The algebraic geometry of Kazhdan-Lusztig-Stanley polynomials

Nicholas Proudfoot
Department of Mathematics, University of Oregon, Eugene, OR 97403
njp@uoregon.edu

Abstract. Kazhdan-Lusztig-Stanley polynomials are a combinatorial generalization of Kazhdan-Lusztig polynomials of Coxeter groups that include g-polynomials of polytopes and Kazhdan-Lusztig polynomials of matroids. In the cases of Weyl groups, rational polytopes, and realizable matroids, one can count points over finite fields on flag varieties, toric varieties, or reciprocal planes to obtain cohomological interpretations of these polynomials. We survey these results and unite them under a single geometric framework.

1 Introduction

The original definition of Kazhdan-Lusztig polynomials for a Coxeter group involves the relationship between two bases for the Hecke algebra, however the polynomials are characterized by a purely combinatorial recursion involving intervals in the Bruhat poset [KL79, Equation (2.2.a)]. Stanley later generalized this recursive definition, replacing the Bruhat poset with an arbitrary locally graded poset [Sta92, Definition 6.2(b)]. Stanley's main motivation was the observation that the g-polynomial of a polytope, which he introduced in [Sta87], arises very naturally in this way [Sta92, Example 7.2]. Brenti went on to generalize this definition slightly further to weakly ranked posets, and dubbed the corresponding polynomials **Kazhdan-Lusztig-Stanley polynomials** [Bre99]. More examples have been studied since then, including Kazhdan-Lusztig polynomials of matroids, which were introduced in [EPW16] and have been the subject of much recent research.

In a sequel to their first paper, Kazhdan and Lusztig proved that their polynomials can be interpreted as Poincaré polynomials for the stalk cohomology groups of the intersection cohomology sheaves of Schubert varieties [KL80, Theorem 4.3]. The idea of the proof is that the combinatorial recursion for the polynomials is precisely the recursion for the Poincaré polynomials that one obtains by applying the Lefschetz fixed point formula to the Frobenius automorphism of certain subvarieties of the flag variety. This technique has subsequently been imported to the study of other classes of Kazhdan-Lusztig-Stanley polynomials, including Kazhdan-Lusztig polynomials of affine Weyl groups [Lus83, Section 11], g-polynomials of rational polytopes [DL91, Theorem 6.2] and [Fie91, Theorem 1.2], h-polynomials of broken circuit complexes of a rational hyperplane arrangements [PW07, Theorem 4.3], and Kazhdan-Lusztig polynomials of hyperplane arrangements [EPW16, Theorem 3.10].

While the various instances of the aforementioned Lefschetz argument are all based on the same idea, they all involve a rather messy induction, and it can be difficult to determine exactly what ingredients are needed to make the argument work. The purpose of this document is to do exactly that. After reviewing the combinatorial theory of Kazhdan-Lusztig-Stanley polynomials (Section 2), we lay out a basic geometric framework for interpreting these polynomials as Poincaré polynomials of stalks of intersection cohomology sheaves on a stratified variety (Section 3). In particular, we show that each of the aforementioned results can be obtained as an application of our general machine (Section 4) without having to redo the inductive argument each time.

Though our main purpose is to survey and unify various old results, there is one new concept that we introduce and study here. When defining Kazhdan-Lusztig-Stanley polynomials, there is a left versus right convention that appears in the definition. The left Kazhdan-Lusztig-Stanley polynomials for a weakly graded poset P coincide with the right Kazhdan-Lusztig-Stanley polynomials for the opposite poset P^* (Remark 2.4). In particular, since the Bruhat poset of a finite Coxeter group is self-opposite and the face poset of a polytope is opposite to the face poset of the dual polytope, the left/right issue (while at times confusing) is not so important. The same statement is not true of the lattice of flats of a matroid, and indeed the right Kazhdan-Lusztig-Stanley polynomials of a matroid are interesting while the left ones are trivial (Example 2.13). We introduce a class of polynomials called ***Z*-polynomials** (Section 2.3) that depend on both the left and right Kazhdan-Lusztig-Stanley polynomials. In the case of the lattice of flats of a matroid, these polynomials coincide with the polynomials introduced in [PXY18].

Under certain assumptions, we use another Lefschetz argument to interpret our Z-polynomials a Poincaré polynomials for the global intersection cohomology of the closure of a stratum in our stratified variety. In particular, in the case of the Bruhat poset of a Weyl group, the Z-polynomials are intersection cohomology Poincaré polynomials of Richardson varieties (Theorem 4.3); in the case of the lattice of flats of a hyperplane arrangements, they are intersection cohomology Poincaré polynomials of arrangement Schubert varieties (Theorem 4.17); and in the case of the affine Grassmannian, they are intersection cohomology Poincaré polynomials of closures of Schubert cells (Corollary 4.8).

It would be interesting to know whether the Z-polynomial of a rational polytope has a cohomological interpretation in terms of toric varieties. These polynomials are closely related to a family of polynomials defined by Batyrev and Borisov (Remark 2.15), but they are not quite the same.

1.1 Things that this paper is not about

There are many interesting questions about Kazhdan-Lusztig-Stanley polynomials that we will mention briefly here but not address in the main part of the paper.

- By giving a cohomological interpretation of a Kazhdan-Lusztig-Stanley polynomial, one can infer that it has non-negative coefficients. There is a rich history of pursuing the non-negativity of certain classes Kazhdan-Lusztig polynomials in the absence of a geometric interpretation. This was achieved by Elias and Williamson for Kazhdan-Lusztig polynomials of Coxeter groups that are not Weyl groups [EW14, Conjecture 1.2(1)] and by Karu for polytopes that are not rational [Kar04, Theorem 0.1] (see also [Bra06, Theorem 2.4(b)]). Braden, Huh, Matherne, Wang, and the author are working to prove an analogous theorem for matroids that are not realizable by hyperplane arrangements.

- For many specific classes of Kazhdan-Lusztig-Stanley polynomials, it is interesting to ask what polynomials can arise. Polo proved that any polynomial with non-negative coefficients and constant term 1 is equal to a Kazhdan-Lusztig polynomial for a symmetric group [Pol99]. In contrast, the g-polynomial of a polytope cannot have internal zeros [Bra06, Theorem 1.4]. If the polytope is simplicial, then the sequence of coefficients is an M-sequence [Sta80], and this is conjecturally the case for all polytopes; see [Bra06, Section 1.2] for a discussion of this conjecture. Kazhdan-Lusztig polynomials of matroids are conjectured to always be log concave with no internal zeros [EPW16, Conjecture 2.5] and even real-rooted [GPY17, Conjecture 3.2], and a similar conjecture has been made for Z-polynomials of matroids [PXY18, Conjecture 5.1].

- Classical Kazhdan-Lusztig polynomials were originally defined in terms of the Kazhdan-Lusztig basis for the Hecke algebra. More generally, Du defines the notion of an **IC basis** for a free $\mathbb{Z}[t, t^{-1}]$-module equipped with an involution [Du94], and Brenti proves that this notion is essentially equivalent to the theory of Kazhdan-Lusztig-Stanley polynomials [Bre03, Theorem 3.2]. Multiplication in the Hecke algebra is compatible with the involution, which Brenti shows is a very special property [Bre03, Theorem 4.1]. Furthermore, the structure constants for multiplication in the Kazhdan-Lusztig basis of the Hecke algebra are positive [EW14, Conjecture 1.2(2)], and Du asks whether this holds in some greater generality [Du94, Section 5]. In the case of Kazhdan-Lusztig polynomials of matroids, a candidate algebra structure was described and positivity was conjectured [EPW16, Conjecture 4.2], but that conjecture turned out to be false (see Section 4.6 of the arXiv version). It is unclear whether this conjecture could be salvaged by changing the definition of the algebra structure, or more generally when a particular collection of Kazhdan-Lusztig-Stanley polynomials comes equipped with a nice algebra structure on its associated module.

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2 Combinatorics

We begin by reviewing the combinatorial theory of Kazhdan-Lusztig-Stanley polynomials, which was introduced in [Sta92, Section 6] and further developed in [Dye93, Bre99, Bre03]. We also introduce Z-polynomials (Section 2.3) and study their basic properties.

2.1 The incidence algebra

Let P be a poset. We say that P is **locally finite** if, for all $x \le z \in P$, the set

$$[x,z] := \{y \in P \mid x \le y \le z\}$$

is finite. Let

$$I(P) := \prod_{x \le y} \mathbb{Z}[t].$$

For any $f \in I(P)$ and $x < y \in P$, let $f_{xy}(t) \in \mathbb{Z}[t]$ denote the corresponding component of f. If P is locally finite, then $I(P)$ admits a ring structure with product given by convolution:

$$(fg)_{xz}(t) := \sum_{x \le y \le z} f_{xy}(t) g_{yz}(t).$$

The identity element is the function $\delta \in I(P)$ with the property that $\delta_{xy} = 1$ if $x = y$ and 0 otherwise.

Let $r \in I(P)$ be a function satisfying the following conditions:

- $r_{xy} \in \mathbb{Z} \subset \mathbb{Z}[t]$ for all $x \le y \in P$ (we will refer to $r_{xy}(t)$ simply as r_{xy})
- if $x < y$, then $r_{xy} > 0$
- if $x \le y \le z$, then $r_{xy} + r_{yz} = r_{xz}$.

Such a function is called a **weak rank function** [Bre99, Section 2]. We will use the terminology **weakly ranked poset** to refer to a locally finite poset equipped with a weak rank function, and we will suppress r from the notation when there is no possibility for confusion.

For any weakly ranked poset P, let $\mathscr{I}(P) \subset I(P)$ denote the subring of functions f with the property that the degree of $f_{xy}(t)$ is less than or equal to r_{xy} for all $x \le y$. The ring $\mathscr{I}(P)$ admits an involution $f \mapsto \bar{f}$ defined by the formula

$$\bar{f}_{xy}(t) := t^{r_{xy}} f_{xy}(t^{-1}).$$

Lemma 2.1. *An element $f \in I(P)$ has an inverse (left or right) if and only if $f_{xx}(t) = \pm 1$ for all $x \in P$. In this case, the left and right inverses are unique and they coincide. If $f \in \mathscr{I}(P) \subset I(P)$ is invertible, then $f^{-1} \in \mathscr{I}(P)$.*

Proof. An element g is a right inverse to f if and only if $g_{xx}(t) = f_{xx}(t)^{-1}$ and

$$f_{xx}(t) g_{xz}(t) = -\sum_{x < y \le z} f_{xy}(t) g_{yz}(t)$$

for all $x < z$. The first equation has a solution if and only if $f_{xx}(t) = \pm 1$, in which case the second equation also has a unique solution. If $f \in \mathscr{I}(P)$, it is clear that $g \in \mathscr{I}(P)$, as well. The argument for left inverses is identical, so it remains only to show that left and right inverses coincide.

Let g be right inverse to f. Then g is also left inverse to some function, which we will denote h. We then have

$$f = f\delta = f(gh) = (fg)h = \delta h = h,$$

so g is left inverse to f, as well. $\square$

2.2 Right and left KLS-functions

An element $\kappa \in \mathscr{I}(P)$ is called a ***P*-kernel** if $\kappa_{xx}(t) = 1$ for all $x \in P$ and $\kappa^{-1} = \bar{\kappa}$. Let

$$\mathscr{I}_{1/2}(P) := \left\{ f \in \mathscr{I}(P) \;\middle|\; f_{xx}(t) = 1 \text{ for all } x \in P \text{ and } \deg f_{xy}(t) < r_{xy}/2 \text{ for all } x < y \in P \right\}.$$

Various versions of the following theorem appear in [Sta92, Corollary 6.7], [Dye93, Proposition 1.2], and [Bre99, Theorem 6.2].

Theorem 2.2. *If $\kappa \in \mathscr{I}(P)$ is a P-kernel, there exists a unique pair of functions $f, g \in \mathscr{I}_{1/2}(P)$ such that $\bar{f} = \kappa f$ and $\bar{g} = g\kappa$.*

Proof. We will prove existence and uniqueness of f; the proof for g is identical. Fix elements $x < w \in P$, and suppose that $f_{yw}(t)$ has been defined for all $x < y \leq w$. Let

$$Q_{xw}(t) := \sum_{x<y\leq w} \kappa_{xy}(t) f_{yw}(t) \in \mathbb{Z}[t].$$

The equation $\bar{f} = \kappa f$ for the interval $[x, w]$ translates to

$$\bar{f}_{xw}(t) - f_{xw}(t) = Q_{xw}(t).$$

It is clear that there is at most one polynomial $f_{xw}(t)$ of degree strictly less than $r_{xw}/2$ satisfying this equation. The existence of such a polynomial is equivalent to the statement

$$t^{r_{xw}} Q_{xw}(t^{-1}) = -Q_{xw}(t).$$

To prove this, we observe that

$$\begin{aligned}
t^{r_{xw}} Q_{xw}(t^{-1}) &= t^{r_{xw}} \sum_{x<y\le w} \kappa_{xy}(t^{-1}) f_{yw}(t^{-1}) \\
&= \sum_{x<y\le w} t^{r_{xy}} \kappa_{xy}(t^{-1}) t^{r_{yw}} f_{yw}(t^{-1}) \\
&= \sum_{x<y\le w} \bar{\kappa}_{xy}(t) \bar{f}_{yw}(t) \\
&= \sum_{x<y\le w} \bar{\kappa}_{xy}(t) (\kappa f)_{yw}(t) \\
&= \sum_{x<y\le w} \bar{\kappa}_{xy}(t) \sum_{y\le z\le w} \kappa_{yz}(t) f_{zw}(t) \\
&= \sum_{x<y\le z\le w} \bar{\kappa}_{xy}(t) \kappa_{yz}(t) f_{zw}(t) \\
&= \sum_{x<z\le w} f_{zw}(t) \sum_{x<y\le z} \bar{\kappa}_{xy}(t) \kappa_{yz}(t) \\
&= \sum_{x<z\le w} f_{zw}(t) \Big((\bar{\kappa}\kappa)_{xz}(t) - \kappa_{xz}(t) \Big) \\
&= - \sum_{x<z\le w} \kappa_{xz}(t) f_{zw}(t) \\
&= -Q_{xw}(t).
\end{aligned}$$

Thus there is a unique choice of polynomial $f_{xw}(t)$ consistent with the equation $\bar{f} = \kappa f$ on the interval $[x, w]$. $\square$

Remark 2.3. Stanley [Sta92] works only with the function g, as does Brenti in [Bre99], while Brenti later switches conventions and works with the function f in [Bre03] (though he notes in a footnote that both functions exist). Dyer [Dye93] defines versions of both functions, but with normalizations that differ from ours.

Brenti refers to g in [Bre99] and f in [Bre03] as the Kazhdan-Lusztig-Stanley function associated with κ. We will refer to f as the **right Kazhdan-Lusztig-Stanley function** associated with κ, and to g as the **left Kazhdan-Lusztig-Stanley function** associated with κ. For any $x \le y$, we will refer to the polynomial $f_{xy}(t)$ or $g_{xy}(t)$ as a (right or left) **Kazhdan-Lusztig-Stanley polynomial**. We will write KLS as an abbreviation for Kazhdan-Lusztig-Stanley.

Remark 2.4. Given a locally finite weakly graded poset P, let P^* denote the **opposite** of P, which means that $y \le x$ in P^* if and only if $x \le y$ in P, in which case $r^*_{yx} = r_{xy}$. For any function $f \in \mathscr{I}(P)$, define $f^* \in \mathscr{I}(P^*)$ by putting $f^*_{yx}(t) := f_{xy}(t)$ for all $x \le y \in P$. If κ is a P-kernel with right KLS-function f and left KLS-function g, then κ^* is a P^*-kernel with left KLS-function f^* and right KLS-function g^*. Thus one can go between left and right KLS-polynomials by reversing the order on the poset.

It will be convenient for us to have a converse to Theorem 2.2. A version of this proposition appears in [Sta92, Theorem 6.5].

Proposition 2.5. *Suppose that $f \in \mathscr{I}_{1/2}(P)$. Then*

1. *f is invertible.*
2. *$\bar{f}f^{-1}$ is a P-kernel with f as its associated right KLS-function.*
3. *$f^{-1}\bar{f}$ is a P-kernel with f as its associated left KLS-function.*

Proof. By Lemma 2.1, f is invertible. We have $(\bar{f}f^{-1})^{-1} = f\bar{f}^{-1} = \overline{\bar{f}f^{-1}}$, so $\bar{f}f^{-1}$ is a P-kernel. Since $\bar{f} = \bar{f}(f^{-1}f) = (\bar{f}f^{-1})f$, the uniqueness part of Theorem 2.2 tells us that f is equal to the associated right KLS-function. The last statement follows similarly. □

2.3 The Z-function

We will call a function $Z \in \mathscr{I}(P)$ **symmetric** if $\bar{Z} = Z$. Let κ be a P-kernel with right KLS-function f and left KLS-function g. Let $Z := g\kappa f \in \mathscr{I}(P)$; we will refer to Z as the ***Z*-function** associated with κ, and to each $Z_{xy}(t)$ as a ***Z*-polynomial**.

Proposition 2.6. *We have $Z = \bar{g}f = g\bar{f}$. In particular, Z is symmetric.*

Proof. Since $\bar{g} = g\kappa$, we have $Z = g\kappa f = \bar{g}f$. Since $\bar{f} = \kappa f$, we have $Z = g\kappa f = g\bar{f}$. □

We have the following converse to Proposition 2.6.

Proposition 2.7. *Suppose that $f, g \in \mathscr{I}_{1/2}(P)$. Then f and g are the right and left KLS-functions for a single P-kernel κ if and only if $\bar{g}f$ is symmetric.*

Proof. Let $\kappa_f := \bar{f}f^{-1}$ and $\kappa_g := g^{-1}\bar{g}$. By Proposition 2.5, f is the right KLS-function of κ_f and g is the left KLS-function of κ_g. Then $\bar{g}f = g\kappa_g f$ and $g\bar{f} = g\kappa_f f$. Multiplying on the left by g^{-1} and on the right by f^{-1}, we see that these two functions are the same if and only if $\kappa_f = \kappa_g$. □

The following version of Proposition 2.7 will be useful in Section 3.4. It allows us to relax both the symmetry assumption and the conclusion of Proposition 2.7.

Proposition 2.8. *Let κ be a P-kernel, and let $f, g \in \mathscr{I}_{1/2}(P)$ be the associated right and left KLS-functions. Suppose we are given $x \in P$ and $h \in \mathscr{I}_{1/2}(P)$ such that, for all $z \geq x$, we have $(\bar{h}f)_{xz}(t) = (h\bar{f})_{xz}(t)$. Then for all $z \geq x$, $h_{xz}(t) = g_{xz}(t)$.*

Proof. We proceed by induction on r_{xz}. When $z = x$, we have $h_{xx}(t) = 1 = g_{xx}(t)$. Now assume that the statement holds for all y such that $x \leq y < z$. We have

$$\sum_{x \leq y \leq z} \bar{g}_{xy}(t) f_{yz}(t) = (\bar{g}f)_{xz}(t) = (g\bar{f})_{xz}(t) = \sum_{x \leq y \leq z} g_{xy}(t) \bar{f}_{yz}(t)$$

and

$$\sum_{x \leq y \leq z} \bar{h}_{xy}(t) f_{yz}(t) = (\bar{h}f)_{xz}(t) = (h\bar{f})_{xz}(t) = \sum_{x \leq y \leq z} h_{xy}(t) \bar{f}_{yz}(t).$$

Subtracting these two equations and applying our inductive hypothesis, we have

$$\bar{g}_{xz}(t) - \bar{h}_{xz}(t) = t^{r_{xz}}\big(g_{xz}(t) - h_{xz}(t)\big).$$

Since $\deg(g_{xz} - h_{xz}) < r_{xz}/2$, this implies that $g_{xz}(t) = h_{xz}(t)$. $\square$

Proposition 2.9. *Let $\kappa \in \mathscr{I}(P)$ be a P-kernel, and let P^* be the opposite of P. Then $Z^* \in \mathscr{I}(P^*)$ is the Z-polynomial associated with the P^*-kernel κ^*.*

Proof. By Remark 2.4, the left KLS-polynomial associated with κ^* is f^*, and the right KLS-polynomial is g^*. Thus the Z-polynomial is $f^*\kappa^*g^* = (g\kappa f)^* = Z^*$. $\square$

Remark 2.10. Let κ be a P-kernel with right KLS-function f, left KLS-function g and Z-function Z. Proposition 2.5 says that, if you know f or g, you can compute κ. Similarly, we observe that if you know Z, you can compute f and g, and therefore κ. This can be proved inductively. Indeed, assume that we can compute f and g on any interval strictly contained in $[x, z]$. Then we have

$$Z_{xz}(t) = \sum_{x \le y \le z} \bar{g}_{xy}(t) f_{yz}(t) = f_{xz}(t) + \bar{g}_{xz}(t) + \sum_{x < y < z} \bar{g}_{xy}(t) f_{yz}(t),$$

and therefore

$$f_{xz}(t) + \bar{g}_{xz}(t) = Z_{xz}(t) - \sum_{x < y < z} \bar{g}_{xy}(t) f_{yz}(t). \tag{1}$$

By our inductive hypothesis, we can compute the right-hand side, which determines the left-hand side. Since $f, g \in \mathscr{I}_{1/2}(P)$, this determines $f_{xz}(t)$ and $g_{xz}(t)$ individually.

On the other hand, it is **not** true that every symmetric function $Z \in \mathscr{I}(P)$ with $Z_{xy}(0) = 1$ for all $x \le y \in P$ is the Z-function associated with some P-kernel. This is because Equation (1) cannot be solved if r_{xz} is even and the coefficient of $t^{r_{xz}/2}$ on the right hand side is nonzero.

2.4 Alternating kernels

Given a function $h \in \mathscr{I}(P)$, we we define $\hat{h} \in \mathscr{I}(P)$ by the formula $\hat{h}_{xy}(t) := (-1)^{r_{xy}} h_{xy}(t)$. The map $h \mapsto \hat{h}$ is an involution of the ring $\mathscr{I}(P)$ that commutes with the involution $h \mapsto \bar{h}$. We will say that h is **alternating** if $\bar{h} = \hat{h}$. A version of the following result appears in [Sta92, Corollary 8.3].

Proposition 2.11. *Let $\kappa \in \mathscr{I}(P)$ be an alternating P-kernel, and let $f, g \in \mathscr{I}_{1/2}(P)$ be the associated right and left KLS-functions. Then $\hat{g} = f^{-1}$ and $\hat{f} = g^{-1}$.*

Proof. Since $\bar{g} = g\kappa$, we have $\hat{\bar{g}} = \hat{g}\hat{\kappa} = \hat{g}\bar{\kappa}$. Then

$$\overline{\hat{g}f} = \bar{\hat{g}}\bar{f} = \hat{\bar{g}}\bar{f} = \hat{g}\bar{\kappa}\kappa f = \hat{g}f,$$

thus $\hat{g}f$ is symmetric. However, since $f, g \in \mathscr{I}_{1/2}(P)$, we have $\deg(\hat{g}f)_{xy}(t) < r_{xy}/2$ for all $x < y$, so this implies that $(\hat{g}f)_{xy}(t) = 0$ for all $x < y$. On the other hand, $(\hat{g}f)_{xx}(t) = \hat{g}_{xx}(t)f_{xx}(t) = 1$. Thus $\hat{g}f = \delta$, and therefore $\hat{g} = f^{-1}$. The second statement follows immediately. $\square$

2.5 Examples

We now discuss a number of examples of P-kernels along their associated KLS-functions and Z-functions. All of these examples will be revisited in Section 4.

Example 2.12. Let W be a Coxeter group, equipped with the Bruhat order and the rank function given by the length of an element of W. The classical ***R*-polynomials** $\{R_{vw}(t) \mid v \leq w \in W\}$ form a W-kernel, and the classical **Kazhdan-Lusztig polynomials** $\{f_{xy}(t) \mid v \leq w \in W\}$ are the associated right KLS-polynomials. These polynomials were introduced by Kazhdan and Lusztig [KL79], and they were one of the main motivating examples in Stanley's work [Sta92, Example 6.9].

If W is finite, then there is a maximal element $w_0 \in W$, and left multiplication by w_0 defines an order-reversing bijection of W with the property that, if $v \leq w$, then $R_{vw}(t) = R_{(w_0 w)(w_0 v)}(t)$ [Lus03, Lemma 11.3]. It follows from Remark 2.4 that $g_{vw}(t) = f_{(w_0 w)(w_0 v)}(t)$. In addition, R is alternating [KL79, Lemma 2.1(i)], hence Proposition 2.11 tells us that $\hat{g} = f^{-1}$ and $\hat{f} = g^{-1}$.

Example 2.13. Let P be any locally finite weakly ranked poset. Define $\zeta \in \mathscr{I}(P)$ by the formula $\zeta_{xy}(t) = 1$ for all $x \leq y \in P$. The element $\mu := \zeta^{-1} \in \mathscr{I}(P)$ is called the **Möbius function**, and the product $\chi := \mu\bar{\zeta} = \zeta^{-1}\bar{\zeta}$ is called the **characteristic function** of P. We then have $\chi^{-1} = \bar{\zeta}^{-1}\zeta = \bar{\chi}$, so χ is a P-kernel. Proposition 2.5(3) tells us that the associated left KLS-function is ζ; this was observed by Stanley in [Sta92, Example 6.8]. However, the associated right KLS-function f can be much more interesting! (In particular, χ is generally not alternating.) For example, if P is the lattice of flats of a matroid M with the usual weak rank function, with minimum element 0 and maximum element 1, then $f_{01}(t)$ is the **Kazhdan-Lusztig polynomial of *M*** as defined in [EPW16], and $Z_{01}(t)$ is the ***Z*-polynomial of *M*** as defined in [PXY18]. In general, the coefficients of $f_{xy}(t)$ can be expressed as alternating sums of multi-indexed Whitney numbers for the interval $[x, y] \subset P$; see [Bre99, Corollary 6.5], [Wak18, Theorem 5.1], and [PXY18, Theorem 3.3] for three different formulations of this result.

Example 2.14. Let P be any locally finite weakly ranked poset. Define $\lambda \in \mathscr{I}(P)$ by the formula $\lambda_{xy}(t) = (t-1)^{r_{xy}}$ for all $x \leq y \in P$. The weakly ranked poset P is called **locally Eulerian** if $\mu_{xy}(t) = (-1)^{r_{xy}}$ for all $x \leq y \in P$, which is equivalent to the condition that λ is a P-kernel [Sta92, Proposition 7.1]. The poset of faces of a polytope, with weak rank function given by relative dimension (where $\dim \emptyset = -1$), is Eulerian. More generally, any fan is an Eulerian poset.

Let Δ be a polytope, let P be the poset of faces of Δ, and let f and g be the associated right and left KLS-functions. Then $g_{\emptyset\Delta}(t)$ is called the ***g*-polynomial** of Δ [Sta92, Example 7.2]. Since the dual polytope Δ^* has the property that its face poset is opposite to P, and since λ depends only on the weak rank function, Remark 2.4 tells us that the right KLS-polynomial $f_{\emptyset\Delta}(t)$ is equal to the g-polynomial of Δ^*. On the other hand, since λ is clearly alternating, Proposition 2.11 tells us that $\hat{g} = f^{-1}$ and $\hat{f} = g^{-1}$ [Sta92, Corollary 8.3].

Remark 2.15. For P locally Eulerian, Batyrev and Borisov define an element $B \in \prod_{x \leq y \in P} \mathbb{Z}[u, v]$ [BB96, Definition 2.7]. Let $B' \in \mathscr{I}(P)$ be the function obtained from B by setting $u = -t$ and $v = -1$. The defining equation for B transforms into the equation $B'\bar{\hat{f}} = f$. Using the fact that

$\hat{f} = g^{-1}$, this means that $B' = f\bar{g}$. Thus B' is similar to $Z = \bar{g}f$, but it is not quite the same. In particular, B' need not be symmetric.

Example 2.16. Let M be a matroid with lattice of flats L. Let $r \in \mathscr{I}(L)$ be the usual weak rank function, and let $\chi \in \mathscr{I}(L)$ be the characteristic function. In this example, we will be interested in the weakly ranked poset $(L, 2r)$, where $2r$ is 2 times the usual weak rank function.

Define $\kappa \in \mathscr{I}(L, 2r)$ by the following formula:

$$\kappa_{FH}(t) := (t-1)^{r_{FH}} \sum_{F \le G \le H} (-1)^{r_{FG}} \chi_{FG}(-1) \chi_{GH}(t).$$

Define $h^{\mathrm{bc}} \in \mathscr{I}_{1/2}(L, 2r)$ by letting

$$h^{\mathrm{bc}}_{FG}(t) := (-t)^{r_{FG}} \chi_{FG}(1 - t^{-1})$$

be the h-polynomial of the broken circuit complex of M^F_G, where M^F_G is the matroid on $G \smallsetminus F$ whose lattice of flats is isomorphic to $[F, G] \subset L$.

Proposition 2.17. *The function κ is an $(L, 2r)$-kernel, and h^{bc} is its associated left KLS-function.*

Proof. By Proposition 2.5(3), it will suffice to show that $\overline{h^{\mathrm{bc}}} = h^{\mathrm{bc}}\kappa$. We follow the argument in the proof of [PW07, Theorem 4.3]. We will write μ_{FG} and δ_{FG} to denote the constant polynomials $\mu_{FG}(t)$ and $\delta_{FG}(t)$. For all $D \le J$, we have

$$\begin{aligned}
(h^{\mathrm{bc}}\kappa)_{DJ}(t) &= \sum_{D \le F \le J} h^{\mathrm{bc}}_{DF}(t) \kappa_{FJ}(t) \\
&= \sum_{D \le F \le H \le J} (t-1)^{r_{FJ}} (-1)^{r_{FH}} \chi_{FH}(-1) \chi_{HJ}(t) (-t)^{r_{DF}} \chi_{DF}(1 - t^{-1}) \\
&= \sum_{D \le E \le F \le G \le H \le I \le J} (t-1)^{r_{FJ}} (-1)^{r_{FH}} \mu_{FG} (-1)^{r_{GH}} \mu_{HI} t^{r_{IJ}} (-t)^{r_{DF}} \mu_{DE} (1 - t^{-1})^{r_{EF}} \\
&= \sum_{D \le E \le F \le G \le H \le I \le J} \mu_{DE} \mu_{FG} \mu_{HI} (-1)^{r_{DG}} t^{r_{DE} + r_{IJ}} (t-1)^{r_{EJ}} \\
&= \sum_{D \le E \le G \le I \le J} \mu_{DE} (-1)^{r_{DG}} t^{r_{DE} + r_{IJ}} (t-1)^{r_{EJ}} \sum_{E \le F \le G} \mu_{FG} \sum_{G \le H \le I} \mu_{HI} \\
&= \sum_{D \le E \le G \le I \le J} \mu_{DE} (-1)^{r_{DG}} t^{r_{DE} + r_{IJ}} (t-1)^{r_{EJ}} \delta_{EG} \delta_{GI} \\
&= \sum_{D \le E \le J} \mu_{DE} (-1)^{r_{DE}} t^{r_{DJ}} (t-1)^{r_{EJ}} \\
&= (-t)^{r_{DJ}} \sum_{D \le E \le J} \mu_{DE} (1-t)^{r_{EJ}} \\
&= t^{2r_{DJ}} (-t^{-1})^{r_{DJ}} \chi_{DJ}(1-t) \\
&= t^{r_{DJ}} h^{\mathrm{bc}}_{DJ}(t^{-1}) \\
&= \overline{h^{\mathrm{bc}}}_{DJ}(t).
\end{aligned}$$

This completes the proof. $\square$

3 Geometry

In this section we give a general geometric framework for interpreting right KLS-polynomials in terms of the stalks of intersection cohomology sheaves on a stratified space. Under some additional assumptions, we also give cohomological interpretations for the associated Z-polynomials. Our primary reference for technical properties of intersection cohomology will be the book of Kiehl and Weissauer [KW01], however, a reader who is learning this material for the first time might also benefit from the friendly discussion in the book of Kirwan and Woolf [KW06, Section 10.4].

3.1 The setup

Fix a finite field $\mathbb{F}_q$, an algebraic closure $\overline{\mathbb{F}}_q$, and a prime ℓ that does not divide q. For any variety Z over $\mathbb{F}_q$, let IC_Z denote the ℓ-adic intersection cohomology sheaf on the variety $Z(\overline{\mathbb{F}}_q)$. We adopt the convention of *not* shifting IC_Z to make it perverse. In particular, if Z is smooth, then IC_Z is isomorphic to the constant sheaf in degree zero.

Suppose that we have a variety Y over $\mathbb{F}_q$ and a stratification

$$Y = \bigsqcup_{x \in P} V_x.$$

By this we mean that each stratum V_x is a smooth connected subvariety of Y and the closure of each stratum is itself a union of strata. We define a partial order on P by putting $x \leq y \iff V_x \subset \bar{V}_y$, and a weak rank function by the formula $r_{xy} = \dim V_y - \dim V_x$. Fix a point $e_x \in V_x$ for each $x \in P$.

Next, suppose that we have a stratification preserving $\mathbb{G}_m$-action $\rho_x : \mathbb{G}_m \to \mathrm{Aut}(Y)$ for each $x \in P$ and an affine $\mathbb{G}_m$-subvariety $C_x \subset Y$ with the following properties:

- C_x is a weighted affine cone with respect to ρ_x with cone point e_x. In other words, the $\mathbb{Z}$-grading on the affine coordinate ring $\mathbb{F}_q[C_x]$ induced by ρ_x is non-negative and the vanishing locus of the ideal of positively graded elements is $\{e_x\}$.
- For all $x, y \in P$, let
$$U_{xy} := C_x \cap V_y \qquad \text{and} \qquad X_{xy} := C_x \cap \bar{V}_y.$$
We require that the restriction of $\mathrm{IC}_{\bar{V}_y}$ to $C_x(\overline{\mathbb{F}}_q)$ is isomorphic to $\mathrm{IC}_{X_{xy}}$.

Note that the variety X_{xy} is a closed $\mathbb{G}_m$-equivariant subvariety of C_x, therefore it is either empty or a weighted affine cone with cone point e_x. We have

$$e_x \in X_{xy} \iff e_x \in \bar{V}_y \iff x \leq y,$$

so X_{xy} is nonempty if and only if $x \leq y$.

Lemma 3.1. *For all $x \leq z$, we have $X_{xz} = \bigsqcup_{x \leq y \leq z} U_{xy}$.*

Proof. We have $X_{xz} = C_x \cap \bar{V}_z = C_x \cap \bigsqcup_{y \leq z} V_y = \bigsqcup_{y \leq z} U_{xy}$. If x is not less than or equal to Y, then X_{xy} is empty, thus so is U_{xy}. $\square$

The condition on restrictions of IC sheaves is somewhat daunting. In each of our families of examples, we will check this condition by means of a group action, using the following lemma.

Lemma 3.2. *Suppose that Y is equipped with an action of an algebraic group G preserving the stratification. Suppose in addition that, for each $x \in P$, there exists a subgroup $G_x \subset G$ such that the composition*

$$\varphi_x : G_x \times C_x \hookrightarrow G \times Y \to Y$$

is an open immersion. Then for all $x \leq y \in P$, the restriction of $\mathrm{IC}_{\bar{V}_y}$ to $C_x(\overline{\mathbb{F}}_q)$ is isomorphic to $\mathrm{IC}_{X_{xy}}$.

Proof. Since φ_x is an open immersion, we have $\varphi_x^{-1}\,\mathrm{IC}_{\bar{V}_y} \cong \mathrm{IC}_{\varphi_x^{-1}(\bar{V}_y)}$ as sheaves on $G_x(\overline{\mathbb{F}}_q) \times C_x(\overline{\mathbb{F}}_q)$ for all $x, y \in P$. Since the action of G on Y preserves the stratification, we have

$$\varphi_x^{-1}(\bar{V}_y) = G_x \times (C_x \cap \bar{V}_y) = G_x \times X_{xy},$$

so $\varphi_x^{-1}\,\mathrm{IC}_{\bar{V}_y} \cong \mathrm{IC}_{G_x} \boxtimes \mathrm{IC}_{X_{xy}}$. Since G_x is smooth, IC_{G_x} is the constant sheaf on G_x. Thus, if we further restrict to $C_x(\overline{\mathbb{F}}_q) \cong \{\mathrm{id}_{G_x}\} \times C_x(\overline{\mathbb{F}}_q)$, we obtain $\mathrm{IC}_{X_{xy}}$. $\square$

Remark 3.3. In some of our examples (Sections 4.2 and 4.3), the $\mathbb{G}_m$-action ρ_x will not actually depend on x. In other examples (Sections 4.1 and 4.4), it will depend on x.

3.2 Intersection cohomology

We will write $\mathrm{IH}^*(Z)$ and $\mathrm{IH}^*_c(Z)$ to denote the ordinary and compactly supported cohomology of IC_Z. Given a point $p \in Z$, we will write $\mathrm{IH}^*_p(Z)$ to denote the cohomology of the stalk of IC_Z at p. Each of these graded $\mathbb{Q}_\ell$-vector spaces has a natural Frobenius automorphism induced by the Frobenius automorphism of Z. We will be interested in the vector spaces $\mathrm{IH}^*_{xy} := \mathrm{IH}^*_{e_x}(\bar{V}_y)$ for all $x \leq y$.

Lemma 3.4. *If $x \leq y \leq z$ and $u \in U_{xy}$, then $\mathrm{IH}^*_{yz} \cong \mathrm{IH}^*_u(X_{xz})$.*

Proof. Since u and e_y lie in the same connected stratum of $\bar{V}_z$, we have an isomorphism of stalks $\mathrm{IC}_{\bar{V}_z, e_y} \cong \mathrm{IC}_{\bar{V}_z, u}$. Since the restriction of $\mathrm{IC}_{\bar{V}_z}$ to $C_x(\overline{\mathbb{F}}_q)$ is isomorphic to $\mathrm{IC}_{X_{xz}}$, we have an isomorphism of stalks $\mathrm{IC}_{\bar{V}_z, u} \cong \mathrm{IC}_{X_{xz}, u}$. Putting these two stalk isomorphisms together, we have

$$\mathrm{IH}^*_{yz} = \mathrm{IH}^*_{e_y}(\bar{V}_z) = \mathrm{H}^*(\mathrm{IC}_{\bar{V}_z, e_y}) \cong \mathrm{H}^*(\mathrm{IC}_{\bar{V}_z, u}) \cong \mathrm{H}^*(\mathrm{IC}_{X_{xz}, u}) = \mathrm{IH}^*_u(X_{xz}).$$

This completes the proof. $\square$

Lemma 3.5. *For all $y \leq z$, $\mathrm{IH}^*_{yz} \cong \mathrm{IH}^*(X_{yz})$.*

Proof. If we apply Lemma 3.4 with $x = y$, we find that $\mathrm{IH}^*_{yz} \cong \mathrm{IH}^*_{e_y}(X_{yz})$. Since X_{yz} is a weighted affine cone with cone point e_y, the cohomology of the stalk of the IC sheaf at e_y coincides with the global intersection cohomology [KL80, Lemma 4.5(a)]. □

We call an intersection cohomology group **chaste** if it vanishes in odd degrees and the Frobenius automorphism acts on the degree $2i$ part by multiplication by q^i [EPW16, Section 3.3]. (This is much stronger than being **pure**, which is a statement about the absolute values of the eigenvalues of the Frobenius automorphism.)

3.3 Right KLS-polynomials

Define $f \in \mathscr{I}(P)$ by putting

$$f_{xy}(t) := \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}_{xy}$$

for all $x \leq y$. We observe that $f \in \mathscr{I}_{1/2}(P)$ by Lemma 3.5 and [EPW16, Proposition 3.4].

Theorem 3.6. *Suppose that we have an element $\kappa \in \mathscr{I}(P)$ such that, for all $x \leq y$ and all positive integers s,*

$$\kappa_{xy}(q^s) = |U_{xy}(\mathbb{F}_{q^s})|.$$

Then IH^*_{xz} is chaste for all $x \leq z$, κ is a P-kernel, and f is the associated right KLS-function.

Remark 3.7. The first time that you read the proof of Theorem 3.6, it is helpful to pretend that we already know that IH^*_{xz} is chaste for all $x \leq z$. In this case, the proof simplifies to a straightforward application of Poincaré duality and the Lefschetz formula, along with Lemmas 3.1, 3.4, and 3.5. The actual proof as it appears is made significantly more subtle by the need to fold the chastity statement into the induction.

Proof of Theorem 3.6: We begin with an inductive proof of chastity. It is clear that IH^*_{xx} is chaste for all $x \in P$. Now consider a pair of elements $x < z$, and assume that IH^*_{yz} is chaste for all $x < y \leq z$. Let s be any positive integer. Applying the Lefschetz formula [KW01, III.12.1(4)], along with Lemmas 3.1 and 3.4, we find that

$$\begin{aligned}
\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^i_c(X_{xz})\Big) &= \sum_{u\in X_{xz}(\mathbb{F}_{q^s})}\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^i_u(X_{xz})\Big)\\
&= \sum_{x\leq y\leq z}\sum_{u\in U_{xy}(\mathbb{F}_{q^s})}\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^i_u(X_{xz})\Big)\\
&= \sum_{x\leq y\leq z}\sum_{u\in U_{xy}(\mathbb{F}_{q^s})}\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^i_{yz}\Big)\\
&= \sum_{x\leq y\leq z}\kappa_{xy}(q^s)\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^i_{yz}\Big).
\end{aligned}$$

By Poincaré duality [KW01, II.7.3], we have

$$\operatorname{tr}\Big(\operatorname{Fr}^s \curvearrowright \operatorname{IH}^i_c(X_{xz})\Big) = q^{sr_{xz}} \operatorname{tr}\Big(\operatorname{Fr}^{-s} \curvearrowright \operatorname{IH}^{2r_{xz}-i}(X_{xz})\Big).$$

By our inductive hypothesis, we have $\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\operatorname{Fr}^s \curvearrowright \operatorname{IH}^*_{yz}\Big) = f_{yz}(q^s)$ for all $x < y \leq z$. Moving the $x = y$ term from the right hand side to the left hand side, the Lefschetz formula becomes

$$\sum_{i\geq 0}(-1)^i \left(q^{sr_{xz}} \operatorname{tr}\Big(\operatorname{Fr}^{-s} \curvearrowright \operatorname{IH}^{2r_{xz}-i}_{xz}\Big) - \operatorname{tr}\Big(\operatorname{Fr}^s \curvearrowright \operatorname{IH}^i_{xz}\Big)\right) = \sum_{x<y\leq z} \kappa_{xy}(q^s) f_{yz}(q^s). \tag{2}$$

We now follow the proof of [EPW16, Theorem 3.7]. Let $b_i = \dim \operatorname{IH}^i_{xz}$. Let $(\alpha_{i,1},\ldots,\alpha_{i,b_i}) \in \overline{\mathbb{Q}}_\ell^{b_i}$ be the eigenvalues of the Frobenius action on IH^i_{xz} (with multiplicity, in any order). Then Equation (2) becomes

$$\sum_{i\geq 0}(-1)^i \sum_{j=1}^{b_i} \left((q^{r_{xz}}/\alpha_{i,j})^s - \alpha^s_{i,j}\right) = \sum_{x<y\leq z} \kappa_{xy}(q^s) f_{yz}(q^s).$$

By Lemma 3.5 and [EPW16, Proposition 3.4], $\operatorname{IH}^i_{xz} = 0$ for $i \geq r_{xz}$, and for any $i < r_{xz}/2$, $\alpha_{i,j}$ has absolute value $q^{i/2} < q^{r_{xz}/2}$. It follows that $q^{r_{xz}}/\alpha_{i,j}$ has absolute value $q^{r_{xz}-i/2} > q^{r_{xz}/2}$, and therefore that the numbers that appear with positive sign on the left-hand side of Equation (2) are pairwise disjoint from the numbers that appear with negative sign. Since the right-hand side is a sum of integer powers of q^s with integer coefficients, [EPW16, Lemma 3.6] tells us that each $\alpha_{i,j}$ must also be an integer power of q. This is only possible if $b_i = 0$ for odd i and $\alpha_{i,j} = q^{i/2}$ for even i, thus IH^*_{xz} is chaste.

Now that we have established chastity, Equation (2) becomes

$$q^{sr_{xz}} f_{xz}(q^{-s}) - f_{xz}(q^s) = \sum_{x<y\leq z} \kappa_{xy}(q^s) f_{yz}(q^s),$$

or equivalently

$$\bar{f}_{xz}(q^s) = q^{sr_{xz}} f_{xz}(q^{-s}) = \sum_{x\leq y\leq z} \kappa_{xy}(q^s) f_{yz}(q^s) = (\kappa f)_{xz}(q^s).$$

Since this holds for all positive s, it must also hold with q^s replaced by the formal variable t, thus $\bar{f} = \kappa f$. The fact that κ is a P-kernel with f as its associated right KLS-function now follows follow from Proposition 2.5(2). □

The same idea used in the proof of Theorem 3.6 can be used to obtain the following converse.

Theorem 3.8. *Suppose that IH^*_{xz} is chaste for all $x \leq z$, and let $\kappa := \bar{f} f^{-1}$. Then for all $s > 0$ and $x \leq z$,*

$$\kappa_{xz}(q^s) = |U_{xz}(\mathbb{F}_{q^s})|.$$

Proof. We proceed by induction. When $x = z$, we have $\kappa_{xz}(t) = 1$ and $U_{xz} = \{e_x\}$, so the statement is clear. Now assume that $\kappa_{xy}(q^s) = |U_{xy}(\mathbb{F}_{q^s})|$ for all $x \le y < z$. By Poincaré duality the Lefschetz formula, we have

$$\bar{f}_{xz}(q^s) = \sum_{x \le y \le z} |U_{xy}(\mathbb{F}_{q^s})| f_{yz}(q^s) = |U_{xz}(\mathbb{F}_{q^s})| + \sum_{x \le y < z} \kappa(q^s) f_{yz}(q^s).$$

By the definition of κ, we have

$$\bar{f}_{xz}(q^s) = \sum_{x \le y \le z} \kappa(q^s) f_{yz}(q^s).$$

Comparing these two equations, we find that $|U_{xz}(\mathbb{F}_{q^s})| = \kappa(q^s)$. □

Remark 3.9. In Section 4.2, we will apply Theorem 3.8 when Y is the affine Grassmannian. Then Y is an ind-scheme rather than a variety, but each $\bar{V}_x$ is an honest variety, and the proof goes through without modification.

3.4 Z-polynomials

In this section we will explain how to give a cohomological interpretation of Z-polynomials under certain more restrictive hypotheses. Specifically, we will assume that IH^*_{xy} is chaste for all $x \le y$, let $\kappa := \bar{f} f^{-1}$, and let g be the **left** KLS-function associated with κ. We will also assume that there is a minimal element $0 \in P$ and a function $h \in \mathscr{I}_{1/2}(P)$ such that $\bar{h}_{0x}(q^s) = |V_x(\mathbb{F}_{q^s})|$ for all $x \in P$ and $s > 0$. Finally, we will assume that $\bar{V}_y$ is proper for all $y \in P$.

Theorem 3.10. *Suppose that all of the above hypotheses are satisfied. Then for all $y \in P$, we have $g_{0y}(t) = h_{0y}(t)$, $\mathrm{IH}^*(\bar{V}_y)$ is chaste, and*

$$\sum_{i \ge 0} t^i \dim \mathrm{IH}^{2i}(\bar{V}_y) = Z_{0y}(t).$$

Proof. Following the proof of Theorem 3.6, we apply the Lefschetz formula to obtain

$$\begin{aligned}
\sum_{i \ge 0} (-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^i_c(\bar{V}_y) \Big) &= \sum_{v \in \bar{V}_y(\mathbb{F}_{q^s})} \sum_{i \ge 0} (-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^*_v(\bar{V}_y) \Big) \\
&= \sum_{x \le y} \sum_{v \in V_x(\mathbb{F}_{q^s})} \sum_{i \ge 0} (-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^*_v(\bar{V}_y) \Big) \\
&= \sum_{x \le y} \bar{h}_{0x}(q^s) \sum_{i \ge 0} (-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^*_{xy} \Big) \\
&= \sum_{x \le y} \bar{h}_{0x}(q^s) f_{xy}(q^s) \\
&= (\bar{h} f)_{0y}(q^s).
\end{aligned}$$

Since $\bar{V}_y$ is proper, compactly supported intersection cohomology coincides with ordinary intersection cohomology. Poincaré duality then tells us that $(\bar{h} f)_{0y}(q^s) = (h \bar{f})_{0y}(q^s)$. Since this is true for all s, we must have $(\bar{h}f)_{0y}(t) = (h\bar{f})_{0y}(t)$. By Proposition 2.8, we may conclude that $h_{0y}(t) = g_{0y}(t)$ for all $y \in P$, and therefore that

$$\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\operatorname{Fr}^s \curvearrowright \operatorname{IH}^i(\bar{V}_y)\Big) = Z_{0y}(q^s). \tag{3}$$

Let $b_i = \dim \operatorname{IH}^i(\bar{V}_y)$. Let $(\alpha_{i,1},\ldots,\alpha_{i,b_i}) \in \overline{\mathbb{Q}}_\ell^{b_i}$ be the eigenvalues of the Frobenius action on $\operatorname{IH}^i(\bar{V}_y)$ (with multiplicity, in any order). Then Equation (3) becomes

$$\sum_{i\geq 0}(-1)^i \sum_{j=1}^{b_i} \alpha_{i,j}^s = Z_{0y}(q^s).$$

By Deligne's theorem [dCM09, Theorems 3.1.5 and 3.1.6], each $\alpha_{i,j}$ has absolute value $q^{i/2}$. Since the right-hand side is a sum of integer powers of q^s with integer coefficients, [EPW16, Lemma 3.6] tells us that each $\alpha_{i,j}$ must also be an integer power of q. This is only possible if $b_i = 0$ for odd i and $\alpha_{i,j} = q^{i/2}$ for even i. This proves that $\operatorname{IH}^i(\bar{V}_y)$ is chaste, and Equation (3) becomes

$$\sum_{i\geq 0} q^{is} \dim \operatorname{IH}^{2i}(\bar{V}_y) = Z_{0y}(q^s).$$

Since this holds for all positive s, it must also hold with q^s replaced by the formal variable t. $\square$

Remark 3.11. We will apply Theorem 3.10 in the case where Y is a flag variety (Section 4.1), an affine Grassmannian (Section 4.2), or the Schubert variety of a hyperplane arrangement (Section 4.3). In the first and third cases, we will be able to make an even stronger statement, namely that

$$\sum_{i\geq 0} t^i \dim \operatorname{IH}^{2i}(\bar{C}_x \cap \bar{V}_y) = Z_{xy}(t)$$

(Theorems 4.3 and 4.17). However, this seems to be true for different reasons in the two cases, and we are unable to find a unified proof; see Remark 4.19 for further discussion.

3.5 Category $\mathcal{O}$

In this section we assume that the hypotheses of Theorem 3.6 are satisfied, and we make the additional assumption that each stratum V_x is isomorphic to an affine space. Though this is a very restrictive assumption, it is satisfied by two of our main families of examples (Sections 4.1 and 4.3).

For each $x \in P$, let $\mathcal{L}_x := \operatorname{IC}_{\bar{V}_x}[\dim V_x]$, and let $\mathcal{O}$ denote the Serre subcategory of $\mathbb{Q}_\ell$-perverse sheaves on $Y(\overline{\mathbb{F}}_q)$ generated by $\{\mathcal{L}_x \mid x \in P\}$. Let $\iota_x : V_x \to Y$ be the inclusion, and define

$$\mathcal{M}_x := (\iota_x)_! \mathbb{Q}_{\ell V_x}[\dim V_x] \qquad \text{and} \qquad \mathcal{N}_x := (\iota_x)_* \mathbb{Q}_{\ell V_x}[\dim V_x].$$

Then $\mathcal{O}$ is a highest weight category in with simple objects $\{\mathcal{L}_x\}$, standard objects $\{\mathcal{M}_x\}$, and costandard objects $\{\mathcal{N}_x\}$ [BGS96, Lemmas 4.4.5 and 4.4.6]. For all $x \leq y \in P$, we have $\operatorname{Ext}^j_{\mathcal{O}}(\mathcal{M}_x, \mathcal{L}_y) = 0$

unless $j + r_{xy}$ is even, and

$$f_{xy}(t) = \sum_{i \geq 0} t^i \dim \operatorname{Ext}_{\mathcal{O}}^{r_{xy}-2i}(\mathcal{M}_x, \mathcal{L}_y). \tag{4}$$

Motivated by the examples in Section 4.1, Beilinson, Ginzburg, and Soergel prove that the category $\mathcal{O}$ admits a grading, and the graded lift $\tilde{\mathcal{O}}$ of $\mathcal{O}$ is Koszul [BGS96, Theorem 4.4.4]. The Grothendieck group of $\tilde{\mathcal{O}}$ is a module over $\mathbb{Z}[t, t^{-1}]$ whose specialization at $t = 1$ is canonically isomorphic to the Grothendieck group of $\mathcal{O}$. If $\tilde{\mathcal{L}}_x$ and $\tilde{\mathcal{N}}_x$ are the natural lifts to $\tilde{\mathcal{O}}$ of $\mathcal{L}_x$ and $\mathcal{N}_x$, then we have [CPS93, Equation (3.0.6)]

$$[\tilde{\mathcal{L}}_y] = \sum_{x \leq y} \bar{f}_{xy}(t^2)[\tilde{\mathcal{N}}_x]. \tag{5}$$

More generally, Cline, Parshall, and Scott study abstract frameworks for obtaining categorical (rather than cohomological) interpretations of Kazhdan-Lusztig-Stanley polynomials [CPS93, CPS97].

4 Examples

In this section we apply the results of Section 3 to a number of different families of examples.

4.1 Flag varieties

Let G be a split reductive algebraic group over $\mathbb{F}_q$. Let $B, B^* \subset G$ be Borel subgroups with the property that $T := B \cap B^*$ is a maximal torus. Let $W := N(T)/T$ be the Weyl group. Let $Y := G/B$ be the **flag variety** of G. For all $w \in W$, let

$$V_w := \{gB \mid g \in BwB\} \qquad \text{and} \qquad C_w := \{gB \mid g \in B^*wB\}.$$

Let $e_w := wB$ be the unique element of $C_w \cap V_w$. The variety V_w is called a **Schubert cell**, and C_w is called an **opposite Schubert cell**. The flag variety is stratified by Schubert cells, and the induced partial order on W is called the **Bruhat order**.

The existence of the homomorphism $\rho_w : \mathbb{G}_m \to T \subset G$ exhibiting C_w as a weighted affine cone is proved in [KL79, Lemma A.6] (see alternatively [KL80, Section 1.5]). Let $N \subset B$ and $N^* \subset B^*$ be the unipotent radicals, and for each $w \in W$, let $N_w := N \cap wN^*w^{-1}$. Then N_w acts freely and transitively on V_w and the action map $N_w \times C_w \to Y$ is an open immersion [KL80, Section 1.4]. In particular, Lemma 3.2 applies.

For all $v \leq w$, let $U_{vw} := C_v \cap V_w$. Kazhdan and Lusztig show that $R_{vw}(q) = |U_{vw}(\mathbb{F}_q)|$ in [KL79, Lemma A.4] (see alternatively [KL80, Section 4.6]), where R is the W-kernel of Example 2.12. We therefore obtain the following corollary to Theorem 3.6, which first appeared in [KL80, Theorem 3.3].

Corollary 4.1. *Let $f \in \mathscr{I}_{1/2}(W)$ be the right KLS-function associated with $R \in \mathscr{I}(W)$. For all*

$v \leq w \in W$, $\mathrm{IH}^*_{e_v}(\bar{V}_w)$ *is chaste and*

$$f_{vw}(t) = \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}_{e_v}(\bar{V}_w).$$

For each $w \in W$, the Schubert cell $V_w \cong N_w$ is isomorphic to an affine space of dimension $\ell(w) = r_{ew}$ (where $e \in W$ is the identity element) [KL80, Section 1.3]. We therefore obtain the following corollary to Theorem 3.10, which originally appeared in [KL80, Corollary 4.8].

Corollary 4.2. *For all* $w \in W$, $g_{ew}(t) = 1$, $\mathrm{IH}^*(\bar{V}_w)$ *is chaste, and*

$$Z_{ew}(t) = \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}(\bar{V}_w).$$

Next, we use features unique to this particular class of examples to describe $Z_{vw}(t)$ for arbitrary $v \leq w \in W$. Let $\tilde{w}_0 \in N(T) \subset G$ be a lift of $w_0 \in W$. Then we have $\tilde{w}_0 V_w = C_{w_0 w}$ and $\tilde{w}_0 C_w = V_{w_0 w}$. In particular, this implies that $\mathrm{IH}^*_{e_w}(\bar{C}_v)$ is chaste for all $v \leq w$, and

$$g_{vw}(t) = f_{(w_0 w)(w_0 v)}(t) = \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}_{e_w}(\bar{C}_v) \tag{6}$$

for all $v \leq w \in W$. Consider the **Richardson variety** $\bar{C}_v \cap \bar{V}_w$.

Theorem 4.3. *For all* $x \leq w \in W$, $\mathrm{IH}^*(\bar{C}_x \cap \bar{V}_w)$ *is chaste and*

$$Z_{xw}(t) = \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}(\bar{C}_x \cap \bar{V}_w).$$

Proof. Knutson, Woo, and Yong [KWY13, Section 3.1] prove that, for all $x \leq y \leq z \leq w \in W$ and $u \in U_{yz}$, we have

$$\mathrm{IH}^*_u(\bar{C}_x \cap \bar{V}_w) \cong \mathrm{IH}^*_u(\bar{C}_x) \otimes \mathrm{IH}^*_u(\bar{V}_w) \cong \mathrm{IH}^*_{e_y}(\bar{C}_x) \otimes \mathrm{IH}^*_{e_z}(\bar{V}_w), \tag{7}$$

and therefore

$$\sum_{i \geq 0} (-1)^i \operatorname{tr}\Big(\mathrm{Fr}^s \curvearrowright \mathrm{IH}^*_u(\bar{C}_x \cap \bar{V}_w)\Big) = g_{xy}(q^s) f_{zw}(q^s).$$

Applying the Lefschetz formula, we have

$$\begin{aligned}
\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\operatorname{Fr}^s \curvearrowright \operatorname{IH}^i(\bar{C}_x\cap\bar{V}_w)\Big) &= \sum_{u\in\bar{C}_w(\mathbb{F}_{q^s})\cap\bar{V}_x(\mathbb{F}_{q^s})}\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\operatorname{Fr}^s \curvearrowright \operatorname{IH}^*_u(\bar{C}_w\cap\bar{V}_x)\Big)\\
&= \sum_{x\leq y\leq z\leq w}\sum_{u\in U_{yz}(\mathbb{F}_{q^s})}\sum_{i\geq 0}(-1)^i \operatorname{tr}\Big(\operatorname{Fr}^s \curvearrowright \operatorname{IH}^*_u(\bar{C}_w\cap\bar{V}_x)\Big)\\
&= \sum_{x\leq y\leq z\leq w} g_{xy}(q^s)R_{yz}(q^s)f_{zw}(q^s)\\
&= (gRf)_{xz}(q^s)\\
&= Z_{xz}(q^s).
\end{aligned}$$

By the same argument employed in the proofs of Theorems 3.6 and 3.10, this implies that $\operatorname{IH}^*(\bar{C}_x\cap\bar{V}_w)$ is chaste and $Z_{xw}(t)=\sum_{i\geq 0} t^i \dim \operatorname{IH}^{2i}(\bar{C}_x\cap\bar{V}_w)$. $\square$

Remark 4.4. By the observation at the end of Example 2.12, we have $g_{xy}(t)=f_{(w_0y)(w_0x)}(t)$, and therefore

$$Z_{xw}(t)=\sum_{x\leq y\leq w}\bar{g}_{xy}(t)f_{yw}(t)=\sum_{x\leq y\leq w}\bar{f}_{(w_0y)(w_0x)}(t)f_{yw}(t).$$

Thus it is possible to express the intersection cohomology Poincaré polynomial of a Richardson variety as a sum of products of classical Kazhdan-Lusztig polynomials (one of which is barred). If $x=e$ (as in Corollary 3.10), then $f_{(w_0y)(w_0x)}(t)=1$, so $\bar{f}_{(w_0y)(w_0x)}(t)=t^{r_{xy}}$ and we obtain the well-known formula for the intersection cohomology Poincaré polynomial of $\bar{V}_w$.

Remark 4.5. Since each V_w is isomorphic to an affine space, the results of Section 3.5 apply. The category $\mathcal{O}$ is equivalent to a regular block of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for the Lie algebra $\operatorname{Lie}(G)$.

4.2 The affine Grassmannian

Let G be a split reductive group over $\mathbb{F}_q$ with maximal torus $T\subset G$, and let $G^\vee$ be the Langlands dual group. Let Λ denote the lattice of coweights of G (equivalently weights of $G^\vee$), and let $\Lambda^\vee$ be the dual lattice. Let $2\rho^\vee\in\Lambda^\vee$ be the sum of the positive roots of G. Let $\Lambda^+\subset\Lambda$ be the set of dominant weights of $G^\vee$, equipped with the partial order $\mu\leq\lambda$ if and only if $\lambda-\mu$ is a sum of positive roots. This makes Λ^+ into a locally finite poset, and we endow it with the weak rank function

$$r_{\mu\lambda}:=\langle\lambda-\mu,2\rho^\vee\rangle.$$

Let $Y:=G((s))/G[[s]]$ be the affine Grassmannian for G. We have a natural bijection between Λ and $T((s))/T[[s]]$. For any $\lambda\in\Lambda^+\subset\Lambda\cong T((s))/T[[s]]$, let $\tilde{\lambda}$ be a lift of λ to $T((s))\subset G((s))$, and let e_λ be the image of $\tilde{\lambda}$ in Y, which is independent of the choice of lift. Let

$$V_\lambda:=\operatorname{Gr}^\lambda:=G[[s]]\cdot e_\lambda\subset Y.$$

This subvariety is smooth of dimension $\langle\lambda, 2\rho^\vee\rangle$, and we have a stratification

$$Y = \bigsqcup_{\lambda\in\Lambda^+} V_\lambda$$

inducing the given weakly ranked poset structure on Λ^+; see, for example, [BF10, Lemma 2.2].

For any $\mu \leq \lambda$, let $L(\lambda)_\mu$ denote the μ weight space of the irreducible representation of $G^\vee$ with highest weight λ. The vector space $L(\lambda)_\mu$ is filtered by the annihilators of powers of a regular nilpotent element of $\mathrm{Lie}(G^\vee)$, and it follows from the work of Lusztig and Brylinski that the intersection cohomology group $\mathrm{IH}^*_{\mu\lambda}$ is canonically isomorphic as a graded vector space to the associated graded of this filtration [BF10, Theorem 2.5]. Moreover, it is chaste [Hai, Theorem 2.0.1]. (The vanishing of $\mathrm{IH}^*_{\mu\lambda}$ in odd degree is originally due to Lusztig [Lus83, Section 11], and the discussion there makes it clear that he was aware that it is chaste, but the full statement of chastity does not appear explicitly.) The polynomial

$$f_{\mu\lambda}(t) := \sum_{i\geq 0} t^i \dim \mathrm{IH}^{2i}_{\mu\lambda}$$

goes by many names, including **spherical affine Kazhdan-Lusztig polynomial**, **Kostka-Foulkes polynomial**, and the ***t*-character of $\boldsymbol{L(\lambda)_\mu}$**. For a detailed discussion of various combinatorial interpretations, see [NR03, Theorem 3.17].

For any $\mu \in \Lambda^+$, let

$$C_\mu := \mathcal{W}_\mu := s^{-1}G[s^{-1}]\cdot e_\lambda \subset Y.$$

The space C_μ is infinite dimensional, but, as in Section 3.1, we will only be interested in the finite dimensional varieties

$$U_{\mu\lambda} := C_\mu \cap V_\lambda \qquad \text{and} \qquad X_{\mu\lambda} := C_\mu \cap \bar{V}_\lambda.$$

These varieties satisfy the two conditions of Section 3.1; that is, each $X_{\mu\lambda}$ is a weighted affine cone with respect to loop rotation, and the restriction of $\mathrm{IC}_{\bar{V}_\lambda}$ to $X_{\mu\lambda}(\overline{\mathbb{F}}_q)$ is isomorphic to $\mathrm{IC}_{X_{\mu\lambda}}$ [BF10, Lemma 2.9] (see also [Zhu, Proposition 2.3.9]). In particular, we have the following corollary to Theorem 3.8.

Corollary 4.6. *Let $\kappa := \bar{f}f^{-1} \in \mathscr{I}(\Lambda^+)$. Then for all $s > 0$ and $\mu \leq \lambda \in \Lambda^+$, $\kappa(q^s) = |U_{\mu\lambda}(\mathbb{F}_{q^s})|$.*

Remark 4.7. We have used the fact that $\mathrm{IH}^*_{\mu\lambda}$ is chaste to determine that $|U_{\mu\lambda}(\mathbb{F}_{q^s})|$ is a polynomial in q^s, and that one can obtain a formula for this polynomial by inverting the matrix of spherical affine Kazhdan-Lusztig polynomials. It would be interesting to prove directly that $U_{\mu\lambda}(\mathbb{F}_{q^s})$ is a polynomial in q^s, both because it would be nice to have an explicit formula for this polynomial, and because it would provide a new proof of chastity.

We now say something about the geometry of the varieties V_λ and Z-polynomials. Let $g, Z \in \mathscr{I}(\Lambda^+)$ be the left KLS-polynomial and the Z-polynomial associated with κ. For each $\lambda \in \Lambda^+$, let $P_\lambda \subset G$ be the parabolic subgroup generated by the root subgroups for roots that pair non-positively with λ. In particular, $P_0 = G$, and $P_\lambda = B$ for generic λ. Let $W_\lambda \subset W$ be the stabilizer of λ in the Weyl group. Then V_λ is an affine bundle over G/P_λ [Zhu, Section 2], which allows us to compute [Lus83, Equation (8.10) and Section 11]

$$|V_\lambda(\mathbb{F}_{q^s})| = q^{\langle\lambda,2\rho^\vee\rangle-\nu_0+\nu_\lambda}\frac{\sum_{w\in W} q^{\ell(w)}}{\sum_{w\in W_\lambda} q^{\ell(w)}}.$$

Then we have the following corollary to Theorem 3.10.

Corollary 4.8. *For all $\lambda \in \Lambda^+$, we have*

$$g_{0\lambda}(t) = t^{\nu_0-\nu_\lambda}\frac{\sum_{w\in W} t^{-\ell(w)}}{\sum_{w\in W_\lambda} t^{-\ell(w)}},$$

$\mathrm{IH}^*(\bar{V}_\lambda)$ *is chaste, and*

$$Z_{0\lambda}(t) = \sum_{i\geq 0} t^i \dim \mathrm{IH}^{2i}(\bar{V}_\lambda).$$

Remark 4.9. Lusztig [Lus83, Equation (8.10)] tells us that

$$Z_{0\lambda}(t) = \prod_{\alpha\in\Delta_+}\frac{t^{\langle\lambda+\rho,\alpha^\vee\rangle}-1}{t^{\langle\lambda,\alpha^\vee\rangle}-1},$$

where $\Delta_+ \subset \Lambda^\vee$ is the set of positive roots for G. Since the geometric Satake isomorphism identifies $\mathrm{IH}^*(\bar{V}_\lambda)$ with $L(\lambda)$, we also obtain the equation $Z_{0\lambda}(1) = \dim L(\lambda)$.

4.3 Hyperplane arrangements

Let V be a vector space over $\mathbb{F}_q$, and let $\mathcal{A} = \{H_i \mid i \in \mathcal{I}\}$ be an essential central arrangement of hyperplanes in V. For each $i \in \mathcal{I}$, let $\Lambda_i := V/H_i$, and let $\mathbb{P}_i := \mathbb{P}(\Lambda_i \oplus \mathbb{F}_q) = \Lambda_i \cup \{\infty\}$ be the projective completion of Λ_i. Let $\Lambda := \bigoplus_{i\in\mathcal{I}}\Lambda_i$ and $\mathbb{P} := \prod_{i\in\mathcal{I}}\mathbb{P}_i$. We have a natural linear embedding $V \subset \Lambda \subset \mathbb{P}$, and we define

$$Y := \bar{V} \subset \mathbb{P}.$$

The variety Y is called the **Schubert variety** of $\mathcal{A}$. The translation action of Λ on itself extends to an action on $\mathbb{P}$, and the subgroup $V \subset \Lambda$ acts on the subvariety $Y \subset \mathbb{P}$.

For any subset $F \subset \mathcal{I}$, let $e_F \in \mathbb{P}$ be the point with coordinates

$$(e_F)_i = \begin{cases} 0 & \text{if } i \in F \\ \infty & \text{if } i \in F^c, \end{cases}$$

and let

$$V_F := \{p \in Y \mid p_i = \infty \iff i \in F^c\}.$$

A subset $F \subset \mathcal{I}$ is called a **flat** if there exists a point $v \in V$ such that $F = \{i \mid v \in H_i\}$. Given a flat F, we define

$$V^F := \bigcap_{i \in F} H_i.$$

Proposition 4.10. *The variety Y is stratified by affine spaces indexed by the flats of $\mathcal{A}$. More precisely:*

1. *For any subset $F \subset \mathcal{I}$, $V_F \neq \emptyset \iff e_F \in Y \iff F$ is a flat.*
2. *For every flat F, $\mathrm{Stab}_V(e_F) = V^F$ and $V_F = V \cdot e_F \;\cong\; V/V^F$.*
3. *For every flat G, $\bar{V}_G = \bigcup_{F \subset G} V_F$.*

Proof. Item 1 is proved in [PXY18, Lemmas 7.5 and 7.6]. For the first part of item 2, we observe that $\mathrm{Stab}_V(e_F)$ is equal to the subgroup of $V \subset \Lambda$ consisting of elements v that are supported on the set $\{i \mid (e_F)_i = \infty\} = F^c$. This is equivalent to the condition that $v \in H_i$ for all $i \in F$, in other words $v \in V^F$. Thus the action of V on e_F defines an inclusion of V/V^F into V_F. The fact that this is an isomorphism follows from [PXY18, Lemma 7.6]. Item 3 is clear from the definition of V_F. □

We have a canonical action of $\mathbb{G}_m$ on Λ by scalar multiplication, which extends to an action on $\mathbb{P}$ and restricts to a stratification-preserving action on Y. For any flat $F \subset \mathcal{I}$, let

$$\mathbb{A}^F := \{p \in \mathbb{P} \mid p_i = 0 \iff i \in F\}.$$

This is isomorphic to a vector space of dimension $|F|$, and the action of $\mathbb{G}_m$ on $\mathbb{P}$ restricts to the action of $\mathbb{G}_m$ on $\mathbb{A}^F$ by inverse scalar multiplication. In particular, the coordinate ring of $\mathbb{A}^F$ is non-negatively graded by the action of $\mathbb{G}_m$, and the vanishing locus of the ideal of positively graded elements is equal to $\{e_F\}$. Let

$$C_F := \mathbb{A}^F \cap Y.$$

This is a closed $\mathbb{G}_m$-equivariant subvariety of $\mathbb{A}^F$ containing e_F, which implies that it is an affine cone with cone point e_F. Let

$$U_{FG} := C_F \cap V_G \qquad \text{and} \qquad X_{FG} := C_F \cap \bar{V}_G.$$

Proposition 4.11. *For all $F \subset G$, the restriction of $\mathrm{IC}_{\bar{V}_G}$ to $C_F(\overline{\mathbb{F}}_q)$ is isomorphic to $\mathrm{IC}_{X_{FG}}$.*

Proof. Fix the flat F, and choose a section $s : V_F \to V$ of the projection from V to V_F. The action map $\varphi_F : s(V_F) \times C_F \hookrightarrow V \times Y \to Y$ is an open immersion [PY17, Section 3], thus we can apply Lemma 3.2. □

Let L be the lattice of flats of $\mathcal{A}$, ordered by inclusion. If F is a flat, the **rank** of F is defined to be the dimension of V_F, and we define a weak rank function r by putting $r_{FG} := \mathrm{rk}\, G - \mathrm{rk}\, F$ for all $F \leq G$. Let $\chi \in \mathscr{I}(L)$ be the characteristic function (Example 2.13).

Proposition 4.12. *For any pair of flats $F \leq G$ and any positive integer s, $\chi_{FG}(q^s) = |U_{FG}(\mathbb{F}_{q^s})|$.*

Proof. When $F = \emptyset$ and $G = \mathcal{I}$, $U_{\emptyset\mathcal{I}} = V \smallsetminus \bigcup_{i\in\mathcal{I}} H_i$ is equal to the complement of the arrangement $\mathcal{A}$ in V. In this case, Crapo and Rota [CR70, Section 16] prove that $\chi_{\emptyset\mathcal{I}}(q^s) = |U_{\emptyset\mathcal{I}}(\mathbb{F}_{q^s})|$.

More generally, for any pair of flats $F \leq G$, consider the hyperplane arrangement

$$\mathcal{A}_G^F := \{(H_i \cap V^F)/V^G \mid i \in G \smallsetminus F\}$$

in the vector space V^F/V^G. The interval $[F, G] \subset L$ is isomorphic as a weakly ranked poset to the lattice of flats of $\mathcal{A}_G^F$, and U_{FG} is isomorphic to the complement of $\mathcal{A}_G^F$ in V^F/V^G. Thus Crapo and Rota's result, applied to the arrangement $\mathcal{A}_G^F$, tells us that $\chi_{FG}(q^s) = |U_{FG}(\mathbb{F}_{q^s})|$. □

The following result originally appeared in [EPW16, Theorem 3.10].

Corollary 4.13. *Let L be the weakly ranked poset of flats of the hyperplane arrangement $\mathcal{A}$, and let $f \in \mathscr{I}(L)$ be the right KLS-function associated with the L-kernel χ. For all $F \leq G \in L$, $\mathrm{IH}^*_{e_F}(\bar{V}_G)$ is chaste, and*

$$f_{FG}(t) = \sum_{i\geq 0} t^i \dim \mathrm{IH}^{2i}_{e_F}(\bar{V}_G) = \sum_{i\geq 0} t^i \dim \mathrm{IH}^{2i}(X_{FG}).$$

Proof. This follows from Lemma 3.5 and Theorem 3.6 via Propositions 4.10-4.12. □

Remark 4.14. The variety X_{FG} is called the **reciprocal plane** of the arrangement $\mathcal{A}_G^F$. Its coordinate ring is isomorphic to the **Orlik-Terao algebra** of $\mathcal{A}_G^F$, which is by definition the subalgebra of rational functions on V^F/V^G generated by the reciprocals of the linear forms that define the hyperplanes.

Remark 4.15. By Proposition 4.10(2), the strata of Y are isomorphic to affine spaces, so Equations (4) and (5) tell us that $f_{xy}(t)$ may also be interpreted as the graded dimension of an Ext group in category $\mathcal{O}$, or as the graded multiplicity of a costandard in a simple in the Grothendieck group of the graded lift.

Turning now to the Z-polynomial $Z \in \mathscr{I}_{1/2}(L)$ associated with χ, we have the following corollary of Theorem 3.10. A version of this result, along with the more general Theorem 4.17, originally appeared in [PXY18, Theorem 7.2].

Corollary 4.16. *For all $F \in L$, $\mathrm{IH}^*(\bar{V}_F)$ is chaste, and*

$$Z_{\emptyset F}(t) = \sum_{i\geq 0} t^i \dim \mathrm{IH}^{2i}(\bar{V}_F).$$

Proof. As we noted in Example 2.13, the L-kernel χ has left KLS-polynomial η, and for all $F \in L$ and $s > 0$, $|V_F(\mathbb{F}_{q^s})| = q^{sr_{\emptyset F}} = \bar{\eta}_{\emptyset F}(q^s)$. Then Theorem 3.10 gives us our result. □

As in Section 4.1, we can give a cohomological interpretation of $Z_{FG}(t)$ for any $F \leq G \in L$.

Theorem 4.17. *For all $F \leq G \in L$, $\mathrm{IH}^*(\bar{C}_F \cap \bar{V}_G)$ is chaste, and*

$$Z_{FG}(t) = \sum_{i\geq 0} t^i \dim \mathrm{IH}^{2i}(\bar{C}_F \cap \bar{V}_G).$$

Proof. The variety $\bar{C}_F \cap \bar{V}_G$ is isomorphic to the variety Y associated with the arrangement $\mathcal{A}_G^F$. Similarly, the interval $[F, G] \subset L$ is isomorphic as a weakly ranked poset to the lattice of flats of $\mathcal{A}_G^F$. Thus the theorem follows from Corollary 4.16 applied to the arrangement $\mathcal{A}_G^F$ and the pair of flats $\emptyset \leq G \smallsetminus F$. □

Remark 4.18. We have chosen to work with arrangements over a finite field in order to apply the techniques of Section 3, but this restriction is not important. First, given a hyperplane arrangement over any field, it is possible to choose a combinatorially equivalent arrangement (one with the same matroid) over a finite field [Rad57, Theorems 4 & 6]. Second, if we are given an arrangement over the complex numbers and we prefer to work with the topological intersection cohomology of the analogous complex varieties, the formulas in the statements of Corollary 4.13 and Theorem 4.17 still hold (see [EPW16, Proposition 3.12] and [PXY18, Theorem 7.2]).

Remark 4.19. The proof of Theorem 4.3 (the analogue of Theorem 4.17 for Richardson varieties) relied on two special facts, namely Equations (6) and (7). In the context of hyperplane arrangements, the analogues of these two equations hold *a posteriori*, but it is not clear how one would prove them directly. In particular, the variety C_F is not smooth, so the decomposition $Y = \bigsqcup_{F \in L} C_F$ is not a stratification, and it is not possible to apply Theorem 3.6 to obtain the analogue of Equation (6). On the other hand, the proof of Theorem 4.17 relies on the fact that any interval in the lattice of flats of an arrangement is isomorphic to the lattice of flats of another arrangement; the analogous statement for the Bruhat order on a Coxeter group is false. Thus the proofs of Theorems 4.3 and 4.17 are truly distinct.

4.4 Toric varieties

Let T be a split algebraic torus over $\mathbb{F}_q$ with cocharacter lattice N and let Σ be a rational fan in $N_{\mathbb{R}}$. We consider Σ to be a weakly ranked poset ordered by *reverse* inclusion, with weak rank function given by relative dimension. We will assume that $\{0\} \in \Sigma$; this is the *maximal* element of Σ, and we will denote it simply by 0.

Let Y be the T-toric variety associated with Σ. The cones of Σ are in bijection with T-orbits in Y and with T-invariant affine open subsets of Y. Given $\sigma \in \Sigma$, let V_σ denote the corresponding orbit, let W_σ denote the corresponding affine open subset, and let $T_\sigma \subset T$ be the stabilizer of any point in V_σ. We then have $\dim V_\sigma = \operatorname{codim} \sigma$, and [CLS11, Theorem 3.2.6]

$$\sigma \leq \tau \iff V_\sigma \subset \bar{V}_\tau \iff W_\sigma \supset W_\tau \iff W_\sigma \supset V_\tau.$$

For each $\sigma \in \Sigma$, we have a canonical identification $V_\sigma \cong T/T_\sigma$, and we define $e_\sigma \in V_\sigma$ to be the identity element of T/T_σ. In particular, we have $T_\sigma \subset T \cong V_0 \subset Y$ for all σ, and we define

$$C_\sigma := W_\sigma \cap \bar{T}_\sigma.$$

The cocharacter lattice of T_σ is equal to $N_\sigma := N \cap \mathbb{R}\sigma$, C_σ is isomorphic to the T_σ-toric variety associated with the cone $\sigma \subset N_{\sigma,\mathbb{R}}$, and $e_\sigma \in C_\sigma$ is the unique fixed point. If $\sigma \leq \tau$, then $U_{\sigma\tau} := C_\sigma \cap V_\tau$ is equal to the T_σ-orbit in C_σ corresponding to the face τ of σ. In particular, this means that

$$|U_{\sigma\tau}(\mathbb{F}_{q^s})| = (q^s - 1)^{r_{\sigma\tau}} = \lambda_{\sigma\tau}(q^s),$$

where $\lambda \in \mathscr{I}(\Sigma)$ is the Σ-kernel of Example 2.14.

For each $\sigma \in \Sigma$, choose a lattice point $n_\sigma \in N$ lying in the relative interior of σ. Then n_σ is a cocharacter of T, and thus defines a homomorphism $\rho_\sigma : \mathbb{G}_m \to T \subset \mathrm{Aut}(Y)$. The fact that σ lies in the relative interior of σ implies that C_σ is a weighted affine cone with respect to ρ_σ with cone point e_σ. Choose in addition a section $s_\sigma : T/T_\sigma \to T$ of the projection. Then the action map $s_\sigma(T/T_\sigma) \times C_\sigma \to Y$ is an open immersion, thus Lemma 3.2 tells us that the hypotheses of Section 3.1 are satisfied. We therefore obtain the following corollary to Theorem 3.6, which originally appeared in [DL91, Theorem 6.2] (see also [Fie91, Theorem 1.2]).

Corollary 4.20. *Let $f \in \mathscr{I}_{1/2}(\Sigma)$ be the right KLS-function associated with λ. For all $\sigma \leq \tau$, $\mathrm{IH}^*_{e_\sigma}(\bar{V}_\tau)$ is chaste and*

$$\sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}_{e_\sigma}(\bar{V}_\tau) = f_{\sigma\tau}(t).$$

Remark 4.21. Let Δ be a lattice polytope, and let Σ be the fan consisting of the cone over Δ along with all of its faces. Then Σ, ordered by reverse inclusion, is isomorphic to the opposite of the face poset of Δ, ordered by inclusion. It follows from Remark 2.4 that, if $g \in \mathscr{I}_{1/2}(\Delta) \cong \mathscr{I}_{1/2}(\Sigma^*)$ is the left KLS-function associated with the Eulerian poset of faces of Δ, then $g^* = f \in \mathscr{I}_{1/2}(\Sigma)$. In particular, the g-polynomial $g_{\emptyset\Delta}(t)$ is equal to $f_{c\Delta 0}(t)$.

4.5 Hypertoric varieties

Let N be a finite dimensional lattice and let $\gamma := (\gamma_i)_{i\in\mathcal{I}}$ be an $\mathcal{I}$-tuple of nonzero elements of N that together span a cofinite sublattice of N. Then γ defines a homomorphism from $\mathbb{Z}^{\mathcal{I}}$ to N, along with a dual inclusion from N^* to $\mathbb{Z}^{\mathcal{I}}$. As in Section 4.3, we define a subset $F \subset \mathcal{I}$ to be a **flat** if there exists an element $m \in N^* \subset \mathbb{Z}^{\mathcal{I}}$ such that $m_i = 0 \iff i \in F$. Given a flat F, we let $\gamma_F := (\gamma_i)_{i\in F}$ and we define $N_F \subset N$ to be the saturation of the span of γ_F. We also define $N^F := N/N_F$, and we define γ^F to be the image of $(\gamma_i)_{i\notin F}$ in N^F.

Choose a prime power q with the property that, for any subset $\mathcal{J} \subset \mathcal{I}$, the multiset $\{\gamma_i \mid i \in \mathcal{J}\}$ is linearly independent only if its image in $N_{\mathbb{F}_q}$ is linearly independent. Let $Q := \mathbb{F}_q[z_i, w_i]_{i\in\mathcal{I}}$. This ring admits a grading by the group $\mathbb{Z}^{\mathcal{I}} = \mathbb{Z}\{x_i \mid i \in \mathcal{I}\}$ in which $\deg z_i = -\deg w_i = x_i$. The degree zero part $Q_0 = \mathbb{F}_q[z_i w_i]_{i\in\mathcal{I}}$ maps to $\mathrm{Sym}\, N_{\mathbb{F}_q}$ by sending $z_i w_i$ to the reduction modulo q of γ_i. Let Q_{N^*} be the subring of Q with basis consisting of $\mathbb{Z}^{\mathcal{I}}$-homogeneous elements whose degrees lie in $N^* \subset \mathbb{Z}^{\mathcal{I}}$, and let $R := Q_{N^*} \otimes_{Q_0} \mathrm{Sym}\, N_{\mathbb{F}_q}$. The variety $Y = Y(\gamma) := \mathrm{Spec}\, R$ is called a **hypertoric variety**.

Let $\mathring{Y} \subset Y$ be the open subvariety defined by the nonvanishing of all elements of R that lift to monomials in Q. Let L be the lattice of flats of γ. We have a stratification

$$Y = \bigsqcup_{F \in L} V_F,$$

with the property that $V_F \cong \mathring{Y}(\gamma^F)$ [PW07, Equation 5]. In particular, the largest stratum is $V_\emptyset$ and the smallest stratum is $V_{\mathcal{I}}$. More generally, the partial order induced by the stratification is the opposite of the inclusion order. For any $F \subset G$, the dimension of V_F minus the dimension of V_G is equal to $2r_{FG}$, where r is the usual weak rank function (as in Example 2.16).

At this point, we are forced to depart from the setup of Section 3.1. We are supposed to define a subvariety $C_F \subset Y$ for each flat F, satisfying certain properties; then for every $F \subset G$, we would consider the varieties $U_{GF} = C_G \cap V_F$ and $X_{GF} = C_G \cap \bar{V}_F$. Morally, we should have $C_F \cong Y(\gamma_F)$, $X_{GF} \cong Y(\gamma_G^F)$, and $U_{GF} \cong \mathring{Y}(\gamma_G^F)$. Unfortunately, we do not know of any natural way to embed $Y(\gamma_F)$ into Y to achieve these isomorphisms. Instead, we will simply define X_{GF} and U_{GF} as above. The conclusion of Lemma 3.1 clearly holds for this definition, while the conclusion of Lemma 3.4 follows from [PW07, Lemma 2.4]. Thus Theorem 3.6 still holds as stated. By [PW07, Proposition 4.2], for all $s > 0$ and all flats $F \subset G$, we have $|U_{GF}(\mathbb{F}_{q^s})| = \kappa_{FG}(q^s)$, where $\kappa \in \mathscr{I}(L, 2r)$ is the $(L, 2r)$-kernel of Example 2.16.

Corollary 4.22. *Let $h^{\mathrm{bc}} \in \mathscr{I}(L, 2r)$ be the left KLS-function associated with the $(L, 2r)$-kernel κ of Example 2.16. For all flats $F \subset G \in L$, $\mathrm{IH}^*(\bar{X}_{GF})$ is chaste, and*

$$h^{\mathrm{bc}}_{FG}(t) = \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}(\bar{X}_{GF}).$$

Proof. As noted above, our stratification of Y induces the weakly ranked poset $(L^*, 2r^*)$. Let f be the right KLS-function associated with the $(L^*, 2r^*)$-kernel κ^*. For all $s > 0$ and all flats $F \subset G$, we have $\kappa^*_{GF}(q^s) = \kappa_{FG}(q^s) = |U_{GF}(\mathbb{F}_{q^s})|$, thus Theorem 3.6 tells us that $\mathrm{IH}^*(\bar{X}_{GF})$ is chaste, and

$$f_{GF}(t) = \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}(\bar{X}_{GF}).$$

By Remark 2.4, we have $h^{\mathrm{bc}} = f^*$, which proves the corollary. □

References

[BB96] Victor V. Batyrev and Lev A. Borisov, *Mirror duality and string-theoretic Hodge numbers*, Invent. Math. **126** (1996), no. 1, 183–203.

[BF10] Alexander Braverman and Michael Finkelberg, *Pursuing the double affine Grassmannian. I. Transversal slices via instantons on A_k-singularities*, Duke Math. J. **152** (2010), no. 2, 175–206.

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[Bra06] Tom Braden, *Remarks on the combinatorial intersection cohomology of fans*, Pure Appl. Math. Q. **2** (2006), no. 4, Special Issue: In honor of Robert D. MacPherson. Part 2, 1149–1186.

[Bre99] Francesco Brenti, *Twisted incidence algebras and Kazhdan-Lusztig-Stanley functions*, Adv. Math. **148** (1999), no. 1, 44–74.

[Bre03] ———, *P-kernels, IC bases and Kazhdan–Lusztig polynomials*, Journal of Algebra **259** (2003), no. 2, 613–627.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[CPS93] Edward Cline, Brian Parshall, and Leonard Scott, *Abstract Kazhdan-Lusztig theories*, Tohoku Math. J. (2) **45** (1993), no. 4, 511–534.

[CPS97] ———, *Graded and non-graded Kazhdan-Lusztig theories*, Algebraic groups and Lie groups, Austral. Math. Soc. Lect. Ser., vol. 9, Cambridge Univ. Press, Cambridge, 1997, pp. 105–125.

[CR70] Henry H. Crapo and Gian-Carlo Rota, *On the foundations of combinatorial theory: Combinatorial geometries*, preliminary ed., The M.I.T. Press, Cambridge, Mass.-London, 1970.

[dCM09] Mark Andrea A. de Cataldo and Luca Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 535–633.

[DL91] J. Denef and F. Loeser, *Weights of exponential sums, intersection cohomology, and Newton polyhedra*, Invent. Math. **106** (1991), no. 2, 275–294.

[Du94] Jie Du, IC *bases and quantum linear groups*, Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, pp. 135–148.

[Dye93] M. J. Dyer, *Hecke algebras and shellings of Bruhat intervals*, Compositio Math. **89** (1993), no. 1, 91–115.

[EPW16] Ben Elias, Nicholas Proudfoot, and Max Wakefield, *The Kazhdan-Lusztig polynomial of a matroid*, Adv. Math. **299** (2016), 36–70.

[EW14] Ben Elias and Geordie Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) **180** (2014), no. 3, 1089–1136.

[Fie91] Karl-Heinz Fieseler, *Rational intersection cohomology of projective toric varieties*, J. Reine Angew. Math. **413** (1991), 88–98.

[GPY17] Katie Gedeon, Nicholas Proudfoot, and Benjamin Young, *Kazhdan-Lusztig polynomials of matroids: a survey of results and conjectures*, Sém. Lothar. Combin. **78B** (2017), Art. 80, 12.

[Hai] Thomas Haines, *A proof of the Kazhdan-Lusztig purity theorem via the decomposition theorem of BBD*, http://www.math.umd.edu/~tjh/KL_purity1.pdf.

[Kar04] Kalle Karu, *Hard Lefschetz theorem for nonrational polytopes*, Invent. Math. **157** (2004), no. 2, 419–447.

[KL79] David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184.

[KL80] David Kazhdan and George Lusztig, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 185–203.

[KW01] Reinhardt Kiehl and Rainer Weissauer, *Weil conjectures, perverse sheaves and l'adic Fourier transform*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 42, Springer-Verlag, Berlin, 2001.

[KW06] Frances Kirwan and Jonathan Woolf, *An introduction to intersection homology theory*, second ed., Chapman & Hall/CRC, Boca Raton, FL, 2006.

[KWY13] Allen Knutson, Alexander Woo, and Alexander Yong, *Singularities of Richardson varieties*, Math. Res. Lett. **20** (2013), no. 2, 391–400.

[Lus83] George Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 208–229.

[Lus03] G. Lusztig, *Hecke algebras with unequal parameters*, CRM Monograph Series, vol. 18, American Mathematical Society, Providence, RI, 2003.

[NR03] Kendra Nelsen and Arun Ram, *Kostka-Foulkes polynomials and Macdonald spherical functions*, Surveys in combinatorics, 2003 (Bangor), London Math. Soc. Lecture Note Ser., vol. 307, Cambridge Univ. Press, Cambridge, 2003, pp. 325–370.

[Pol99] Patrick Polo, *Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups*, Represent. Theory **3** (1999), 90–104.

[PW07] Nicholas Proudfoot and Ben Webster, *Intersection cohomology of hypertoric varieties*, J. Algebraic Geom. **16** (2007), no. 1, 39–63.

[PXY18] Nicholas Proudfoot, Yuan Xu, and Ben Young, *The Z-polynomial of a matroid*, Electron. J. Combin. **25** (2018), no. 1, Paper 1.26, 21.

[PY17] Nicholas Proudfoot and Ben Young, *Configuration spaces,* $\mathrm{FS}^{\mathrm{op}}$*-modules, and Kazhdan-Lusztig polynomials of braid matroids*, New York J. Math. **23** (2017), 813–832.

[Rad57] R. Rado, *Note on independence functions*, Proc. London Math. Soc. (3) **7** (1957), 300–320.

[Sta80] Richard P. Stanley, *The number of faces of a simplicial convex polytope*, Adv. in Math. **35** (1980), no. 3, 236–238.

[Sta87] Richard Stanley, *Generalized H-vectors, intersection cohomology of toric varieties, and related results*, Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 187–213.

[Sta92] Richard P. Stanley, *Subdivisions and local h-vectors*, J. Amer. Math. Soc. **5** (1992), no. 4, 805–851.

[Wak18] Max Wakefield, *A flag Whitney number formula for matroid Kazhdan-Lusztig polynomials*, Electron. J. Combin. **25** (2018), no. 1, Paper 1.22, 14.

[Zhu] Xinwen Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence*, arXiv:1603.05593.