Configuration spaces, $\mathrm{FS}^{\mathrm{op}}$-modules, and Kazhdan-Lusztig polynomials of braid matroids

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Abstract. The equivariant Kazhdan-Lusztig polynomial of a braid matroid may be interpreted as the intersection cohomology of a certain partial compactification of the configuration space of n distinct labeled points in $\mathbb{C}$, regarded as a graded representation of the symmetric group S_n. We show that, in fixed cohomological degree, this sequence of representations of symmetric groups naturally admits the structure of an FS-module, and that the dual $\mathrm{FS}^{\mathrm{op}}$-module is finitely generated. Using the work of Sam and Snowden, we give an asymptotic formula for the dimensions of these representations and obtain restrictions on which irreducible representations can appear in their decomposition.

1 Introduction

Given a matroid M, the Kazhdan-Lusztig polynomial $P_M(t)$ was defined in [EPW16]. More generally, if M is equipped with an action of a finite group W, one can define the W-equivariant Kazhdan-Lusztig polynomial $P_M^W(t)$ [GPY]. By definition, $P_M^W(t)$ is a graded virtual representation of W, and taking dimension recovers the non-equivariant polynomial. These representations have been computed when M is a uniform matroid [GPY, Theorem 3.1] and conjecturally for certain graphical matroids [Ged, Conjecture 4.1]. However, in the case of the braid matroid (the matroid associated with the complete graph on n vertices), very little is known. The non-equivariant version of this problem was taken up in [EPW16, Section 2.5] and the S_n-equivariant version in [GPY, Section 4], but with few concrete results or even conjectures.

In this paper we use an interpretation of the equivariant Kazhdan-Lusztig polynomial of the braid matroid M_n as the intersection cohomology of a certain partially compactified configuration space to show that, in fixed cohomological degree, it admits the structure of an FS-module, as studied in [Pir00, CEF15, SS16]. Applying the results of Sam and Snowden [SS16], we use the FS-module structure (or, more precisely, the dual $\mathrm{FS}^{\mathrm{op}}$-module structure) to improve our understanding of this sequence of representations. In particular, we obtain the following results (Corollary 6.2):

- For fixed i, we prove that the generating function for the i^{th} non-equivariant Kazhdan-Lusztig coefficient of M_n (with n varying) is a rational function with poles lying in a prescribed set.
- For fixed i, we derive an asymptotic formula for the i^{th} non-equivariant Kazhdan-Lusztig coefficient of M_n in terms of another Kazhdan-Lusztig coefficient that depends only on i.
- We show that, if λ is a partition of n and the associated Specht module V_λ appears as a summand of the i^{th} equivariant Kazhdan-Lusztig coefficient of M_n, then λ has at most $2i$ rows.

We also produce relative versions of these results in which we start with an arbitrary graph Γ and consider the sequence of graphs whose n^{th} element is obtained from Γ by adding n new vertices and connecting them to everything (including each other). The original problem is the special case where Γ is the empty graph.

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2 Kazhdan-Lusztig polynomials and configuration spaces

Let M be a matroid on the ground set $\mathcal{I}$, equipped with an action of a finite group W. This means that W acts on $\mathcal{I}$ by permutations and that the action of W takes bases to bases. An **equivariant realization** of $W \curvearrowright M$ is W-subrepresentation $V \subset \mathbb{C}^{\mathcal{I}}$ such that $B \subset \mathcal{I}$ is a basis for M if and only if V projects isomorphically onto $\mathbb{C}^B$. Given such a V, we define $U_V := V \cap (\mathbb{C}^\times)^{\mathcal{I}}$, and we define the **reciprocal plane**

$$X_V := \overline{\{z \in (\mathbb{C}^\times)^{\mathcal{I}} \mid z^{-1} \in U_V\}} \subset \mathbb{C}^{\mathcal{I}},$$

which is a partial compactification of U_V.

Let $C^W_{M,i}$ denote the coefficient of t^i in the equivariant Kazhdan-Lusztig polynomial $P^W_M(t)$ of $W \curvearrowright M$. The following theorem appears in [GPY, Corollary 2.12] as an application of the work in [PWY16, Section 3].

Theorem 2.1. *If $V \subset \mathbb{C}^{\mathcal{I}}$ is an equivariant realization of $W \curvearrowright M$, then $C^W_{M,i}$ is isomorphic as a representation of W to the intersection cohomology group $IH^{2i}(X_V; \mathbb{C})$. In particular, $C^W_{M,i}$ is an honest (not just virtual) representation.*

Let $\mathcal{I}_n := \{(i,j) \mid i \neq j \in [n]\}$, and let M_n be the matroid on the ground set $\mathcal{I}_n$ whose bases consist of oriented spanning trees for the complete graph on n vertices. We will refer to M_n as the **braid matroid**, which comes equipped with a natural action of the symmetric group S_n.

Remark 2.2. It is more standard to define the braid matroid on the ground set of *unordered* pairs of elements of $[n]$. The two matroids are not the same (the ground set of our version is twice as large), but they have S_n-equivariantly isomorphic lattices of flats, so they have the same equivariant Kazhdan-Lusztig polynomials. We prefer the ordered version because it is equivariantly realizable (as we explain below), thus we may apply Theorem 2.1.

Consider the linear map $f : \mathbb{C}^n \to \mathbb{C}^{\mathcal{I}}$ given by $f_{ij}(z_1, \ldots, z_n) = z_i - z_j$. The kernel of f is equal to the diagonal line $\mathbb{C}_\Delta \subset \mathbb{C}^n$, so f descends to an inclusion of $V_n := \mathbb{C}^n/\mathbb{C}_\Delta$ into $\mathbb{C}^{\mathcal{I}}$, which gives an equivariant realization of $\mathbb{C}^n$. The space $U_n := U_{V_n}$ may be identified with the configuration space of n distinct labeled points in $\mathbb{C}$, modulo simultaneous translation, and the reciprocal plane

$X_n := X_{V_n}$ is a partial compactification of this configuration space. Informally, V_n is obtained from U_n by allowing the distances between points to go to zero, while X_n is obtained from U_n by allowing the distances between points to go to infinity.

Remark 2.3. The reciprocal plane X_n may also be described as the spectrum of the subring $\mathbb{C}\left[\frac{1}{x_i - x_j} \mid i \neq j\right]$ of the ring of rational functions on $\mathbb{C}^n$. This ring is called the **Orlik-Terao algebra** of $V_n \subset \mathbb{C}^{\mathcal{I}}$.

The non-equivariant Kazhdan-Lusztig polynomial of M_n for $n \leq 20$ appears in [EPW16, Section A.2]. The first few coefficients of this polynomial can be expressed in terms of Stirling numbers [EPW16, Corollary 2.24 and Proposition 2.26]. The same can be said of all of the terms, but the expressions become increasingly complicated. Indeed, we will show in an future paper that the i^{th} coefficient can be expressed as an alternating sum of i-fold products of Stirling numbers, where the number of summands is equal to $2\cdot 3^{i-1}$. We also made a conjecture about the leading term when n is even [EPW16, Section A]. The degree of the Kazhdan-Lusztig polynomial is by definition strictly less than half of the rank of the matroid, so the largest possible degree of $P_{M_{2i}}(t)$ is $i-1$.

Conjecture 2.4. *For all $i > 0$, $C_{M_{2i},i-1} = (2i-3)!!(2i-1)^{i-2}$, the number of labeled triangular cacti on $(2i-1)$ nodes [Slo14, Sequence A034941].*

The equivariant Kazhdan-Lusztig polynomial of the braid matroid is even more difficult to understand. The linear term is computed in [GPY, Proposition 4.4], and we also compute the remaining coefficients for $n \leq 9$ [GPY, Section 4.3]. We also give a functional equation that characterizes the generating function for the Frobenius characteristics of the equivariant Kazhdan-Lusztig polynomials [GPY, Equation (7)], but we do not know how to solve this equation.

3 The spectral sequence

In this section we explain how to construct a spectral sequence to compute the intersection cohomology of the reciprocal plane, which we will later use to endow the Kazhdan-Lusztig coefficients of braid matroids with an FS-module structure. Let W be a finite group acting on a finite set $\mathcal{I}$, and let $V \subset \mathbb{C}^{\mathcal{I}}$ be a subrepresentation. A subset $F \subset \mathcal{I}$ is called a **flat** if there exists a point $v \in V$ such that $F = \{i \mid v_i = 0\}$. Given a flat V, let $V^F := V \cap \mathbb{C}^{F^c} \subset \mathbb{C}^{F^c}$ and let $V_F \subset \mathbb{C}^F$ be the image of V along the projection $\mathbb{C}^{\mathcal{I}} \twoheadrightarrow \mathbb{C}^F$. The dimension of V_F is called the **rank** of F, while the dimension of V^F is called the **corank**. Let $U_F := U_{V_F}$ and $X^F := X_{V^F}$.

Theorem 3.1. *There exists a first quadrant cohomological spectral sequence E in the category of W-representations with*

$$E_1^{p,q} = \bigoplus_{\operatorname{crk} F = p} H^{2i-p-q}(U_F;\mathbb{C}) \otimes IH^{2(i-q)}(X^F;\mathbb{C})$$

converging to

$$\bigoplus_{p+q=2i} E_\infty^{p,q} = IH^{2i}(X_V) \qquad \textit{and} \qquad \bigoplus_{p+q\neq 2i} E_\infty^{p,q} = 0.$$

Proof. In the special case where $V = \{v \in \mathbb{C}^{\mathcal{I}} \mid \sum v_i = 0\}$ and W is the group of all permutations of $\mathcal{I}$, this theorem was proved in [PWY16, Proposition 3.3]. Almost all of the argument written there goes through unchanged in the more general setting. The only exception is the proof of [PWY16, Lemma 3.3], which is specific to that example. However, the more general version of this lemma follows from [EPW16, Theorem 3.3], along with the fact that étale maps are local isomorphisms in the analytic topology. □

Remark 3.2. Theorem 3.1 is the main ingredient of the proof of Theorem 2.1.

We now unpack Theorem 3.1 in the special case where $\mathcal{I} = \mathcal{I}_n$ and $V = V_n$. In this case, flats are in bijection with set-theoretic partitions of $[n]$. More precisely, given a partition of $[n]$, the set of all ordered pairs (i, j) such that i and j lie in the same block of the partition is a flat, and every flat arises in this way. A flat of corank p corresponds to a partition into $p+1$ (unlabeled) blocks $P_1, \ldots, P_{p+1}$. Given such a flat F, we have $U_F \cong U_{|P_1|} \times \cdots \times U_{|P_{p+1}|}$ and $X^F \cong X_{p+1}$. In order to clarify the issue of labeled versus unlabeled partitions, we make the following definitions:

$$A_i^{p,q}(n) := \bigoplus_{f:[n] \twoheadrightarrow [p+1]} H_{2i-p-q}\Big(U_{|f^{-1}(1)|} \times \cdots \times U_{|f^{-1}(p+1)|}; \mathbb{C}\Big) \otimes IH_{2(i-q)}(X_{p+1}; \mathbb{C})$$

and

$$B_i^{p,q}(n) := A_i^{p,q}(n)^{S_{p+1}},$$

where S_{p+1} acts on $[p+1]$. Thus we have the following corollary of Theorem 3.1.

Corollary 3.3. *There exists a first quadrant cohomological spectral sequence E in the category of S_n-representations with $E_1^{p,q} = B_i^{p,q}(n)^*$ converging to*

$$\bigoplus_{p+q=2i} E_\infty^{p,q} = IH^{2i}(X_n) \qquad \text{and} \qquad \bigoplus_{p+q\neq 2i} E_\infty^{p,q} = 0.$$

Remark 3.4. The reason for using homology rather than cohomology in the definition of $A_i^{p,q}(n)$ (and then undoing this by dualizing in Corollary 3.3) will become clear in Section 6. Briefly, the explanation is that intersection cohomology admits the structure of an FS-module and intersection homology admits the structure of an $\mathrm{FS}^{\mathrm{op}}$-module, and it is the $\mathrm{FS}^{\mathrm{op}}$-module structure that will prove to be more useful.

4 FS-modules and $\mathrm{FS}^{\mathrm{op}}$-modules

Let FS be the category whose objects are nonempty finite sets and whose morphisms are surjective maps. An FS-module is a covariant functor from FS to the category of complex vector spaces, and an $\mathrm{FS}^{\mathrm{op}}$-module is a contravariant functor from FS to the category of complex vector spaces. If N is an FS-module or an $\mathrm{FS}^{\mathrm{op}}$-module, we write $N(n) := N([n])$, which we regard as a representation of the symmetric group $S_n = \mathrm{Aut}_{\mathrm{FS}}([n])$. Let FA be the category whose objects are nonempty finite sets and whose morphisms are all maps.

For any positive integer m, let $P_m := \mathbb{C}\{\mathrm{Hom}_{\mathrm{FS}}(-,[m])\}$ be the $\mathrm{FS}^{\mathrm{op}}$-module that takes a finite set E to the vector space with basis given by surjections from E to $[m]$; this is a projective $\mathrm{FS}^{\mathrm{op}}$-module called the **principal projective** at m. We say that an $\mathrm{FS}^{\mathrm{op}}$-module N is **finitely generated** if it is isomorphic to the quotient of a finite sum of principal projectives, and we say that it is **finitely generated in degrees $\leq d$** if one only needs to use P_m for $m \leq d$. This is equivalent to the statement that, for any finite set E and any vector $v \in N(E)$, we can write v as a finite linear combination of elements of the form $f^*(x)$, where $f : E \twoheadrightarrow [m]$ and $x \in N(m)$ for some $m \leq d$.

We call an $\mathrm{FS}^{\mathrm{op}}$-module ***d*-small** if it is a subquotient of a module that is finitely generated in degrees $\leq d$. A d-small $\mathrm{FS}^{\mathrm{op}}$-module is always finitely generated [SS16, Corollary 8.1.3], but not necessarily in degrees $\leq d$.

For any partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \vdash n$, let V_λ be the corresponding irreducible representation of S_n. If λ is a partition of k and $n \geq k + \lambda_1$, let $\lambda(n)$ be the partition of n obtained by adding a part of size $n - k$. For any $\mathrm{FS}^{\mathrm{op}}$-module N, consider the ordinary generating function

$$H_N(u) := \sum_{n=1}^{\infty} u^n \dim N(n),$$

and the exponential generating function

$$G_N(u) := \sum_{n=1}^{\infty} \frac{u^n}{n!} \dim N(n).$$

For any natural number d, let

$$r_d(N) := \lim_{n\to\infty} \frac{\dim N(n)}{d^n},$$

which may or may not exist. The statements and proofs of the following results were communicated to us by Steven Sam.

Theorem 4.1. *Let N be a d-small $\mathrm{FS}^{\mathrm{op}}$-module.*

1. *If $\lambda \vdash n$ and $\mathrm{Hom}_{S_n}(V_\lambda, N(n)) \neq 0$, then $\ell(\lambda) \leq d$.*

2. *For any partition λ with $n \geq |\lambda| + \lambda_1$, $\dim \mathrm{Hom}_{S_n}\big(V_{\lambda(n)}, N(n)\big)$ is bounded by a polynomial in n of degree at most $d-1$.*

3. *The ordinary generating function $H_N(u)$ is a rational function whose poles are contained in the set $\{1/j \mid 1 \leq j \leq d\}$.*

4. *There exists polynomials $p_0(u), \ldots, p_d(u)$ such that the exponential generating function $G_N(u)$ is equal to*

$$\sum_{j=0}^{d} p_j(u) e^{ju}.$$

5. *The function $H_N(u)$ has at worst a simple pole at $1/d$. Equivalently, the limit $r_d(N)$ exists, and the polynomial $p_d(u)$ in statement 4 is the constant function with value $r_d(N)$.*

Proof. To prove statements 1 and 2, it is sufficient to prove them for the principal projective P_m for all $m \leq d$. Let $Q_m(-) := \mathbb{C}\{\mathrm{Hom}_{\mathrm{FA}}(-,[m])\}$, so that P_m is a submodule of Q_m. Then $Q_m(n) \cong (\mathbb{C}^m)^{\otimes n}$, and Schur-Weyl duality tells us that the multiplicity of V_λ in this representation is equal to the dimension of the representation of $\mathrm{GL}(m;\mathbb{C})$ indexed by λ. In particular, it is zero unless λ has at most m parts, and the dimension of the representation indexed by $\lambda(n)$ is bounded by a polynomial in n of degree at most $m-1$. Statements 1 and 2 follow for Q_m, and therefore for P_m.

If N' is finitely generated in degrees $\leq d$, then statement 3 holds for N' by [SS16, Corollary 8.1.4]. If N is a subquotient of N', then it is still finitely generated in degrees $\leq r$ for some r, so statement 3 holds for N with d replaced by r. But, since N is a subquotient of N', we have $\dim N(n) \leq \dim N'(n)$ for all n, which implies that $e_j = 0$ for all $j \leq r$. Statement 4 follows from statement 3 by finding a partial fractions decomposition of the ordinary generating function, as observed in [SS16, Remark 8.1.5].

To prove statement 5, it is again sufficient to consider P_m for all $m \leq d$. We have

$$\dim P_m(n) = |\,\mathrm{Hom}_{\mathrm{FS}}([n],[m])| \leq |\,\mathrm{Hom}_{\mathrm{FA}}([n],[m])| = m^n \leq d^n.$$

Since N is a subquotient of a finite direct sum of modules of this form, the dimension of $N(n)$ is bounded by a constant times d^n. □

We now record a pair of lemmas that say that certain natural constructions preserve smallness.

Lemma 4.2. *Fix a natural number k, a k-tuple of natural numbers $(d_1,\ldots,d_k)$, and a collection of $\mathrm{FS}^{\mathrm{op}}$-modules $N_1,\ldots,N_k$ such that N_i is d_i-small. Let $d = d_1+\cdots+d_k$. Then the $\mathrm{FS}^{\mathrm{op}}$-module N given by the formula*

$$N(E) := \bigoplus_{f:E\twoheadrightarrow[k]} N_1(f^{-1}(1)) \otimes \cdots \otimes N_k(f^{-1}(k))$$

is d-small.

Proof. Since d-smallness is preserved by taking direct sums and passing to subquotients, we may assume that $N_i = P_{m_i}$ for some $m_i \leq d_i$. Then

$$\begin{aligned}
N(E) &\cong \bigoplus_{f:E\twoheadrightarrow[k]} P_{m_1}(f^{-1}(1)) \otimes \cdots \otimes P_{m_k}(f^{-1}(k)) \\
&\cong \bigoplus_{f:E\twoheadrightarrow[k]} \mathbb{C}\left\{\mathrm{Hom}_{\mathrm{FS}}\big(f^{-1}(1),[m_1]\big)\right\} \otimes \cdots \otimes \mathbb{C}\left\{\mathrm{Hom}_{\mathrm{FS}}\big(f^{-1}(k),[m_k]\big)\right\} \\
&\cong \bigoplus_{f:E\twoheadrightarrow[k]} \mathbb{C}\Big\{\mathrm{Hom}_{\mathrm{FS}}\big(f^{-1}(1),[m_1]\big) \times \cdots \times \mathrm{Hom}_{\mathrm{FS}}\big(f^{-1}(k),[m_k]\big)\Big\} \\
&\cong \mathbb{C}\Big\{\mathrm{Hom}_{\mathrm{FS}}\big(E,[m_1]\sqcup\cdots\sqcup[m_k]\big)\Big\} \\
&\cong \mathbb{C}\Big\{\mathrm{Hom}_{\mathrm{FS}}\big(E,[m_1+\cdots+m_k]\big)\Big\} \\
&\cong P_{m_1+\cdots+m_k}(E),
\end{aligned}$$

so N is d-small. □

Lemma 4.3. *Let N be d-small and let S be any set. Let N_S be the FS-module defined by putting $N_S(E) := N(S \sqcup E)$ for all E, with maps defined in the obvious way. Then N_S is also d-small.*

Proof. As in the proof of Lemma 4.2, we may reduce to the case where $N = P_m$ for $m \leq d$. In this case, it is sufficient to show that every surjection $f : S \sqcup E \to [m]$ factors as $g \circ (\mathrm{id}_S \sqcup h)$, where g is a surjection from $S \sqcup [j]$ to $[m]$ for some $j \leq m$ and h is a surjection from $[m]$ to $[j]$. It is clear that we can do this by taking j to be the cardinality of $f(E)$. □

Finally, the following lemma will be needed in the proof of Theorem 6.1.

Lemma 4.4. *Suppose that $N \to N' \to N''$ is a complex of d-small $\mathrm{FS}^{\mathrm{op}}$-modules, and let H denote its homology in the middle. If $r_d(N) = 0 = r_d(N'')$, then $r_d(H) = r_d(N')$.*

Proof. This follows from the fact that $\dim N'(n) - \dim N(n) - \dim N''(n) \leq \dim H(n) \leq \dim N(n)$ and the definition of r_d. □

5 Configurations of points in the plane

Let $\mathrm{Conf}(E)$ be the space of injective maps from E to $\mathbb{R}^2$. Arnol'd [Arn69] proved that

$$H^*(\mathrm{Conf}(E);\mathbb{C}) \cong \Lambda_{\mathbb{C}}\left[x_{ij} \mid i,j \in E\right] \Big/ \Big\langle x_{ii},\ \ x_{ij} - x_{ji},\ \ x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} \Big\rangle.$$

Let $H^i(E) := H^i(\mathrm{Conf}(E);\mathbb{C})$ and $H_i(E) := H_i(\mathrm{Conf}(E);\mathbb{C}) \cong H^i(\mathrm{Conf}(E);\mathbb{C})^*$. Given a map $f : E \to F$, we have a map $H^*(\mathrm{Conf}(E);\mathbb{C}) \to H^*(\mathrm{Conf}(F);\mathbb{C})$ taking x_{ij} to $x_{f(i)f(j)}$. This gives H^i the structure of an FA-module and H_i the structure of an $\mathrm{FA}^{\mathrm{op}}$-module. Since FS is a subcategory of FA, we may regard H^i as an FS-module and H_i as an $\mathrm{FS}^{\mathrm{op}}$-module.

Proposition 5.1. *The $\mathrm{FS}^{\mathrm{op}}$-module H_0 is 1-small. If $i \geq 1$, then H_i is $2i$-small and $r_{2i}(H_i) = 0$.*

Proof. We have $H_0 \cong P_1$, which is by definition 1-small. Since $H^*(E)$ is generated in degree 1, $H^i(E)$ is a quotient of $H^1(E)^{\otimes i}$. This means that $H_i(E)$ is a subspace of $H_1(E)^{\otimes i}$, thus to prove $2i$-smallness it will suffice to show that H_1 is finitely generated in degrees ≤ 2. We begin by showing that H_1 is finitely generated in degrees ≤ 3. Let E be any set; the group $H_1(E)$ has a basis $\{e_{ij}\}$, dual to the basis $\{x_{ij}\}$ for $H^1(E)$. Let $i \neq j$ be elements of E, and consider the map $E \to \{1,2,3\}$ taking i to 1, j to 2, and everything else to 3. The induced map $H_1(\{1,2,3\}) \to H_1(E)$ takes e_{12} to e_{ij}, so we obtain a surjective map from the projective module $P_{\{1,2,3\}}$ to $H_1(E)$.

To get down from 3 to 2, consider the parity map $\{1,2,3\} \to \{1,2\}$. The induced map $H_1(\{1,2\}) \to H_1(\{1,2,3\})$ takes e_{12} to $e_{12} + e_{23}$. By symmetry, we can vary the map and obtain $e_{13} + e_{23}$ and $e_{12} + e_{13}$ as images of induced maps from $H_1(\{1,2\})$ to $H_1(\{1,2,3\})$. Since these three vectors span $H_1(\{1,2,3\})$, H_1 is generated in degree 2.

For the last statement, we begin by noting that $\dim H_1(n) = \binom{n}{2}$, therefore

$$r_2(H_1) = \lim_{n\to\infty} 2^{-n}\binom{n}{2} = 0.$$

This implies $r_{2i}(H_1^{\otimes i}) = r_2(H_1)^i = 0$. Since $H_i \subset H_1^{\otimes i}$, we have $r_{2i}(H_i) = 0$, as well. □

For any $p \geq 0$, let

$$\begin{aligned}
\mathrm{Comp}_{p,i}(E) &:= \bigoplus_{f:E\twoheadrightarrow[p+1]} \Big(H_\bullet(f^{-1}(1)) \otimes \cdots \otimes H_\bullet(f^{-1}(p+1))\Big)_i \\
&\cong \bigoplus_{\substack{f:E\twoheadrightarrow[p+1] \\ i_1+\cdots+i_{p+1}=i}} H_{i_1}(f^{-1}(1)) \otimes \cdots \otimes H_{i_{p+1}}(f^{-1}(p+1)).
\end{aligned}$$

It is clear that $\mathrm{Comp}_{p,i}$ comes endowed with a natural $\mathrm{FS}^{\mathrm{op}}$-module structure.

Proposition 5.2. *The* $\mathrm{FS}^{\mathrm{op}}$*-module* $\mathrm{Comp}_{p,0}$ *is* $(p+1)$*-small, and* $\mathrm{Comp}_{p,i}$ *is* $(p+2i)$*-small for all* $i \geq 1$.

Proof. By Lemma 4.2 and Proposition 5.1 the summand of $\mathrm{Comp}_{p,i}$ corresponding to the tuple $(i_1, \ldots, i_{p+1})$ is $(d+2i)$-small, where d is the number of k such that $i_k = 0$. When $i = 0$, we have $d = p+1$. When $i > 0$, the maximum value of d is p. □

6 The main theorem

For any finite set E, let $D_i(E)$ be the i^{th} $\mathrm{Aut}(E)$-equivariant Kazhdan-Lusztig coefficient of the matroid associated with the complete graph on the vertex set E. In particular, if we take $E = [n]$, we have $D_i(n) = C^{S_n}_{M_n,i}$.

For any non-negative integers p, q, define

$$A^{p,q}_i(E) := \mathrm{Comp}_{p,2i-p-q}(E) \otimes D^*_{i-q}(p+1).$$

Since $\mathrm{Comp}_{p,2i-p-q}$ is an $\mathrm{FS}^{\mathrm{op}}$-module with an action of the symmetric group S_{p+1} (given by permuting the pieces of the composition) and $D_{i-q}(p+1)^*$ is a fixed vector space equipped with an action of S_{p+1}, $A_i^{p,q}$ inherits the structure of an $\mathrm{FS}^{\mathrm{op}}$-module with an action of the symmetric group S_{p+1}. Let $B_i^{p,q} := (A_i^{p,q})^{S_{p+1}}$ be the invariant submodule, and let $(B_i^{p,q})^*$ be the dual FS-module. By Corollary 3.3, we have a first quadrant cohomological spectral sequence with E_1 page $B_i^{p,q}(E)^*$ that converges to $D_i(E)$. Since each $(B_i^{p,q})^*$ is an FS-module and the differentials in the spectral sequence are compatible with the module structure, D_i inherits the structure of an FS-module. Dually, D_i^* is an $\mathrm{FS}^{\mathrm{op}}$-module, and we have a first quadrant homological spectral sequence of $\mathrm{FS}^{\mathrm{op}}$-modules with E_1 page $B_i^{p,q}$ that converges to D_i^*.

Theorem 6.1. *For all $i \geq 1$, the* $\mathrm{FS}^{\mathrm{op}}$*-module D_i^* is $2i$-small, and we have*

$$r_{2i}(D_i^*) = \frac{\dim D_{i-1}(2i)}{(2i)!}.$$

Proof. We first prove that D_i^* is $2i$-small. Since smallness is preserved under taking subquotients, it suffices to prove that $B_i^{p,q}$ is $2i$-small for all p and q. Since $B_i^{p,q} \subset A_i^{p,q}$, it suffices to prove it for $A_i^{p,q}$. By Proposition 5.2 and the fact that smallness is preserved by taking a tensor product with a fixed vector space, $A_i^{p,q}$ is $(p+1)$-small when $p+q=2i$ and $(p+2(2i-p-q))$-small otherwise.

Consider the case where $p+q=2i$. By definition of the equivariant Kazhdan-Lusztig polynomial, $D_i(E)=0$ unless $2i < |E|-1$ or $|E|=1$ and $i=0$. In particular, if $p=2i$ and $q=0$, then $D_{i-q}(p+1) = D_i(2i) = 0$, and therefore $A_i^{p,q}=0$. Thus we may assume that $p<2i$. Since $A_i^{p,q}$ is $(p+1)$-small it is also $2i$-small.

Next, consider the case where $p+q<2i$, so $A_i^{p,q}$ is $(p+2(2i-p-q))$-small. By the above vanishing property for $D_i(E)$, we have $D_{i-q}(p+1)=0$ unless $2(i-q)<p$ or $p=0$ and $q=i$. Thus we may conclude that $A_i^{p,q}=0$ unless

$$p+2(2i-p-q)+p = 2(i-q)-p+2i < 2i \qquad \text{or} \qquad p=0 \text{ and } q=i.$$

In particular, $A_i^{p,q}$ is $2i$-small, and therefore so is D_i^*.

This argument in fact proves that $A_i^{p,q}$ is $(2i-1)$-small unless $(p,q)=(0,i)$ or $(2i-1,1)$, and the same is therefore true for $B_i^{p,q}$. Furthermore, we have $B_i^{0,i} \cong H_i$, and Proposition 5.1 tells us that $r_{2i}(H_i)=0$. Thus $r_{2i}(B_i^{p,q})=0$ unless $(p,q)=(2i-1,1)$, and Lemma 4.4 therefore tells us that $r_{2i}(D_i^*) = r_{2i}(B_i^{2i-1,1})$.

We have $B_i^{2i-1,1} \cong (\mathrm{Comp}_{2i-1,0})^{S_{2i}} \otimes D_{i-1}^*(2i)$, where $(\mathrm{Comp}_{2i-1,0})^{S_{2i}}$ is the $\mathrm{FS}^{\mathrm{op}}$-module that takes E to a vector space with basis given by partitions of E into $2i$ nonempty pieces. This means that $\dim(\mathrm{Comp}_{2i-1,0})^{S_{2i}}(n)$ is equal to the Stirling number of the second kind $S(n,2i)$, thus

$$r_{2i}(D_i^*) = r_{2i}(B_i^{2i-1,1}) = \lim_{n\to\infty} \frac{\dim B_i^{2i-1,1}(n)}{(2i)^n} = \lim_{n\to\infty} \frac{S(n,2i)\dim D_{i-1}(2i)}{(2i)^n} = \frac{\dim D_{i-1}(2i)}{(2i)!},$$

and the theorem is proved. □

Let $H_i(u) := H_{D_i^*}(u)$ and $G_i(u) := G_{D_i^*}(u)$. Note that, since representations of finite groups are self-dual, $H_i(u)$ and $G_i(u)$ may be regarded as generating functions (ordinary and exponential) for the degree i Kazhdan-Lusztig coefficients of braid matroids. The following corollary follows immediately from Theorems 4.1 and 6.1.

Corollary 6.2. *Let i be a positive integer.*

1. *If $\lambda \vdash n$ and $\operatorname{Hom}_{S_n}(V_\lambda, D_i(n)) \neq 0$, then $\ell(\lambda) \leq 2i$.*
2. *For any partition λ with $n \geq |\lambda| + \lambda_1$, $\dim \operatorname{Hom}_{S_n}\big(V_{\lambda(n)}, D_i(n)\big)$ is bounded by a polynomial in n of degree at most $2i - 1$.*
3. *The ordinary generating function $H_i(u)$ is a rational function whose poles are contained in the set $\{1/j \mid 1 \leq j \leq 2i\}$. Furthermore, $H_i(u)$ has at worst a simple pole at $1/2i$.*
4. *There exists polynomials $p_0(u), \ldots, p_{2i}(u)$ such that the exponential generating function $G_i(u)$ is equal to*
$$\sum_{j=0}^{d} p_j(u) e^{ju}.$$
Furthermore, $p_{2i}(u)$ is equal to the constant polynomial with value $r_{2i}(D_i^*) = \frac{\dim D_{i-1}(2i)}{(2i)!}$.

Remark 6.3. Theorem 6.1 and Conjecture 2.4 combine to say that
$$r_{2i}(D_i^*) = \frac{(2i-3)!!(2i-1)^{i-2}}{(2i)!} = \frac{(2i-1)^{i-3}}{2^i\, i!}.$$
In particular, if Conjecture 2.4 is true (or more generally if $D_{i-1}(2i) \neq 0$), then $H_i(u)$ does have a pole at $1/2i$.

Remark 6.4. Let X_E be the reciprocal plane associated with the set E. As explained in Section 6, this is a partial compactification of $\operatorname{Conf}(E)$ in which the distances between pairs of points are allowed to go to infinity. By Theorem 2.1, we have canonical isomorphisms $D_i(E) \cong IH^{2i}(X_E; \mathbb{C})$ and $D_i(E)^* \cong IH_{2i}(X_E; \mathbb{C})$. For every surjection $f : E \to F$, we obtain a normally nonsingular inclusion of X_F into X_E as a slice to the stratum of X_E given by the partition of E into the fibers of f [EPW16, Theorem 3.3]. This induces a pullback map in intersection cohomology and a pushforward map in intersection homology, giving a geometric interpretation of the FS-module structure on D_i and the $\mathrm{FS}^{\mathrm{op}}$-module structure on D_i^*. However, we know of no way to use this perspective directly to prove Theorem 6.1, or even to prove that D_i^* is finitely generated.

7 Examples

We now example the cases when $i = 1$ or 2 in greater detail.

Example 7.1. We first consider the case when $i = 1$. In [GPY, Proposition 4.3], we showed that $\mathrm{Hom}_{S_n}(V_\lambda, D_1(n)) = 0$ for all λ with more than 2 rows, and that $\dim \mathrm{Hom}_{S_n}\left(V_{[k](n)}, D_1(n)\right)$ is bounded by $n/2+1-k$. By [EPW16, Corollary 2.24], we have $\dim D_1(n) = 2^{n-1} - 1 - \binom{n}{2}$, which implies that

$$H_1(u) = \frac{u^4}{(1-u)^3(1-2u)}$$

and

$$G_1(u) = \frac{1}{2} + \left(\frac{u^2}{2} - 1\right) e^u + \frac{1}{2} e^{2u}.$$

In particular, $r_2(D_1^*) = 1/2 = \dim D_0(2)/2!$.

Example 7.2. We next consider the case when $i = 2$. By [EPW16, Corollary 2.24], we have

$$\dim D_2(n) = s(n, n-2) - S(n, n-1)S(n-1, 2) + S(n, 3) + S(n, 4),$$

where $s(n, k)$ and $S(n, k)$ are Stirling numbers of the first and second kind, respectively. We have well-known generating function identities

$$\sum_{n\geq 1} S(n,k)u^n = \frac{u^k}{\prod_{j=1}^{k}(1-ju)},$$

as well as [Slo14, A000914]

$$\sum_{n\geq 1} s(n,n-2)u^n = \frac{2u^3+u^4}{(1-u)^5}.$$

Since $S(n, n-1)S(n-1, 2) = \binom{n}{2}\left(2^{n-2} - 1\right)$, it is not hard to show that

$$\sum_{n\geq 1} S(n,n-1)S(n-1,2)u^n = \frac{u^2}{(1-2u)^3} - \frac{u^2}{(1-u)^3}.$$

Putting it all together, we get

$$\begin{aligned} H_2(u) &= \frac{2u^3+u^4}{(1-u)^5} - \left(\frac{u^2}{(1-2u)^3} - \frac{u^2}{(1-u)^3}\right) \\ &\quad + \frac{u^3}{(1-u)(1-2u)(1-3u)} + \frac{u^4}{(1-u)(1-2u)(1-3u)(1-4u)} \\ &= \frac{15u^6 - 50u^7 + 40u^8 + 4u^9}{(1-u)^5(1-2u)^3(1-4u)}. \end{aligned}$$

After performing a partial fractions decomposition we find that $r_4(D_2^*) = 1/24 = \dim D_1(4)/4!$.

We do not have a general formula for the dimension of $\mathrm{Hom}_{S_n}(V_\lambda, D_2(n))$, but we have computed $D_2(n)$ for all $n \leq 9$ [GPY, Section 4.4], and it is indeed the case in these examples that the multiplicity of V_λ is $D_2(n)$ is zero whenever λ has more than 4 rows.

8 The relative case

Let Γ be a finite graph with vertex set V. For any finite set E, let $\Gamma(E)$ be the graph with vertex set $V \sqcup E$ such that two elements of V are adjacent if and only if they were adjacent in Γ, and elements of E are adjacent to everything. We will define an FS-module structure on the i^{th} $\operatorname{Aut}(E)$-equivariant Kazhdan-Lusztig coefficient $D_i^\Gamma(E)$ of the matroid associated with the graph $\Gamma(E)$, and prove that the dual $\mathrm{FS}^{\mathrm{op}}$-module is $2i$-small. If Γ is the empty graph, then $\Gamma(E)$ is just the complete graph on E, so we have $D_i^\Gamma = D_i$.

We begin by generalizing the material in Section 5. Let $\Gamma = (V, Q)$ be a finite graph with vertex set V and edge set Q, and let $\operatorname{Conf}(\Gamma)$ be the set of maps from V to $\mathbb{R}^2$ that send adjacent vertices to distinct points. We have the following description of the cohomology ring of $\operatorname{Conf}(\Gamma)$ [OT92, Theorems 3.126 and 5.89]:

$$\begin{aligned} H^*(\operatorname{Conf}(\Gamma);\mathbb{C}) &\cong \Lambda_{\mathbb{C}}[x_q]_{q\in Q} \Big/ \left\langle \sum_{j=1}^{k} (-1)^j x_{q_1}\cdots \hat{x}_{q_j}\cdots x_{q_k} \;\middle|\; (q_1,\ldots,q_k) \text{ a closed path} \right\rangle \\ &\cong \text{the subring of all meromorphic differential forms on } \mathbb{C}^V \\ &\ \text{generated by } \frac{dz_i - dz_j}{z_i - z_j} \text{ for all } \{i,j\} \in Q. \end{aligned}$$

By definition, a map from $\Gamma = (V, Q)$ to $\Gamma' = (V', Q')$ is a map from V to V' that takes Q to Q'. Given a map $f : \Gamma \to \Gamma'$, we obtain a map $H^*(\operatorname{Conf}(\Gamma);\mathbb{C}) \to H^*(\operatorname{Conf}(\Gamma');\mathbb{C})$ taking x_q to $x_{f(q)}$. In particular, we obtain an FA-module $H^i_\Gamma(E) := H^i(\operatorname{Conf}(\Gamma(E));\mathbb{C})$ and a dual $\mathrm{FA}^{\mathrm{op}}$-module $H_i^\Gamma(E) := H_i(\operatorname{Conf}(\Gamma(E));\mathbb{C})$. As in the case where Γ is empty, we can regard H^i_Γ as an FS-module and H_i^Γ as an $\mathrm{FS}^{\mathrm{op}}$-module. The proof of the following proposition is identical to the proof of Proposition 5.1.

Proposition 8.1. *The* $\mathrm{FS}^{\mathrm{op}}$*-module* H_0^Γ *is 1-small. If* $i \geq 1$*, then* H_i^Γ *is* $2i$*-small and* $r_{2i}(H_i^\Gamma) = 0$.

Given a graph Γ with vertex set V and a subset $S \subset V$, let Γ_S be the induced subgraph with vertex set S. Given a surjective map $f : V \to V'$, let Γ^f be the graph with vertex set V' whose edges are the images of edges of Γ (ignoring loops and multiple edges). Fix a graph Δ with vertex set $[p+1]$, and define

$$\operatorname{Comp}_{p,i}^{\Gamma,\Delta}(E) := \bigoplus_{\substack{f: V\sqcup E \twoheadrightarrow [p+1] \\ \Gamma(E)^f = \Delta \\ \Gamma(E)_{f^{-1}(j)} \text{ connected } \forall j}} H_i\Big(\operatorname{Conf}\big(\Gamma(E)_{f^{-1}(1)}\big) \times \cdots \times \operatorname{Conf}\big(\Gamma(E)_{f^{-1}(p+1)}\big);\mathbb{C}\Big).$$

Given surjective maps $g : E \to F$ and $f : V \sqcup F \to [p+1]$ such that $\Gamma(E)_{f^{-1}(j)}$ is connected for all j, we can compose f with g to obtain a surjective map $g^*f : V \sqcup E \to [p+1]$ with the property that $\Gamma(E)_{(g^*f)^{-1}(i)}$ is connected for all j and $\Gamma(E)^{g^*f} = \Gamma(F)^f$. This observation allows us to define an $\mathrm{FS}^{\mathrm{op}}$-module structure on $\operatorname{Comp}_{p,i}^{\Gamma,\Delta}$. Taking Γ to be the empty graph and Δ the complete graph, we have $\operatorname{Comp}_{p,i}^{\Gamma,\Delta} = \operatorname{Comp}_{p,i}$. The following proposition generalizes Proposition

5.2.

Proposition 8.2. *The* $\mathrm{FS}^{\mathrm{op}}$*-module* $\mathrm{Comp}_{p,0}^{\Gamma,\Delta}$ *is* $(p+1)$*-small, and* $\mathrm{Comp}_{p,i}^{\Gamma,\Delta}$ *is* $(p+2i)$*-small for all* $i \geq 1$.

Proof. Let $\mathrm{Comp}_{p,i}^{\Gamma} := \bigoplus_\Delta \mathrm{Comp}_{p,i}^{\Gamma,\Delta}$. We will prove that $\mathrm{Comp}_{p,i}^{\Gamma}$ is $(p+1)$-small when $i = 0$ and $(p+2i)$-small when $i \geq 1$, and therefore so is each of its summands. The above description of the cohomology ring of $\mathrm{Conf}(\Gamma)$ in terms of meromorphic differential forms makes it clear that $H^*(\mathrm{Conf}(\Gamma);\mathbb{C})$ is a subring of $H^*(\mathrm{Conf}(V);\mathbb{C})$, and therefore that the f-summand of $\mathrm{Comp}_{p,i}^{\Gamma,\Delta}(E)$ is a quotient of the f-summand of $\mathrm{Comp}_{p,i}(V \sqcup E)$. The proposition then follows from Proposition 5.2 and Lemma 4.3. □

We next generalize the material in Section 6. For any finite set E and any non-negative integers p, q, define

$$A_{\Gamma,i}^{p,q}(E) \;:=\; \bigoplus_{\Delta} \mathrm{Comp}_{p,2i-p-q}^{\Gamma,\Delta}(E) \;\otimes\; D_{i-q}^{\Delta}(\emptyset)^*.$$

As in the case where Γ is the empty graph, $A_{\Gamma,i}^{p,q}$ is an $\mathrm{FS}^{\mathrm{op}}$-module with an action of S_{p+1}, and we define the invariant $\mathrm{FS}^{\mathrm{op}}$-module $B_{\Gamma,i}^{p,q} := (A_i^{p,q})^{S_{p+1}}$ along with its dual FS-module $(B_{\Gamma,i}^{p,q})^*$. There is again a first quadrant cohomological spectral sequence with E_1 page $B_{\Gamma,i}^{p,q}(E)^*$ that converges to $D_i^{\Gamma}(E)$, inducing an FS-module structure on D_i^{Γ}.

Theorem 8.3. *Let* Γ *be a graph with vertex set* V*. For all* $i \geq 1$*, the* $\mathrm{FS}^{\mathrm{op}}$*-module* $(D_i^{\Gamma})^*$ *is* $2i$*-small, and we have*

$$r_{2i}\big((D_i^{\Gamma})^*\big) = \frac{(2i)^{|V|} \dim D_{i-1}(2i)}{(2i)!} = (2i)^{|V|} r_{2i}(D_i^*).$$

Proof. The same argument that we used in the proof of Theorem 6.1 shows that $(D_i^{\Gamma})^*$ is $2i$-small and $r_{2i}\big((D_i^{\Gamma})^*\big) = r_{2i}(B_{\Gamma,i}^{2i-1,1})$. Explicitly, we have

$$B_{\Gamma,i}^{2i-1,1}(E) \;=\; \left(\bigoplus_{f: V \sqcup E \twoheadrightarrow [2i]} D_{i-1}^{\Gamma(E)^f}(\emptyset)^* \right)^{S_{2i}}.$$

When E is large, $\Gamma(E)_{f^{-1}(j)}$ is connected for all j and $\Gamma(E)^f$ is equal to K_{2i} for almost all maps $f : V \sqcup E \twoheadrightarrow [2i]$, and the number of such maps is asymptotic to $(2i)^{|V|+n}$. We therefore have

$$r_{2i}(B_{\Gamma,i}^{2i-1,1}) = \lim_{n\to\infty} \frac{(2i)^{|V|+n} \dim D_{i-1}(2i)}{(2i)^n (2i)!} = \frac{(2i)^{|V|} \dim D_{i-1}(2i)}{(2i)!},$$

and the theorem is proved. □

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