states": S is the canonical quantization of an operator s. The polar decomposition of s gives that of S and was computed directly with Jones by two methods: by an analytic continuation argument à la Bisognano-Wichmann; or by considering representations of the algebra generated by two projections. The local algebra Cliff$(I)''$ is manifestly hyperfinite. Moreover Cliff$(I)''$ is a type III_1 factor by the ergodicity of σ_t, because Δ^{it} is the direct sum of the trivial representation and copies of the regular representation of $\mathbb{R}$.

Loop Group Subfactors (Jones-Wassermann). Let L_IG be the local loop group consisting of loops concentrated in I, i.e. loops equal to 1 off I, and let π_i be an irreducible positive energy representation of level ℓ. Haag-Araki duality and the fermionic construction of π_i imply that operators in $\pi(L_IG)$ and $\pi(L_{I^c}G)$, defined up to a phase, actually commute ("locality"). Thus, we get the canonical inclusion:

$$\pi_i(L_IG)'' \subseteq \pi_i(L_{I^c}G)'. \tag{2}$$

Consequences of Takesaki Devissage. Because the modular operators for the fermionic free field theory are geometric and the loop group representations are constructed as subtheories, Takesaki devissage can be applied to the geometric inclusion of local algebras on $\mathcal{F}_P^{\otimes \ell}$, $\pi^{\otimes \ell}(L_IG)'' \subset (\mathrm{Cliff}(I)^{\otimes \ell})''$. It has the following consequences:

Haag Duality in the Vacuum Sector. If π_0 is the vacuum representation at level ℓ (so that the lowest energy subspace, generated by the vacuum vector, gives the trivial representation of G), then $\pi_0(L_IG)'' = \pi_0(L_{I^c}G)'$. Moreover an argument of Reeh-Schlieder shows that the vacuum vector is cyclic for $\pi_0(L_IG)''$, and hence $\pi_0(L_IG)'$. The corresponding modular operators are geometric. So in general the inclusion (2) *measures the failure of Haag duality.*

Local Equivalence. $\pi_0|_{L_IG} \cong \pi_i|_{L_IG}$, so that all positive energy representations at level ℓ become unitarily equivalent when restricted to the local loop groups. (Note that smeared vector primary fields give explicit bounded intertwiners.)

Type of Local Algebras. $\pi_i(L_IG)''$ is isomorphic to the hyperfinite type III_1 factor. Hyperfiniteness can also be deduced more directly, independently of the Connes-Haagerup classification, by the *factorization property.* This property, inherited from fermions, means that the representations π_0 and $\pi_0 \otimes \pi_0$ of $L_{I \cup J}G = L_IG \times L_JG$ are unitarily equivalent if I and J are nontouching disjoint intervals. So if $I_n \uparrow I$, there is a type I factor B_n lying between $\pi_0(L_{I_n}G)''$ and $\pi(L_{I_{n+1}}G)''$. $B_n \uparrow \pi_0(L_IG)''$ forces hyperfiniteness (the "Dick trick").

Generalized Haag Duality. Let $\pi = \bigoplus \pi_i$ on $H = \bigoplus H_i$, the direct sum of all the level ℓ representations and let ϕ be the primary field for the vector representation. Then $\pi(L_I G)'' = \langle \phi(f), \phi(f)^* : f \in C_c^\infty(I) \otimes V\rangle' \cap (\bigoplus B(H_i))$.

Von Neumann Density. Let I_1 and I_2 be *touching* intervals obtained by removing a point from the interval I. Then $\pi(L_{I_1}G)'' \vee \pi(L_{I_2}G)'' = \pi(L_I G)''$ ("irrelevance of points"). Jones and I first deduced this from a stronger result: the pullback of the quotient strong operator topology on LG under the map $LG \to PU(\mathcal{F}_P)$ makes $L_{I_1}G \times L_{I_2}G$ *dense* in $L_I G$. Von Neumann density also follows by taking commutants in generalized Haag duality and noting that, because of its L^2 bounds, ϕ "does not see points".

Irreducibility. If $L^{\pm 1}G = L_I G \times L_{I^c} G$ is the subgroup of LG consisting of loops trivial to all orders at ± 1, then irreducible positive energy representations of LG stay irreducible and inequivalent when restricted to $L^{\pm 1}G$.

11 Connes Fusion and Braiding

Connes defined an associative tensor operation ("Connes fusion") on bimodules over (type III) von Neumann algebras. Let $X = {}_A X_B$ be an (A, B)-bimodule and $Y = {}_B Y_C$ a (B, C)-bimodule. Let (H_0, Ω) be a "trivial" (B, B)-bimodule defined by modular theory. Let $\mathcal{X} = \mathrm{Hom}_{B^{\mathrm{op}}}(H_0, X)$, $\mathcal{Y} = \mathrm{Hom}_B(H_0, Y)$ and define $X \boxtimes Y$ as the Hilbert space completion of $\mathcal{X} \otimes \mathcal{Y}$ with inner product $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = (x_2^* x_1 y_2^* y_1 \Omega, \Omega)$. It is naturally an (A, C)-bimodule. If ${}_A X_B$ and ${}_B Y_A$ are irreducible, Y is called *conjugate* to X iff $X \boxtimes Y$ and $Y \boxtimes X$ both contain the trivial bimodule at least once. Y is then unique up to isomorphism and the trivial bimodule appears exactly once. Any homomorphism $\rho : A \to B$ defines an (A, B)-bimodule, because ρ makes H_0 an A-module. Connes fusion corresponds to composition of homomorphisms. Because all modules over a type III factor are equivalent, every bimodule arises this way. Many properties of Connes fusion can be proved in the homomorphism picture.

Definition of Fusion (State-Field Correspondence). For representations of LG, the bimodule point of view comes through restricting to $L_I G \times L_{I^c} G$ and Connes fusion can be defined without explicit reference to von Neumann algebras. Let X, Y be positive energy representations of LG at level ℓ. Replace states $\xi \in X, \eta \in Y$ by intertwiners $x \in \mathcal{X} = \mathrm{Hom}_{L_{I^c}G}(H_0, X)$, $y \in \mathcal{Y} = \mathrm{Hom}_{L_I G}(H_0, Y)$. The "fields" x, y create the states $\xi = x\Omega$, $\eta = y\Omega$ from the vacuum. The inner product on $\mathcal{X} \otimes \mathcal{Y}$ is given by the *four-point formula* $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_2^* x_1 y_2^* y_1\rangle$ (vacuum expectation). The Hilbert space completion $X \boxtimes Y$ naturally supports a projective representation of $L_I G \times L_{I^c} G$.

Braiding Properties of Bounded Intertwiners. By hermiticity, the braiding relations (1) for smeared primary fields can be written symbolically as

$$a_{f0} b_{\Box 0}^* = \sum \lambda_g b_{gf}^* a_{g\Box}, \quad a_{g\Box} b_{\Box 0} = \varepsilon_g b_{gf} a_{f0}, \tag{3}$$

where $|\varepsilon_g| = 1$ in the second Abelian relation. We call $a = a_{f0}$ and $b = b_{\Box 0}$ the *principal parts.* Letting $A_1 = a_{f0}$, $A_2 = \bigoplus |\lambda_g|^{1/2} a_{g\Box}$, $B_1 = b_{\Box 0}$, $B_2 = \bigoplus \varepsilon_g |\lambda_g|^{1/2} b_{gf}$, the braiding relations (3) take the form $A_1 B_1^* = B_2^* A_2$, $A_2 B_1 =$

B_2A_1: these equations are unchanged if the A_i's or B_j's are replaced by their phases. The a_{ij}'s become bounded after such a "phase correction". Each set of intertwiners (c_{ij}) can be modified in three steps so that (3) still holds but with a and b *unitary*:(i) replace c_{ij} by $\sum 2^{-n}\pi_i(g_n)c_{ij}\pi_j(u_n)$, with (g_n) a dense subgroup of L_IG and u_n partial isometries in $\pi_0(L_IG)''$ with $u_iu_j^* = \delta_{ij}I$, $\sum u_i^*u_i = I$; (ii) make a phase correction on (c_{ij}) so that the principal part c satisfies $cc^* = I$; (iii) replace c_{ij} by $c_{ij}\pi_j(u)$ where u is a partial isometry in $\pi_0(L_IG)''$ with $u^*u = I$, $uu^* = c^*c$. If now $x : H_0 \to H_f$ and $y : H_0 \to H_\square$ are arbitrary intertwiners, their *nonprincipal* parts are defined by $x_{ij} = a_{ij}\pi_j(a_{f0}^*x)$ and $y_{pq} = b_{pq}\pi_q(b_{\square 0}^*y)$. They satisfy the analogues of (3).

Computation (Braiding-Fusion Duality). To prove the fusion rules $H_f \boxtimes H_\square \cong \oplus H_g$, where now $g \in \mathbf{Y}_\ell$, it suffices to define an explicit isometry U of $H_f \boxtimes H_\square$ into $\oplus H_g$, which is an intertwiner for $L^{\pm 1}G = L_IG \times L_{I^c}G$; for by Schur's lemma and irreducibility for $L^{\pm 1}G$, U must be unitary making $H_f \boxtimes H_\square$ a positive energy representation. By the braiding relation for intertwiners,

$$\|x \otimes y\|^2 = \langle x_{f0}^*x_{f0}y_{\square 0}^*y_{\square 0}\rangle = \sum \lambda_g \langle x_{f0}^*y_{gf}^*x_{g\square}y_{\square 0}\rangle = \sum |\lambda_g| \langle y_{\square 0}^*x_{g\square}^*x_{g\square}y_{\square 0}\rangle.$$

The coefficients have to be positive, as the equation can be interpreted as writing a vector state as a linear combination of inequivalent pure states. Thus, only the non-vanishing of the λ_g's is important. Now define $U(x \otimes y) = \oplus|\lambda_g|^{1/2}x_{g\square}y_{\square 0}\Omega$.

Braiding. The braiding map $b : X \boxtimes Y \to Y \boxtimes X$ is the unitary given by $b(x \otimes y) = e^{-\pi i L_0} \cdot (e^{i\pi L_0}ye^{-i\pi L_0} \otimes e^{i\pi L_0}xe^{-i\pi L_0})$. Under the "concrete" isomorphism U on $H_\square \boxtimes H_\square$, $UbU^*(\oplus|\lambda_g|^{1/2}x_{g\square}y_{\square 0}\Omega) = \oplus|\lambda_g|^{1/2}y_{g\square}x_{\square 0}\Omega = \oplus|\lambda_g|^{1/2}\mu_g x_{g\square}y_{\square 0}\Omega$, so that $UbU^* = \mu_g I$ on H_g. In general $H_1 \boxtimes \cdots \boxtimes H_n$ can also be defined and computed using a $2n$-point function, after having divided S^1 into n intervals. The b's have a very simple concrete form, especially on $H_\square^{\boxtimes n}$ where only vector primary fields are invoked. This realization makes manifest their braiding and cabling properties.

Closure under Fusion and Conjugation. By associativity and induction: each irreducible positive energy representation H_i appears in some $H_\square^{\boxtimes n}$; the H_i's are closed under Connes fusion; each H_i has a (unique) conjugate $\overline{H_i}$.

General Fusion Rules (Faltings' Trick). The fusion coefficients N_{ij}^k are given by $H_i \boxtimes H_j = \oplus N_{ij}^k H_k$. Braiding shows that $H_i \boxtimes H_j \cong H_j \boxtimes H_i$. Let $\mathcal{R}$ be the representation ring of formal sums $\sum m_iH_i$. $\mathcal{R}$ is commutative with an identity and an involution. Thus, the complexification $\mathcal{R}_\mathbb{C}$ is a finite-dimensional *-algebra with a nondegenerate positive trace $\mathrm{tr}(\sum c_iH_i) = c_0$. So $\mathcal{R}_\mathbb{C} \cong \mathbb{C}^M$, where $M = |\mathbf{Y}_\ell|$. The fusion rules for $H_{\lambda^k V}$ are deduced by combining the method used for $H_\square$ with properties of $\mathcal{R}$. From these fusion rules, the characters of $\mathcal{R}_\mathbb{C}$ are given by $[H_f] \mapsto \mathrm{ch}(H_f, h) = \chi_f(D(h))$ where $h \in \mathbf{Y}_\ell$ and $D(h) \in SU(N)$ is the diagonal matrix with $D(h)_{kk} = \exp(2\pi i(h_k + N - k - H)/(N + \ell))$ where $H = (\sum h_k + N - k)/N$. Thus, the N_{ij}^k's can be computed using the multiplication rules for the basis $\mathrm{ch}(H_f, \cdot)$ of $C(\mathbf{Y}_\ell)$. They agree with the Verlinde formulas in Kac's book.

Summary. The positive energy representations $H_0, \ldots, H_M$ at a fixed level ℓ become a braided ribbon C* tensor category.

12 Subfactors

Let $N \subset M$ be an inclusion of type II_1 factors in $B(H)$, so that H becomes an (M, N^{op})-bimodule. They act on $L^2(M, \mathrm{tr})$. Let $e = e_1$ be the projection onto $L^2(N)$ and $M_1 = \langle M, e_1 \rangle''$, the *Jones basic construction.*

Definition. N is of *finite index* in M iff M_1 is a type II_1 factor. Its *(Jones) index* is given by $[M : N] = \mathrm{tr}(e_1)^{-1} = \dim_N L^2(M)$. So M is finite dimensional as an N-module. There is an equivalent probabilistic definition due to Pimsner-Popa. The projection $e : L^2(M) \to L^2(N)$ restricts to a "conditional expectation" $E : M \to N$ satisfying $E(x) \geq \lambda x$ for $x \geq 0$ where $\lambda = [M : N]^{-1}$. The index yields the best possible value of $\lambda > 0$.

Higher Relative Commutants (Subfactor Invariants). It turns out that $[M_1 : M] = [M : N]$, so the basic construction can be iterated to get a tower:

$$N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset^{e_3} \cdots$$

The higher relative commutants are $A_n = M' \cap M_n$, $B_n = M_1' \cap M_n$. They are finite-dimensional von Neumann algebras, so direct sums of matrix algebras. The inclusions $B_n \subset A_n$ increase to an inclusion of type II_1 factors $B \subset A$. The inclusion $N \subset M$ is said to have *finite depth* if the centers of A_n and B_n have uniformly bounded dimension. The inclusion is *irreducible* iff $N' \cap M = \mathbb{C}$.

Bimodule Picture. $L^2(M)$ is a bimodule over (M, M), (M, N), (N, M), and (N, N). The algebras A_n and B_n encode the decomposition and branching rules for the bimodules $L^2(M)^{\boxtimes m}$, fused over N.

Popa's Finite Depth Classification Theorem. If the inclusion of hyperfinite type II_1 factors $N \subset M$ has finite depth and is irreducible, then $N \subset M \cong B^{\mathrm{op}} \subset A^{\mathrm{op}}$. A version of the same theorem also holds in the hyperfinite type III_1 case, provided that the Pimsner-Popa inequality is taken as the definition of finite index and the inclusion $B^{\mathrm{op}} \subset A^{\mathrm{op}}$ is replaced by its tensor product with the hyperfinite type III_1 factor.

13 Quantum Invariant Theory Subfactors

Classical Invariant Theory Subfactors. If V is a representation of G, we get an inclusion of type II_1 factors $(\cup_m \mathbb{C} \otimes \mathrm{End}_G V^{\otimes m})'' \subset (\cup_m \mathrm{End}_G V^{\otimes m+1})''$ with Jones index $\dim(V)^2$. When $G = SU(N)$ and $V = \mathbb{C}^N$, the right-hand side is generated by $S_\infty = \cup S_n$ and the left-hand side is obtained by applying the shift endomorphism $\rho(s_i) = s_{i+1}$ where $s_i = (i, i+1)$. The higher relative commutants are given by:

$$\begin{array}{lccccccc} A_n: & \mathrm{End}_G V & \subset & \mathrm{End}_G V \otimes \overline{V} & \subset & \mathrm{End}_G V \otimes \overline{V} \otimes V & \subset \\ & \cup & & \cup & & \cup & \\ B_n: & \mathbb{C} & \subset & \mathrm{End}_G \overline{V} & \subset & \mathrm{End}_G \overline{V} \otimes V & \subset \end{array} \tag{4}$$

Braid Group Subfactors (Jones-Wenzl). Let tr be a positive definite trace on the infinite braid group $B_\infty = \cup B_n$, generated by $g_1, g_2, g_3, \ldots$ with relations $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i - j| \leq 2$. Suppose that tr has the *Jones-Markov* property $\mathrm{tr}(a g_n^{\pm 1}) = \mu \mathrm{tr}(a)$ for $a \in B_n = \langle g_1, \ldots, g_{n-1} \rangle$. Form

$L^2(\mathbb{C}B_\infty, \mathrm{tr})$ and let π be the left unitary action of B_∞. Assume in addition that the algebras $\pi(\mathbb{C}B_n)$ are finite-dimensional and that the dimensions of their centers are uniformly bounded ("finite depth"). The braid group subfactor is given by the inclusion $\pi(g_2, g_3, \ldots)'' \subset \pi(g_1, g_2, \ldots)''$ and as for the symmetric group arises from a shift $\rho(g_i) = g_{i+1}$. It has index $|\mu|^{-2}$. More generally, Wenzl considered the irreducible parts of the inclusions $\pi(g_{m+1}, g_{m+2}, \ldots)'' \subset \pi(g_1, g_2, \ldots)''$, obtained by reducing by minimal projections in the relative commutant. The first examples arose by taking $g_i = ae_i + b$ with the e_i's Jones projections and a, b constants. Most other examples arose from the solutions of the quantum Yang-Baxter equations associated with quantum groups at roots of unity and restricted solid-on-solid models in statistical mechanics. Unfortunately, the positivity of the trace here only followed after a very detailed analysis of $\pi(\mathbb{C}B_n)$ using q-algebraic combinatorics.

Quantum Invariant Theory Subfactors. There is a more direct construction of the braid group subfactors. It is more conceptual, manifestly unitary, and allows a direct computation of the higher relative commutants. The data $(G, V, \otimes)$ is replaced by $(LG, H, \boxtimes)$: $(\cup_m \mathbb{C}\otimes \mathrm{End}_{LG} H^{\boxtimes m})'' \subset (\cup_m \mathrm{End}_{LG} H^{\boxtimes m+1})''$. If H corresponds to the vector representation, the right-hand side is generated by B_∞ and the left-hand side is obtained from the shift $\rho(g_i) = g_{i+1}$. The Jones index equals the square of the *quantum dimension* of H. This is given by $d(H_f) = \mathrm{ch}(H_f, 0)$ and is the unique positive character of $\mathcal{R}$. Thanks to "Wenzl's lemma", the higher relative commutants are obtained by replacing $(G, V, \otimes)$ by $(LG, H, \boxtimes)$ in (4).

14 Doplicher-Haag-Roberts Formalism

Algebraic QFT gives a translation from the bimodule to the homomorphism point of view. For fixed H_i, let $I \subset\subset J$ and $I^c \subset\subset K$ and take unitary intertwiners $U : H_0 \to H_i$ for $L_J G$ and $V : H_0 \to H_i$ for $L_K G$. Set $M = \pi_0(L_I G)''$. Then $\rho_i(x) = V^* U x U^* V$ defines a *DHR endomorphism* of M and the loop group inclusion $\pi_i(L_I G)'' \subset \pi_i(L_{I^c} G)'$ is isomorphic to the inclusion $\rho_i(M) \subset M$. The endomorphism ρ_i is *localized* in $I_1 = S^1 \backslash \overline{K} \subset\subset I$, in the sense that it fixes loop group elements supported in $I \backslash I_1$. Let T be a diffeomorphism, supported in I, with $T(I_1)$ disjoint from I_1 in a clockwise sense. Define the statistics operator by $g = u^* \rho_i(u)$, where $u = T^* \rho_i(T)$. Then $g\rho_i(g)g = \rho_i(g)g\rho_i(g)$ and g lies in $\rho_i^2(M)' \cap M$. Hence $g_k = \rho_i^{k-1}(g)$ gives a unitary representation of B_∞. Under the bimodule-endomorphism correspondence, the results on Connes fusion imply: $\rho_i^{k+1}(M)' \cap M \cong \mathrm{End}_{LG} H_i^{\boxtimes k+1}$, with $g_1, \ldots, g_k$ identified with the Connes braiding; the Jones index of the loop group subfactor is $d(H_i)^2$; and the higher relative commutants for the loop group subfactor agree with those of the corresponding quantum invariant theory subfactor.

15 The Main Result on Subfactors

Because the higher relative commutants agree, Popa's finite depth classification theorem implies:

Theorem (Jones-Wassermann Conjecture). *The loop group inclusion of hyperfinite type III_1 factors $\pi_i(L_I G)'' \subset \pi_i(L_{I^c} G)'$ is isomorphic to the tensor product of the* *hyperfinite type III_1 factor with the quantum invariant theory inclusion of type II_1 factors* $N_0 = (\cup \mathbb{C} \otimes \mathrm{End}_{LG} H_i^{\boxtimes m})'' \subset (\cup \mathrm{End}_{LG} H_i^{\boxtimes m+1})'' = M_0$.

This result may be sharpened using the inclusion $M \hookrightarrow M_2(M)$, $x \mapsto x \oplus \rho(x)$.

THEOREM. *There is an automorphism α of M and a unitary $u \in M$ such that $\alpha\rho = \mathrm{Ad}\, u\, \rho\alpha$ and, if $\rho_1 = \alpha\rho$ and $M_1 = (\cup \rho_1^m(M)' \cap M)''$, then $M = M_1 \overline{\otimes} M^{\rho_1}$, $N = \rho_1(M_1) \otimes M^{\rho_1}$. M^{ρ_1} is isomorphic to the hyperfinite type III_1 factor and the inclusion $N_1 \subset M_1$ is isomorphic to $N_0 \subset M_0$ by an isomorphism preserving endomorphisms and braid group operators.*

16 Future Directions

WZW Models. Other constructions of subfactors can be obtained by taking other compact simple groups G, not necessarily simply connected. The theory for the B, C, D series seems to follow from the (3×3)-matrix ODE of Fateev-Dotsenko.

Minimal Models. The theory has been developed along similar lines by Loke for discrete series representations of Diff S^1 for central charge $c < 1$.

Conformal Inclusions. A subgroup H of G gives a conformal inclusion if the level one representations of LG remain finitely reducible when restricted to H. The basic construction M_1 for the inclusion $N = \pi_0(L_I H)'' \subset \pi_0(L_I G)'' = M$ can be identified with $\pi_0(L_{I^c} H)'$, so $N \subset M_1$ is a loop group inclusion. So $N \subset M$ has finite index and depth. For example the conformal inclusion $SU(2) \subset SO(5)$ gives the Jones subfactor of index $3 + \sqrt{3}$.

Disjoint Intervals. If the circle is divided up into $2n$ disjoint intervals and I is the union of n alternate intervals, the inclusion $\pi_i(L_I G)'' \subset \pi_i(L_{I^c} G)'$ has finite index and probably finite depth. It is related to higher genus CFT.

Fusion. Connes fusion can be viewed as glueing together two circles along a common semicircle. This picture seems to be a unitary boundary value of Graeme Segal's holomorphic proposal for fusion, based on a disc with two smaller discs removed. When the discs shrink to points on the Riemann sphere, Segal's definition should degenerate to the algebraic geometric fusion of Kazhdan-Lusztig et al.

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