GALE DUALITY AND KOSZUL DUALITY

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ABSTRACT. Given an affine hyperplane arrangement with some additional structure, we define two finite-dimensional, noncommutative algebras, both of which are motivated by the geometry of hypertoric varieties. We show that these algebras are Koszul dual to each other, and that the roles of the two algebras are reversed by Gale duality. We also study the centers, deformations, and representation categories of our algebras, which are in many ways analogous to integral blocks of category \( \mathcal{O} \).

1. INTRODUCTION

In this paper, we define and study a class of finite dimensional graded algebras which are related to the combinatorics of hyperplane arrangements and to the geometry of hypertoric varieties. To define these algebras, we take as our input what we call a polarized arrangement, consisting of a linear subspace \( V \) of a coordinate vector space \( \mathbb{R}^I \), a vector \( \eta \in \mathbb{R}^I/V \), and a covector \( \xi \in V^* \). Geometrically, it is convenient to think of this data as describing an affine space \( (V + \eta \subseteq \mathbb{R}^I) \) along with an affine linear functional (given by \( \xi \)) and a finite hyperplane arrangement \( \mathcal{H} \) (the restrictions of the coordinate hyperplanes in \( \mathbb{R}^I \)). We will usually denote a polarized arrangement by the triple \( \mathcal{V} = (V, \eta, \xi) \), with the understanding that the inclusion of \( V \) into \( \mathbb{R}^I \) is part of the data.

If \( \mathcal{V} \) is rational, meaning that \( V, \eta, \) and \( \xi \) are all defined over \( \mathbb{Q} \), then we may associate to \( \mathcal{V} \) a symplectic complex algebraic variety called a hypertoric variety, equipped with a Hamiltonian \( \mathbb{C}^* \) action. The variety itself depends only on the arrangement \( \mathcal{H} \) (that is, on \( V \) and \( \eta \)), and is denoted \( \mathcal{M}_\mathcal{H} \). It is defined as a hyperkähler quotient of the quaternionic vector space \( \mathbb{H}^n \) by an \( (n - \text{dim } V) \)-dimensional torus determined by \( V \), where the quotient parameter is specified by \( \eta \). This hypertoric variety carries a natural Hamiltonian action of an algebraic torus with Lie algebra \( V_\mathbb{C}^* \), and \( \xi \) determines a one-dimensional subtorus. The definitions and results of this paper do not require any knowledge of hypertoric varieties (indeed, they will hold even if \( \mathcal{V} \) is not rational, in which case there are no varieties in the picture). They will, however, be strongly motivated by hypertoric geometry, and we will take every
opportunity to point out this motivation. The authors plan to make this connection more precise in future work. The interested reader can learn more about hypertoric varieties in the survey [Pro].

Given a polarized arrangement $\mathcal{V}$, we define two quadratic graded algebras, which we denote by $A = A(\mathcal{V})$ and $B = B(\mathcal{V})$. The algebra $A$ has a presentation in terms of the combinatorics and linear algebra of $\mathcal{V}$; if $V$ is rational, its category of modules may be identified with a certain category of sheaves on $\mathcal{M}_H$, supported on a collection of Lagrangian subvarieties determined by the $\mathbb{C}^\times$-action. The algebra $B$ is defined using a convolution product on the direct sum of the cohomology groups of the toric varieties associated to certain faces of $\mathcal{H}$, which appear as pairwise intersections of the aforementioned Lagrangian subvarieties of $\mathcal{M}_H$. This definition can be extended to the non-rational case using the fact that the cohomology ring of a toric variety has a purely combinatorial description in terms of its polytope.

There are two forms of duality lurking in this picture, one of which comes from linear programming and the other from ring theory. In linear programming, we can define the Gale dual of a polarized arrangement $\mathcal{V} = (V, \eta, \xi)$ as the triple

$$\mathcal{V}^\vee = (V^\perp, -\xi, -\eta),$$

where $V^\perp \subseteq (\mathbb{R}^I)^*$ is the space of linear forms on $\mathbb{R}^I$ that vanish on $V$. On the other hand, to any quadratic algebra $E$ we may associate another quadratic algebra $E^!$ which is known as its quadratic dual. We will prove that the algebras $A$ and $B$ are dual to each other in both of these senses:

**Theorem (A).** We have isomorphisms

$$A(\mathcal{V})^! \cong A(\mathcal{V}^\vee),$$

$$B(\mathcal{V}) \cong B(\mathcal{V}^\vee)^!.$$

The module categories of the algebras $A$ and $B$ are similar in many ways to integral blocks of various flavors of category $O$, which is a category of representations of a semi-simple Lie algebra. In particular, we prove the following three facts, all of

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1Here and elsewhere in the paper we use the term "combinatorial" loosely to refer to constructions involving finite operations on linear algebra data.
which are analogous to results in representation theory related to category $O$ and
the geometry of the Springer resolution [Spa76, Irv85, Bru08, SW].

**Theorem (B).**

1. The algebras $A$ and $B$ are quasi-hereditary and Koszul (and thus are Koszul dual).
2. If $V$ is rational, then the center of $B$ is canonically isomorphic to the cohomology
   ring of $\mathcal{M}_H$.
3. There is a canonical bijection between indecomposable projective-injective $B$-modules
   and compact chambers of the hyperplane arrangement $\mathcal{H}$; if $V$ is rational, these are
   in bijection with the set of all irreducible projective Lagrangian subvarieties of $\mathcal{M}_H$.

Part (2) of Theorem (B) says that the centers of our algebras are isomorphic to
the cohomology rings of hypertoric varieties, which are entirely determined by the
subspace $V \subset \mathbb{R}^I$ [Kon00, HS02, Pro]. This leads us to ask to what extent the algebras
themselves depend on $\eta$ and $\xi$. The answer turns out two be that two polarized
arrangements with the same underlying vector space are neither isomorphic nor
Morita equivalent, but they are derived Morita equivalent.

**Theorem (C).** The bounded derived category of graded modules over $A(V)$ or $B(V)$ depends
only on the subspace $V \subset \mathbb{R}^I$.

The paper is structured as follows. In Section 2, we lay out the combinatorics and
linear algebra of polarized arrangements, introducing definitions and constructions
upon which we will rely throughout the paper. Section 3 is devoted to the algebra
$A$, and contains a proof of the isomorphism on the top edge of the square in Theorem (A). Section 4 is about the algebra $B$; in it we complete the proof of Theorem
(A), as well as part (2) of Theorem (B). Section 5 begins with a general overview of
quasi-hereditary and Koszul algebras, culminating in the proofs of parts (1) and (3)
of Theorem (B). The purpose of Section 6 is to prove Theorem (C), and along the
way we study Ringel duality, Serre functors, and mutations of exceptional collections
in the category of right $A$-modules. Finally, in Section 7, we consider natural
deformations of the algebras $A$ and $B$, which we show are both special cases of a
completely general construction that may be applied to any quadratic algebra. This
construction is (as far as we know) new, and will be studied in greater detail in a
future paper.

Much of this work is motivated by our belief that Gale duality for hypertoric
varieties should be thought of as a special case of a more general phenomenon called
symplectic duality between the varieties $\mathcal{M}_H$ and $\mathcal{M}_{H'}$ [BLP, BLPW], where $H'$ is the hyperplane arrangement associated to $V'$. Other examples of symplectic dual pairs include Springer resolutions for Langlands dual groups, and certain pairs of moduli spaces of instantons on surfaces. These examples all appear as the Higgs branches of the moduli space of vacua for mirror dual 3-d $\mathcal{N} = 4$ super-conformal field theories, or as the Higgs and Coulomb branches of a single such theory. For hypertoric varieties, this was shown by Strassler and Kapustin [KS99]. We anticipate that our results on symplectic duality will ultimately be related to the structure of these field theories.

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2. LINEAR PROGRAMMING

2.1. Polarized arrangements. Let $I$ be a finite set.

Definition 2.1. A polarized arrangement indexed by $I$ is a triple $\mathcal{V} = (V, \eta, \xi)$ consisting of

- a vector subspace $V \subset \mathbb{R}^I$,
- a vector $\eta \in \mathbb{R}^I/V$, and
- a covector $\xi \in V^* = (\mathbb{R}^I)^*/V^\perp$,

such that

(a) every element of $V + \eta$ has at least $|I| - \dim V$ nonzero entries, and
(b) every element of $V^\perp + \xi$ has at least $\dim V$ nonzero entries.

If $V$, $\eta$, and $\xi$ are all defined over $\mathbb{Q}$, then $\mathcal{V}$ is called rational.

Associated to a (not necessarily rational) polarized arrangement $\mathcal{V} = (V, \eta, \xi)$ is an arrangement $\mathcal{H}$ of $|I|$ hyperplanes in the affine space $V + \eta \subset \mathbb{R}^I$, where the $i^{th}$
hyperplane is given by the equation
\[ H_i = \{ x \in V + \eta \mid x_i = 0 \}. \]
For any subset \( S \subset I \), we let
\[ H_S = \bigcap_{i \in S} H_i \]
be the flat spanned by the set \( S \). Condition (a) implies that \( H \) is simple, meaning that \( \text{codim} H_S = |S| \) whenever \( H_S \) is nonempty. Observe that \( \xi \) may be regarded as an affine-linear functional on \( V + \eta \); it does not have well-defined values, but it may be used to compare any pair of points. Condition (b) implies that \( \xi \) is generic with respect to the arrangement, in the sense that it is not constant on any positive-dimensional \( H_S \).

Note that we do not exclude the possibility that \( V \) is contained in some hyperplane \( \{ x_i = 0 \} \), in which case the corresponding hyperplane \( H_i \) is empty. Such an index \( i \) is referred to as a loop of \( V \), since it is a loop in the associated matroid.

2.2. Boundedness and feasibility. Given a sign vector \( \alpha \in \{\pm 1\}^I \), let
\[ \Delta_\alpha = (V + \eta) \cap \{ x \in \mathbb{R}^I \mid \alpha(i)x_i \geq 0 \text{ for all } i \} \]
and
\[ \Sigma_\alpha = V \cap \{ x \in \mathbb{R}^I \mid \alpha(i)x_i \geq 0 \text{ for all } i \}. \]
If \( \Delta_\alpha \) is nonempty, it is the closed chamber of \( H \) obtained by setting the defining equations of the hyperplanes to inequalities according to the signs in \( \alpha \). The cone \( \Sigma_\alpha \) is always nonempty, as it contains 0. If \( \Delta_\alpha \) is nonempty, then \( \Sigma_\alpha \) is simply the cone of unbounded directions in \( \Delta_\alpha \). If not, then \( \Sigma_\alpha \) may be understood as the cone of unbounded directions in a chamber \( \Delta'_\alpha \) associated to a different polarized arrangement \( (V, \eta', \xi) \). Put differently, we translate the hyperplanes in some generic way so that the polyhedron \( \Delta_\alpha \) becomes nonempty, and then take its cone of unbounded directions.

We now define subsets \( \mathcal{F}, \mathcal{B}, \mathcal{P} \subset \{\pm 1\}^I \) as follows. First we let
\[ \mathcal{F} = \{ \alpha \in \{\pm 1\}^I \mid \Delta_\alpha \neq \emptyset \}. \]
Elements of \( \mathcal{F} \) are called feasible. It is clear that \( \mathcal{F} \) depends only on \( V \) and \( \eta \). Next, we let
\[ \mathcal{B} = \{ \alpha \in \{\pm 1\}^I \mid \xi(\Sigma_\alpha) \text{ is bounded above} \}. \]
Elements of $\mathcal{B}$ are called **bounded**, and it is clear that $\mathcal{B}$ depends only on $V$ and $\xi$. Elements of the intersection

$$\mathcal{P} := \mathcal{F} \cap \mathcal{B} = \{ \alpha \in \{\pm 1\}^I \mid \xi(\Delta_\alpha) \text{ is nonempty and bounded above} \}$$

are called **bounded feasible**; here $\xi(\Delta_\alpha)$ is regarded as a subset of the affine line. Our use of these terms comes from linear programming, where we consider $\alpha$ as representing the linear program “find the maximum of $\xi$ on the polyhedron $\Delta_\alpha$”.

### 2.3. Gale duality.

Here we introduce one of the two main dualities of this paper.

**Definition 2.2.** The **Gale dual** $V^\vee$ of a polarized arrangement $V$ is given by the triple $(V^\perp, -\xi, -\eta)$. We denote by $F^\vee$, $B^\vee$, and $P^\vee$ the feasible, bounded, and bounded feasible sign vectors for $V^\vee$.

This definition agrees with the notion of duality in linear programming: the linear programs for $V$ and $V^\vee$ and a fixed sign vector $\alpha$ are dual to each other. The following key result is a form of the strong duality theorem of linear programming.

**Theorem 2.3.** $F^\vee = B$, $B^\vee = F$, and therefore $P^\vee = P$.

**Proof.** It is enough to show that $\alpha = (+1, \ldots, +1)$ is feasible for $V$ if and only if it is bounded for $V^\vee$. The Farkas lemma [Zie95, 1.8] says that exactly one of the following statements holds:

1. there exists a lift of $\eta$ to $\mathbb{R}^I$ which lies in $\mathbb{R}^I_{\geq 0}$,
2. there exists $c \in V^\perp \subset (\mathbb{R}^I)^*$ which is positive on $\mathbb{R}^I_{\geq 0}$ and negative on $\eta$.

Statement (1) is equivalent to $\alpha \in \mathcal{F}$, while a vector $c$ satisfying (2) lies in $\Sigma^\vee_\alpha$, so $c(\eta) < 0$ means that $-\eta$ is not bounded above on $\Sigma^\vee_\alpha$. \qed

### 2.4. Restriction and deletion.

We define two operations which reduce the number of hyperplanes in the system as follows. First, consider a subset $S \subset I$ such that $V + \mathbb{R}^{I\smallsetminus S} = \mathbb{R}^I$. Since $\eta$ is assumed to be generic, this condition is equivalent to saying that $H_S \neq \emptyset$. Consider the natural isomorphism

$$i : \mathbb{R}^{I\smallsetminus S}/(V \cap \mathbb{R}^{I\smallsetminus S}) \rightarrow \mathbb{R}^I/V$$
induced by the inclusion of $\mathbb{R}^{I \setminus S}$ into $\mathbb{R}^I$. We define a new polarized arrangement $\mathcal{V}^S = (V^S, \xi^S, \eta^S)$, indexed by the set $I \setminus S$, as follows:

\[
V^S := V \cap \mathbb{R}^{I \setminus S} \subset \mathbb{R}^{I \setminus S} \\
\xi^S := \xi|_{V^S} \\
\eta^S := i^{-1}(\eta).
\]

The arrangement $\mathcal{V}^S$ is called the restriction of $\mathcal{V}$ to $S$, since the associated hyperplane arrangement is isomorphic to the hyperplane arrangement obtained by restricting to the subspace $H_S$.

Dually, suppose that $S \subset I$ is a subset such that $\mathbb{R}^S \cap V = \{0\}$, and let

\[
\pi : \mathbb{R}^I \rightarrow \mathbb{R}^{I \setminus S}
\]

be the coordinate projection, which restricts to an isomorphism

\[
\pi|_{\mathcal{V}} : V \longrightarrow \pi(V).
\]

We define the another polarized arrangement $\mathcal{V}_S = (V_S, \eta_S, \xi_S)$, also indexed by $I \setminus S$, as follows:

\[
V_S = \pi(V) \subset \mathbb{R}^{I \setminus S} \\
\eta_S = \pi(\eta) \\
\xi_S = \xi \circ \pi|_{\mathcal{V}}^{-1}.
\]

The arrangement $\mathcal{V}_S$ is called the deletion of $S$ from $\mathcal{V}$, since the associated hyperplane arrangement is obtained by removing the hyperplanes $\{H_i\}_{i \in S}$ from the arrangement associated to $\mathcal{V}$. The following lemma is an easy consequence of the definitions.

**Lemma 2.4.** $(\mathcal{V}^S)^\vee$ is equal to $(\mathcal{V}^\vee)_S$.

2.5. **The adjacency relation.** Define a relation $\leftrightarrow$ on $\{\pm 1\}^I$ by setting

\[
\alpha \leftrightarrow \beta
\]

if and only if $\alpha$ and $\beta$ differ in exactly one place; if $\alpha, \beta \in \mathcal{F}$, this means that $\Delta_\alpha$ and $\Delta_\beta$ are obtained from each other by flipping across a single hyperplane. We will write $\alpha \leftrightarrow^i \beta$ to indicate that $\alpha$ and $\beta$ differ in the $i$th component of $\{\pm 1\}^I$. The following lemma says that an infeasible neighbor of a bounded feasible sign vector is feasible for the Gale dual system.
Lemma 2.5. Suppose that $\alpha \notin F$, $\beta \in P$, and $\alpha \leftrightarrow \beta$. Then $\alpha \in F^\vee$.

Proof. Suppose that $\alpha \not\leftrightarrow \beta$. The fact that $\beta \in P \subseteq F$ tells us that $\Delta_\beta \neq \emptyset$, while $\Delta_\alpha = \emptyset$, thus

$$H_1 \cap \Delta_\beta = \Delta_\alpha \cap \Delta_\beta = \emptyset.$$  

From this we can conclude that

$$\Sigma_\alpha = \Sigma_\beta \cap \{x_i = 0\} \subset \Sigma_\beta.$$  

The fact that $\beta \in P \subset B$, tells us that $\xi(\Sigma_\beta)$ is bounded above, thus so is $\xi(\Sigma_\alpha)$. This in turn tells us that $\alpha \in B$, which is equal to $F$ by Theorem 2.3. □

2.6. Bases and the partial order. Let $\text{Bas}$ be the set of subsets $b \subset I$ of order $\dim V$ such that $H_b \neq \emptyset$. Such a subset is called a basis for the matroid associated to $V$, and in fact this property depends only on the subspace $V \subset R^I$. A set $b$ is a basis if and only if the composition $V \hookrightarrow \mathbb{R}^I \to \mathbb{R}^b$ is an isomorphism, which is equivalent to saying that we have a direct sum decomposition $\mathbb{R}^I = V \oplus \mathbb{R}^{b^c}$, where we put $b^{I} = I \setminus b$. We have a bijection

$$\mu : \text{Bas} \to P$$

taking $b$ to the unique sign vector $\alpha$ such that $\xi$ attains its maximum on $\Delta_\alpha$ at the point $H_b$.

The covector $\xi$ induces a partial order $\leq$ on $\text{Bas} \cong P$. It is the transitive closure of the relation $\preceq$, where $b_1 \prec b_2$ if $|b_1 \cap b_2| = |b_1| - 1 = \dim V - 1$ and $\xi(H_{b_1}) < \xi(H_{b_2})$. The first condition means that $H_{b_1}$ and $H_{b_2}$ lie on the same one-dimensional flat, so $\xi$ cannot take the same value on these two points.

Let $\text{Bas}^\vee$ denote the set of bases of $V^\vee$. We have a bijection $b \mapsto b^c$ from $\text{Bas}$ to $\text{Bas}^\vee$, since $\mathbb{R}^I = V \oplus \mathbb{R}^{b^c}$ if and only if $\mathbb{R}^I = V^\perp \oplus \mathbb{R}^{b^c}$. The next result says that this bijection is compatible with the equality $P = P^\vee$ and the bijections $\mu : \text{Bas} \to P$ and $\mu^\vee : \text{Bas}^\vee \to P^\vee$.

Lemma 2.6. For all $b \in \text{Bas}$, $\mu(b) = \mu^\vee(b^c)$.

Proof. Let $b \in \text{Bas}$. It will be enough to show that $\mu(b) = \alpha$ if and only if

1. the projection of $\alpha$ to $\{\pm 1\}^{b^c}$ is feasible for the restriction $V^{b^c}$, and
2. the projection of $\alpha$ to $\{\pm 1\}^b$ is bounded for the deletion $V_{b^c}$,

since these conditions are clearly interchanged by swapping $b$ and $b^c$, and Theorem 2.3 and Lemma 2.4 tell us that they are also interchanged by Gale duality.
Note that $V^b$ represents the restriction of the arrangement to $H_b$, which is a point. All of the remaining hyperplanes are therefore loops, whose positive and negative sides are either all of $H_b$ or empty. Condition (1) just says then that $H_b$ lies in the chamber $\Delta_\alpha$. Given that, in order for $H_b$ to be the $\xi$-maximum on $\Delta_\alpha$, it is enough that when all the hyperplanes not passing through $H_b$ are removed from the arrangement, the chamber containing $\alpha$ is bounded. But this is exactly the statement of condition (2), which completes the proof. □

The following lemma demonstrates that this bijection is compatible with our partial order.

**Lemma 2.7.** Under the bijection $b \mapsto b^c$, the partial order on $\text{Bas}^V$ is the opposite of the partial order on $\text{Bas}$. That is, $b_1 \leq b_2$ if and only if $b_1^c \geq b_2^c$.

**Proof.** It will be enough to show that the generating relations $\prec$ are reversed. This means checking that whenever $\mu(b_1) \leftrightarrow \mu(b_2)$, we have $\xi(H_{b_1}) < \xi(H_{b_2})$ if and only if $\xi^{V^b}(H_{b_1}^c) > \xi^{V^b}(H_{b_2}^c)$. The fact that $\mu(b_1) \leftrightarrow \mu(b_2)$ implies that there exists $i_1 \in b_1$ and $i_2 \in b_2$ such that $b_2 = b_1 \cup \{i_2\} \setminus \{i_1\}$. We may then reduce the lemma to the (easy) case where $|I| = 2$ by replacing $V$ with the polarized arrangement $V_{(b_1 \setminus b_2)}^b$, obtained by restricting to the 1-dimensional flat spanned by $H_{b_1}$ and $H_{b_2}$ and then deleting all hyperplanes but $H_{i_1}$ and $H_{i_2}$. □

For $b \in \text{Bas}$, let

$$B_b = \{ \alpha \in \{\pm 1\}^I \mid \alpha(i) = \mu(b)(i) \text{ for all } i \in b \}.$$ 

Note that $B_b \subset B$, and $B_b$ depends only on $V$ and $\xi$. Geometrically, the feasible sign vectors in $B_b$ are those such that $\Delta_\alpha$ lies in the “negative cone” defined by $\xi$ with vertex $H_b$. Dually, we define

$$F_b = \{ \alpha \in \{\pm 1\}^I \mid \alpha(i) = \mu(b)(i) \text{ for all } i \notin b \} = B_b^c$$

to be the set of sign vectors such that $H_b \in \Delta_\alpha$. In particular, $F_b \subset F$. We will need the following lemma in Section 5.

**Lemma 2.8.** If $\alpha \in B_b$ and $\mu(a) = \alpha$, then $a \leq b$.

**Proof.** Let $a = \mu^{-1}(\alpha)$, and let $C$ be the negative cone of $\mu(b)$. Then $\alpha \in B_b$ means that $H_a \in C$. Let $C'$ be the smallest face of $C$ on which $H_a$ lives. We will prove the lemma by induction on $d = \dim C'$. If $d = 0$, then $a = b$ and we are done. Otherwise, there is a 1-dimensional flat which is contained in $C'$ and passes through $H_a$. Following it
in the $\xi$-positive direction, it must leave $C'$, since $C'$ is $\xi$-bounded. The point where it exits will be $H_c$ for some basis $c$, and $a < c$ by construction. But $H_c$ lies on a face of $C$ of smaller dimension, so the inductive hypothesis gives $c \leq b$. □

3. THE ALGEBRA $A$

3.1. Definition of $A$. Fix a polarized arrangement $V$, and consider the quiver $Q = Q(V)$ with vertex set $F$ and arrows $\{ (\alpha, \beta) \mid \alpha \leftrightarrow \beta \}$. Note in particular that there is an arrow from $\alpha$ to $\beta$ if and only if there is an arrow from $\beta$ to $\alpha$. Let $P(Q)$ be the algebra of real linear combinations of paths in the quiver $Q$, generated by pairwise orthogonal idempotents $\{ e_\alpha \mid \alpha \in F \}$ along with edge paths $\{ p(\alpha, \beta) \mid \alpha \leftrightarrow \beta \}$. We use the following notation: if $\alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \ldots \leftrightarrow \alpha_k$ is a path in the quiver, then we write\footnote{Note that with this convention, a representation of $Q$ is a right module over $P(Q)$.}

$$p(\alpha_1, \alpha_2, \ldots, \alpha_k) := p(\alpha_1, \alpha_2) \cdot p(\alpha_2, \alpha_3) \cdot \ldots \cdot p(\alpha_{k-1}, \alpha_k).$$

Let $t_i \in V^*$ be the restriction to $V$ of the $i^{th}$ coordinate function on $\mathbb{R}^I$.

**Definition 3.1.** We define $A = A(V)$ to be the quotient of $P(Q) \otimes_{\mathbb{R}} \text{Sym} V^*$ by the two-sided ideal generated by the following relations:

- **A1:** If $\alpha \in F \setminus P$, then $e_\alpha = 0$.

- **A2:** If four distinct elements $\alpha, \beta, \gamma, \delta \in F$ satisfy $\alpha \leftrightarrow \beta \leftrightarrow \gamma \leftrightarrow \delta \leftrightarrow \alpha$, then

$$p(\alpha, \beta, \gamma) = p(\alpha, \beta, \gamma).$$

- **A3:** If $\alpha, \beta \in F$ and $\alpha \leftrightarrow \beta$, then

$$p(\alpha, \beta) = t_i e_\alpha.$$ 

We put a grading on this algebra by letting $\deg(e_\alpha) = 0$ for all $\alpha \in F$, $\deg(p(\alpha, \beta)) = 1$ for all $\alpha \leftrightarrow \beta$, and $\deg(t_i) = 2$ for all $i \in I$.

**Remark 3.2.** We have made the generating set of $A$ considerably larger than it needs to be in order to simplify the relations. First, we note that the relations of type A3 allow us to express any element of $\text{Sym} V^*$ in terms of paths, thus $A$ really is a quotient of the path algebra $P(Q)$. This follows from the fact that for any $\alpha \in F$, the set $\{ t_i \mid i \in I_\alpha \}$ generates $V^*$, where

$$I_\alpha := \{ i \in I \mid \Delta_\alpha \cap H_i \neq \emptyset \}.$$
Second, the relations of type $A_1$ tells us that we could have written $A$ as a quotient of $P(Q_P)$, where $Q_P$ is the subquiver of $Q$ consisting only of those vertices in $P$. In this case, however, the relations of types $A_2$ and $A_3$ would each have bifurcated into two cases, depending on whether or not $\beta \in P$.

It is important to note that if we write $A$ as a quotient of $P(Q_P)$, the relations are generated in degree 2. Thus $A$ is what is called a **quadratic algebra**.

**Remark 3.3.** In a future paper we will show that if $V$ is rational, the category of representations of $A$ is equivalent to a category of sheaves on the hypertoric variety $\mathcal{M}_H$. In the special case where $\mathcal{M}_H$ is a cotangent bundle, these sheaves will be microlocal $\mathcal{D}$-modules whose characteristic varieties are contained in the conormal varieties to a certain stratification of the base. More generally, the sheaves will be supported on the relative core of $\mathcal{M}_H$ (see Remark 4.2), a finite collection of Lagrangian subvarieties of $\mathcal{M}_H$ defined by $\xi$.

**Remark 3.4.** The quiver $Q$ has appeared in the literature before; it is the Cayley graph of the **Deligne groupoid** of $\mathcal{H}$ [Par00]. The precise relationship between the algebra $A$ and the Deligne groupoid will be explained in Remark 7.2.

### 3.2. Taut paths

We next establish a series of results that allow us to understand the elements of $A$ more explicitly. Though we do not need these results for the remainder of Section 3, they will be used in Sections 4 and 5.

**Definition 3.5.** We say that a path $\alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \ldots \leftrightarrow \alpha_k$ in $Q$ is **taut** if the sign vectors $\alpha_1$ and $\alpha_k$ differ in exactly $k - 1$ places; in other words, each place changes sign at most once along the path.

**Proposition 3.6.** Let $\alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \ldots \leftrightarrow \alpha_k$ be a path in $Q$. Then there is a taut path $\alpha_1 = \beta_1 \leftrightarrow \beta_2 \leftrightarrow \ldots \leftrightarrow \beta_d = \alpha_k$ and a polynomial $f(t) \in \text{Sym} V^*$ such that

$$p(\alpha_1, \ldots, \alpha_k) = p(\beta_1, \ldots, \beta_d) \cdot f(t)$$

in the algebra $A$.

**Proof.** We can represent paths in the quiver geometrically by paths in the affine space $V + \eta$ in which $\mathcal{H}$ lives. Let $\phi : [0, 1] \to V + \eta$ be a piecewise linear path with the property that for any $t \in [0, 1]$, the point $\phi(t)$ lies in at most one hyperplane $H_i$. 

and the endpoints $\phi(0)$ and $\phi(1)$ lie in no hyperplanes. Such a path determines an element $p(\phi) = p(\gamma_1, \ldots, \gamma_r)$ in the algebra $A$, where $\Delta_{\gamma_1}, \ldots, \Delta_{\gamma_r}$ are the successive chambers visited by $\phi$.

We can represent our given element $p(\alpha_1, \ldots, \alpha_k)$ in this way by choosing points $x_1, \ldots, x_k$ in each chamber and letting $\phi$ be the concatenation of the line segments $x_jx_{j+1}$. By choosing the $x_j$ generically, we can also assume that (1) the line segment $x_1x_j$ only passes through one hyperplane at a time for $2 \leq j \leq k$, and (2) for all $1 < j < k$, the plane containing $x_1, x_j$, and $x_{j+1}$ contains no point which lies in more than two hyperplanes.

We then contract each point $x_j$ successively to the basepoint $x_1$ along a straight line segment. The sequence of chambers visited by the path can change in two possible ways. First, when the line segment $x_jx_{j+1}$ passes through the intersection of two hyperplanes, a sequence $\cdots \alpha \leftrightarrow \beta \leftrightarrow \gamma \cdots$, $\alpha \neq \gamma$ is replaced by another sequence $\cdots \alpha \leftrightarrow \delta \leftrightarrow \gamma \cdots$. The quiver relation $A_2$ implies that the element in the algebra $A$ represented by the path does not change. Second, when the point $x_j$ passes through a hyperplane $H_i$, either the sequence of chambers visited by the path is left unchanged, or a loop $\cdots \alpha \leftrightarrow \beta \leftrightarrow \alpha \cdots$ is replaced by $\cdots \alpha \cdots$. In this case, the old path is equal to the new path times $t_i$, by relation $A_3$ in the definition of the algebra $A$.

The final result is the product of a polynomial and a straight-line path from $x_1$ to $x_k$, which evidently represents a taut path, since it cannot cross any hyperplane more than once. \hfill $\Box$

**Corollary 3.7.** Let $\alpha = \alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \ldots \leftrightarrow \alpha_d = \beta$ and $\alpha = \beta_1 \leftrightarrow \beta_2 \leftrightarrow \ldots \leftrightarrow \beta_d = \beta$ be two taut paths between fixed elements $\alpha, \beta \in \mathcal{P}$. Then

$$p(\alpha_1, \ldots, \alpha_k) = p(\beta_1, \ldots, \beta_d).$$

**Proof.** The proof of Proposition 3.6 tells us that any taut path is equivalent to one which is represented by a single line segment joining a point of $\Delta_\alpha$ to a point of $\Delta_\beta$. So it is enough to show that two such line segments give rise to the same element of $A$. This, too, follows from the proof of the proposition, as we can add line segments
which stay within $\Delta_\alpha$ and $\Delta_\beta$ to the beginning and end of one of these paths without changing the sequence of visited chambers.

□

**Corollary 3.8.** Consider an element

$$a = p \cdot \prod_{i \in I} t_i^{d_i} \in e_\alpha A e_\beta,$$

where $p$ is a taut path from $\alpha$ to $\beta$. Let $S \subset I$ be the union

$$S = \{i \in I \mid p \text{ crosses } H_i\} \cup \{i \in I \mid d_i > 0\}.$$

Then for any $\gamma \in \mathcal{F}$ which agrees with $\alpha$ at all positions in $I \setminus S$, there is an expression for $a$ which passes through $\gamma$. In particular, if $\gamma \in \mathcal{F} \setminus \mathcal{P}$, then $a = 0$.

**Proof.** First suppose that $d_i = 0$ for all $i$. In this case, we can define a taut path (in $\mathcal{F}$) which goes through $\gamma$ by composing taut paths from $\alpha$ to $\gamma$ and from $\gamma$ to $\beta$. The result then follows from Corollary 3.7.

Using this special case and the centrality of the elements $t_i \in A$, we can reduce to the case in which $d_i > 0$ for every $i$ such that $H_i$ separates $\Delta_\beta$ from $\Delta_\gamma$. In this case, we observe that $e_\beta \cdot \prod_{i \in I} t_i^{d_i}$ can be represented by a taut path from $\beta$ to $\gamma$ followed by the reverse path, which completes the proof. □

3.3. **Quadratic duality.** We conclude this section by establishing the first part of Theorem (A) from the Introduction, namely that the algebras $A(V)$ and $A(V^\vee)$ are quadratic duals of each other. First a quick review of quadratic duality.

Let $R = \mathbb{R}\{e_\alpha \mid \alpha \in \mathcal{I}\}$ be a ring spanned by finitely many pairwise orthogonal idempotents, and let $M$ be an $R$-bimodule. Let $T_R(M)$ be the tensor algebra of $M$ over $R$, and let $W$ be an $R$-bimodule equipped with an inclusion

$$\iota : W \hookrightarrow M \otimes_R M = T_R(M)_2.$$

Let

$$E = T_R(M) / T_R(M) \cdot \iota(W) \cdot T_R(M)$$

be the associated quadratic algebra. We will write $W_{\alpha \beta} = e_\alpha W e_\beta$ to denote the space of relations between nodes $\alpha$ and $\beta$.

The **quadratic dual** $E^!$ of $E$ is defined as the quotient

$$E^! = T_R(M^*) / T_R(M^*) \cdot \iota(W)^\perp \cdot T_R(M^*),$$

In Section 7 we will need to make a distinction between $W$ and its image in $M \otimes_R M$. For consistency we will make the distinction here, as well, even though it is somewhat unnecessary in this context.
where $M^*$ is the vector space dual of $M$, and
\[ \iota(W)^\perp \subset M^* \otimes_R M^* \cong (M \otimes_R M)^* \]
is the space of elements that vanish on $\iota(W)$. Note that dualizing $M$ interchanges the left and right $R$-actions, so that $(M_{\alpha \beta})^* = (M^*)_{\beta \alpha}$. It is clear from this definition that this is a true duality, meaning that there is a natural isomorphism $E^\perp \cong E$.

In the case of the algebra $A(V)$, we have $I = \mathcal{P}$ (as explained in Remark 3.2) and $M = \mathbb{R}\{p(\alpha, \beta) \mid \alpha \leftrightarrow \beta\}$. Relations of type $A2$ from Definition 3.1 come from $W_{\alpha \gamma}$, while relations of type $A3$ come from $W_{\alpha \alpha}$. Since $\mathcal{P}^\perp = \mathcal{P}$, the algebras $A(V)$ and $A(V^\perp)$ have the same $R$ and the same $M$. Thus, in order to make sense of the statement that they are quadratic duals, we must define a perfect pairing
\[ M \otimes M \rightarrow \mathbb{R} \]
to identify $M$ with its dual. The obvious way to do this would be to make the edge set into an orthonormal basis, but this does not quite give us what we need; instead, we need to twist this pairing by a sign.

Choose a subset $X$ of edges in the underlying undirected graph of $Q$ with the property that for any square $\alpha \leftrightarrow \beta \leftrightarrow \gamma \leftrightarrow \delta \leftrightarrow \alpha$ of distinct elements, an odd number of the edges of the square are in $X$. The existence of such an $X$ follows from the fact that our graph is a subgraph of the edge graph of an $n$-cube. Then define a pairing $\langle \cdot, \cdot \rangle$ by putting
\[ \langle p(\alpha, \beta), p(\beta, \alpha) \rangle = \begin{cases} -1 & \text{if } \{\alpha, \beta\} \in X \\ 1 & \text{if } \{\alpha, \beta\} \notin X \end{cases} \]
and
\[ \langle p(\alpha, \beta), p(\delta, \gamma) \rangle = 0 \]
unless $\alpha = \gamma$ and $\beta = \delta$.

**Remark 3.9.** Note that in the pairing that we have defined, paths from a node $\alpha$ to another node $\beta$ pair nontrivially only with paths from $\beta$ to $\alpha$; this is consistent with our general treatment of quadratic duality above. We warn the reader that this will become a little bit confusing in the proof of Theorem 3.10 due to the fact that $A(V)$ admits an anti-automorphism reversing the directions of paths. In other words, the paths from $\alpha$ and $\beta$ and the relations between them look exactly like paths from $\beta$ to $\alpha$ and the relations between them.
Theorem 3.10. With respect to the above pairing, $A(V^\vee) \cong A(V)^\dagger$.

Proof. We will analyze $M \otimes_R M$ piece by piece by considering every pair $\alpha, \gamma \in \mathcal{P}$ which admit paths of length two connecting them. If $\alpha \neq \gamma$, then they must differ in two places, and there are exactly two elements $\delta^1, \delta^2$ in $\{\pm 1\}^I$ so that $\alpha \leftrightarrow \delta^j \leftrightarrow \gamma$, $j = 1, 2$. Since we are assuming that there is a path from $\alpha$ to $\gamma$ in $\mathcal{P}$, at least one of the $\delta_j$ must be in $\mathcal{P}$. If both $\delta^1$ and $\delta^2$ are in $\mathcal{P}$, then $e_\alpha M \otimes_R M e_\gamma$ is the two-dimensional vector space spanned by the paths $p(\alpha, \delta^1, \gamma)$ and $p(\alpha, \delta^2, \gamma)$, while $e_\gamma M \otimes_R M e_\alpha$ is the two-dimensional vector space spanned by the paths $p(\gamma, \delta^1, \alpha)$ and $p(\gamma, \delta^2, \alpha)$. The algebra $A(V)$ has the unique relation $p(\alpha, \delta^1, \gamma) - p(\alpha, \delta^2, \gamma)$ between $\alpha$ and $\gamma$, and the algebra $A(V^\vee)$ has the unique relation $p(\gamma, \delta^1, \alpha) - p(\gamma, \delta^2, \alpha)$ between $\gamma$ and $\alpha$, thus the orthogonality property holds.

If $\delta^1 \in \mathcal{P}$ and $\delta^2 \notin \mathcal{P}$, then by Lemma 2.5 we must either have $\delta^1 \in \mathcal{F} \setminus \mathcal{P}$ and $\delta^2 \notin \mathcal{F}^\vee$ or vice-versa. In the first case, $W_{\alpha \gamma} = \{0\}$ and $W_{\gamma \alpha}^\vee = (e_\gamma M \otimes_R M e_\alpha)_2$ by relation A3, and thus are mutual annihilators. In the second, these are reversed.

We have now dealt with the case where $\alpha \neq \gamma$, and must consider the case where $\alpha = \gamma$. The length two paths from $\alpha$ to itself can be indexed by the set

$$J_\alpha := \{i \in I \mid \alpha_i \in \mathcal{P}\},$$

where $\alpha_i$ denotes the element of $\{\pm 1\}^I$ which differs from $\alpha$ in precisely the position $i$. Note that $J_\alpha$ is a subset of the set $I_\alpha$ that we defined in Remark 3.2, consisting exactly of those $i$ for which $\xi$ is bounded above on the chamber opposite $\Delta_\alpha$ across $H_i$. Let $I_\alpha^\vee$ and $J_\alpha^\vee$ be the corresponding sets for $V^\vee$, and note that $J_\alpha^\vee = J_\alpha$.

The space $t(W_{\alpha \alpha})$ of relations among the loops at $\alpha$ can therefore be considered as a vector subspace of $\mathbb{R}^{J_\alpha}$, which can be computed as follows. The relations among the elements $\{t_i\}_{i \in I}$ of $\text{Sym}(V^\ast)$ are given by $V^\perp \subset (\mathbb{R}^I)^\ast \cong \mathbb{R}^I$. The relations $t(W_{\alpha \alpha})$ are given by first intersecting $V^\perp$ with $\mathbb{R}^{J_\alpha}$ and then projecting (isomorphically) onto $\mathbb{R}^{J_\alpha}$. Alternatively, we can first project $V^\perp$ onto $\mathbb{R}^{I_\alpha \cup J_\alpha}$ and then intersect with $\mathbb{R}^{J_\alpha}$.

On the dual side, the space of relations among loops at $\alpha$ is a vector subspace of $\mathbb{R}^{J_\alpha^\vee} = \mathbb{R}^{J_\alpha}$, and the pairing is the standard one without any signs. Here we obtain our relations by first intersecting $V = V^\perp$ with $\mathbb{R}^{I_\alpha^\vee}$ and then projecting onto $\mathbb{R}^{J_\alpha}$. The fact that we obtain the orthogonal complement of the relations $t(W_{\alpha \alpha})$ follows from the fact that $I_\alpha \setminus J_\alpha = I \setminus I_\alpha^\vee$, which is a consequence of Lemma 2.5. \qed

4. The algebra $B$
4.1. Toric varieties. Let \( \mathcal{V} = (V, \eta, \xi) \) be a rational polarized arrangement. Then \( V \) inherits an integer lattice \( V_\mathbb{Z} := V \cap \mathbb{Z}^I \), and the dual vector space \( V^* \) inherits a dual lattice which is a quotient of \( (\mathbb{Z}^I)^* \subset (\mathbb{R}^I)^* \). Let
\[
T^I = (\mathbb{R}^I)^*/(\mathbb{Z}^I)^* \quad \text{and} \quad T = V^*/V_\mathbb{Z}^*,
\]
so that we have a surjective homomorphism
\[
T^I \twoheadrightarrow T.
\]
Then for every \( \alpha \in \mathcal{F} \), the polyhedron \( \Delta_\alpha \) determines a toric variety with an action of the complexification \( T_\mathbb{C} \). The construction that we give below is originally due to Cox [Cox97].

Let \( T^I_\mathbb{C} \) be the complexification of the coordinate torus \( T^I \), and let \( C^I_\alpha \) be the representation of \( T^I_\mathbb{C} \) in which the \( i \)th coordinate of \( T^I_\mathbb{C} \) acts on the \( i \)th coordinate of \( C^I_\alpha \) with weight \( \alpha(i) \in \{ \pm 1 \} \). Let
\[
G = \ker(T^I_\mathbb{C} \twoheadrightarrow T_\mathbb{C}),
\]
and let
\[
Z_\alpha = C^I_\alpha \setminus \bigcup_{S \subseteq I \text{ such that } H_S \cap \Delta_\alpha = \emptyset} \{ z \in C^I \mid z_i = 0 \text{ for all } i \in S \},
\]
which is acted upon by \( T^I_\mathbb{C} \) and therefore by \( G \). The toric variety \( X_\alpha \) associated to \( \Delta_\alpha \) is defined as the quotient
\[
X_\alpha = Z_\alpha/G,
\]
and inherits an action of \( T_\mathbb{C} = T^I_\mathbb{C}/G \). The action of the subtorus \( T \subset T_\mathbb{C} \) is hamiltonian with respect to a natural symplectic structure on \( X \), and \( \Delta_\alpha \) is the moment polyhedron. If \( \Delta_\alpha \) is compact\(^4\), then \( X_\alpha \) is projective. More generally, \( X_\alpha \) is projective over the affine toric variety whose coordinate ring is the semi-group ring of the semi-group \( \Sigma_\alpha \cap V_\mathbb{Z} \).

Since the polyhedron \( \Delta_\alpha \) is simple by our assumption on \( \eta \), the toric variety \( X_\alpha \) has at worst finite quotient singularities. This implies that the ordinary and \( T \)-equivariant cohomology rings of \( X_\alpha \) may be presented purely in terms of the combinatorics and linear algebra of the polyhedron \( \Delta_\alpha \). Specifically, we have a natural

\(^4\)We use the word "compact" rather than "bounded" to avoid confusion with the fact that \( \alpha \) is called bounded if \( \xi \) is bounded above on \( \Delta_\alpha \).
isomorphism\textsuperscript{5}

\[ H^*_T(X_\alpha) \cong \mathbb{R}[u_i]_{i \in I} / \langle u_S \mid S \subset I \text{ such that } H_S \cap \Delta_\alpha = \emptyset \rangle, \]

where \( u_S = \prod_{i \in S} u_i \). Here the element \( u_i \) represents \( \alpha(i) \) times the class of the toric divisor corresponding to the (possibly empty) facet \( H_i \cap \Delta_\alpha \) of \( \Delta_\alpha \). In particular, it lives in degree two. This isomorphism identifies \( H^*_T(X) \) not just as an \( \mathbb{R} \)-algebra, but as an algebra over the polynomial ring \( \text{Sym} V = H^*_T(pt) \), where \( \text{Sym} V \) acts on the right-hand side via the inclusion of \( V \) into \( \mathbb{R}^I = \mathbb{R}\{u_i \mid i \in I\} \). We then obtain

\[ H^*(X_\alpha) \cong H^*_T(X_\alpha) \otimes_{\text{Sym} V} \mathbb{R} \]

by killing the positive degree elements of \( \text{Sym} V \).

**Remark 4.1.** Note that we can use the above presentations to make sense of the rings \( H^*_T(X_\alpha) \) and \( H^*(X_\alpha) \) even when \( V \) is not rational, despite the fact that there is no toric variety \( X_\alpha \) in this situation.

### 4.2. The relative core.

Again suppose that \( V \) is rational, and let

\[ \tilde{\mathcal{X}} = \bigsqcup_{\alpha \in \mathcal{P}} X_\alpha \]

be the disjoint union of the toric varieties associated to the bounded feasible sign vectors, that is, to the chambers of \( \mathcal{H} \) on which \( \xi \) is bounded above. We will define a quotient \( \mathcal{X} \) of \( \tilde{\mathcal{X}} \) which, informally, is obtained by gluing the components of \( \tilde{\mathcal{X}} \) together along the toric subvarieties corresponding to the faces at which the corresponding polyhedra intersect.

More precisely, let \( \mathcal{C}' \) be the standard coordinate representation of \( T^I_{\mathcal{C}} \), and let

\[ \mathbb{H}' = \mathcal{C}' \times (\mathcal{C}')^* \]

be the product of \( \mathcal{C}' \) with its dual. For every \( \alpha \in \{\pm 1\}^I \), \( \mathcal{C}'_{\alpha} \) can be found in a unique way as a subrepresentation of \( \mathbb{H}' \). Then

\[ \mathcal{X} := \left( \bigcup_{\alpha \in \mathcal{P}} Z_\alpha \right) / G, \]

where the union is taken inside of \( \mathbb{H}' \). Then \( T_{\mathcal{C}} \) acts on \( \mathcal{X} \), and we have a \( T_{\mathcal{C}} \)-equivariant projection

\[ \pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}. \]

\textsuperscript{5}All cohomology groups in this paper will be taken with coefficients in \( \mathbb{R} \).
The restriction to this map to each $X_\alpha$ is obviously an embedding. As a result, any face of a polyhedron $\Delta_\alpha$ corresponds to a $T$-invariant subvariety which is itself a toric variety for a subtorus of $T_C$.

**Remark 4.2.** The singular variety $X$ sits naturally as a closed subvariety of the hypertoric variety $\mathcal{M}_H$, which is defined as an algebraic symplectic quotient of $\mathbb{H}^I$ by $G$ (or, equivalently, as a hyperkähler quotient of $\mathbb{H}^I$ by the compact form of $G$). See [Pro] for more details. The subvariety $X$ is Lagrangian, and is closely related to two other Lagrangian subvarieties of $\mathcal{M}_H$ which have appeared before in the literature. The projective components of $X$ (meaning the components whose corresponding polyhedra are compact) form a complete list of irreducible projective Lagrangian subvarieties of $\mathcal{M}_H$, and their union is called the core of $\mathcal{M}_H$. On the other hand, if we had defined $X$ as a union over all $\alpha \in \mathcal{F}$ rather than restricting to those $\alpha \in \mathcal{P}$, we would have obtained a larger subvariety of $\mathcal{M}_H$ called the extended core, which we will denote $X_\eta$. The subvariety $X_\eta$ can also be described as the zero level of the moment map for the Hamiltonian $T_C$-action on $\mathcal{M}_H$.

Our variety $X$, which sits in between the core and the extended core, will be referred to as the relative core of $\mathcal{M}_H$ with respect to the $C \times \xi$-action defined by $\xi$. It may be characterized as the set of points $x \in \mathcal{M}_H$ such that $\lim_{\lambda \to \infty} \lambda \cdot x$ exists.

**Example 4.3.** Suppose that $\mathcal{H}$ consists of $n$ points in a line. Then $\mathcal{M}_H$ is isomorphic to the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_n$, and its core is equal to the exceptional fiber of this resolution, which is a chain of $n - 1$ projective lines. The extended core is larger; it includes two affine lines attached to the projective lines at either end of the chain. The relative core lies half-way in between, containing exactly one of the two affine lines. This reflects the fact that $\xi$ is bounded above on exactly one of the two unbounded chambers of $\mathcal{H}$.

The core, relative core, and extended core of $\mathcal{M}_H$ are all $T$-equivariant deformation retracts of $\mathcal{M}_H$, which allows us to give a combinatorial description of their ordinary and equivariant cohomology rings [Pro, 3.2.2].

**Theorem 4.4.** We have natural isomorphisms

$$H^*_T(X) \cong H^*_T(\mathcal{M}_H) \cong \mathbb{R}[u_{i\in I}] / \langle u_S \mid S \subset I \text{ such that } H_S = \emptyset \rangle$$

and

$$H^*(X) \cong H^*(\mathcal{M}_H) \cong H^*_T(\mathcal{M}_H) \otimes_{\text{Sym}_V} \mathbb{R}.$$
Remark 4.5. As in Remark 4.1, we can use Theorem 4.4 to make sense of the rings $H^*_T(X)$ and $H^*(X)$ even if $V$ is not rational, in which case the variety $X$ does not itself exist.

4.3. Definition of $B$. For $(\alpha, \beta) \in P \times P$ and $(\alpha, \beta, \gamma) \in P \times P \times P$, let

$$X_{\alpha\beta} = X_\alpha \cap X_\beta \quad \text{and} \quad X_{\alpha\beta\gamma} = X_\alpha \cap X_\beta \cap X_\gamma,$$

where the intersections are taken inside of $X$. Then

$$\tilde{X} \times_\pi \tilde{X} \cong \bigsqcup_{(\alpha, \beta) \in P \times P} X_{\alpha\beta},$$

and

$$\tilde{X} \times_\pi \tilde{X} \times_\pi \tilde{X} \cong \bigsqcup_{(\alpha, \beta, \gamma) \in P \times P \times P} X_{\alpha\beta\gamma}.$$

Let $p_{12}, p_{13}, p_{23} : \tilde{X} \times_\pi \tilde{X} \times_\pi \tilde{X} \to \tilde{X} \times_\pi \tilde{X}$ denote the natural projections. Note that these maps are proper; they are finite disjoint unions of closed immersions of toric subvarieties.

Definition 4.6. Let $d_{\alpha\beta}$ denote the complex codimension of $X_{\alpha\beta}$ in $X$ (equivalently the real codimension of $\Delta_\alpha \cap \Delta_\beta$ in $V$), and consider the graded vector space

$$B = B(V) = \bigoplus_{(\alpha, \beta) \in P \times P} H^*(X_{\alpha\beta})[-d_{\alpha\beta}].$$

If we ignore the grading, then $B$ may be identified with the cohomology ring of $\tilde{X} \times_\pi \tilde{X}$. We may therefore define a multiplication $\star : B \otimes B \to B$ by the formula

$$a \star b = p_{13*}(p_{12}^*(a) \cup p_{13}^*(b)).$$

Proposition 4.7. $B$ is a graded ring with respect to the above convolution product.

Proof. Suppose that $a \in H^k(X_{\alpha\beta})[-d_{\alpha\beta}]$ and $b \in H^\ell(X_{\beta\gamma})[-d_{\beta\gamma}]$, so that

$$\deg(a) = k + d_{\alpha\beta} \quad \text{and} \quad \deg(b) = \ell + d_{\beta\gamma}.$$

Let $d_{\alpha\beta\gamma}$ denote the complex codimension of $X_{\alpha\beta\gamma}$ in $X$ (equivalently the real codimension of $\Delta_\alpha \cap \Delta_\beta \cap \Delta_\gamma$ in $V$). The push-forward $p_{13*}$ raises cohomological degree by twice the complex codimension of $X_{\alpha\beta\gamma}$ in $X_{\alpha\gamma}$, that is, by $2(d_{\alpha\beta\gamma} - d_{\alpha\gamma})$. Thus

$$\deg(a \star b) = k + \ell + 2(d_{\alpha\beta\gamma} - d_{\alpha\gamma}) + d_{\alpha\gamma} = k + \ell + 2d_{\alpha\beta\gamma} - d_{\alpha\gamma}.$$
It is an easy exercise in linear algebra to check that
\[ 2d_{\alpha\beta\gamma} = d_{\alpha\beta} + d_{\beta\gamma} + d_{\alpha\gamma}, \]
therefore
\[ \deg(a \ast b) = k + \ell + d_{\alpha\beta} + d_{\beta\gamma} = \deg(a) + \deg(b). \]
\[ \square \]

**Remark 4.8.** As in Remarks 4.1 and 4.5, we can make sense of the definition of \( B \) regardless of whether or not \( V \) is rational. Indeed, the cohomology of \( X_{\alpha\beta} \) is expressed combinatorially as
\[ \left(3\right) H^*(X_{\alpha\beta}) \cong \mathbb{R}[u_i]_{i \in I}/\langle u_S \mid S \subset I \text{ such that } H_S \cap \Delta_{\alpha} \cap \Delta_{\beta} = \emptyset \rangle \otimes \text{Sym}_V \mathbb{R}. \]
Denote this ring by \( R_{\alpha\beta} \). The cohomology ring \( R_{\alpha\beta\gamma} \) of the triple intersections has a similar expression. Then the pullback map \( R_{\alpha\beta} \to R_{\alpha\beta\gamma} \) is induced by the identity on \( \mathbb{R}[u_i]_{i \in I} \), while the pushforward \( R_{\alpha\beta\gamma} \to R_{\alpha\gamma} \) is induced by multiplication by \( u_S \), where
\[ S = \{ i \in I \mid \alpha(i) = \gamma(i) \neq \beta(i) \}. \]
This allows us to put a ring structure on \( \bigoplus_{(\alpha,\beta)} R_{\alpha\beta}[-d_{\alpha\beta}] \). Although we will use geometric arguments from toric geometry in a few places, they can all be reformulated purely combinatorially, and all of our results about the ring \( B \) hold for arbitrary arrangements unless otherwise noted (as in Theorem 4.11).

**Remark 4.9.** Suppose that \( V \) is rational. As we have observed in Remark 4.2, each component \( X_{\alpha} \) of \( X \) can be thought of as an irreducible Lagrangian subvariety of the hypertoric variety \( \mathcal{M}_H \), which allows us to interpret the cohomology groups of their intersections as Floer cohomology groups. With this identification, \( B \) can be thought of an Ext-algebra in the Fukaya category of \( \mathcal{M}_H \). This description should be related to the description in Remark 3.3 by taking homomorphisms to the canonical coisotropic brane, as described by Kapustin and Witten [KW07].

### 4.4. A and B

We now state and prove the main theorem of this section, which, along with Theorem 3.10, comprises Theorem (A) from the Introduction.

**Theorem 4.10.** There is a natural isomorphism \( A(V^\vee) \cong B(V) \).

**Proof.** We define a map \( \phi: A(V^\vee) \to B(V) \) by
- sending the idempotent \( e_\alpha \) to the unit element \( 1_{\alpha\alpha} \in H^0(X_{\alpha\alpha}) \) for all \( \alpha \in \mathcal{P} \),
• sending \( p(\alpha, \beta) \) to the unit \( 1_{\alpha\beta} \in H^0(X_{\alpha\beta}) \) for all \( \alpha, \beta \in \mathcal{P} \) with \( \alpha \leftrightarrow \beta \), and
• sending \( t_i \in (V^\perp)^* \cong \mathbb{R}^I / V \) to the sum of its images in each \( H^2(X_{\alpha\beta}) \), which by Equations (1) and (2) is a quotient of \( \mathbb{R}^I / V \).

To show that this is a homomorphism, we need to check that these elements satisfy the relations \( A2 \) and \( A3 \) from Definition 3.1 (for \( V^\vee \)). In order to check that relation \( A2 \) holds, suppose that \( \alpha, \beta, \gamma, \delta \in \mathcal{F}^\vee \) satisfy \( \alpha \leftrightarrow \beta \leftrightarrow \gamma \leftrightarrow \delta \leftrightarrow \alpha \), and \( \alpha, \gamma \in \mathcal{P}^\vee = \mathcal{P} \). There are two possibilities: first, if \( \beta \) and \( \delta \) also lie in \( \mathcal{P}^\vee \), then
\[
1_{\alpha\beta} \ast 1_{\beta\gamma} = 1_{\alpha\gamma} = 1_{\alpha\delta} \ast 1_{\delta\gamma},
\]
so relation \( A2 \) is satisfied in \( B \). The other possibility is that only one of \( \beta \) and \( \delta \) lies in \( \mathcal{P}^\vee \), and the other is in \( \mathcal{F}^\vee \setminus \mathcal{P}^\vee = B \setminus \mathcal{P} \). Suppose \( \beta \in \mathcal{P} \) and \( \delta \in B \setminus \mathcal{P} \). Then \( e_\delta = 0 \) in \( A(V^\vee) \), hence the relation \( A2 \) tells us that
\[
p(\alpha, \beta)p(\beta, \gamma) = p(\alpha, \delta)p(\delta, \gamma) = 0.
\]
On the other hand, the fact that \( \delta \in B \setminus \mathcal{P} \) implies that \( X_{\alpha\gamma} = \emptyset \), hence \( 1_{\alpha\beta} \ast 1_{\beta\gamma} = 0 \) in \( B(V) \).

Now suppose that \( \alpha \in \mathcal{P}^\vee \), \( \beta \in \mathcal{F}^\vee \), and \( \alpha \leftrightarrow \beta \). Relation \( A3 \) breaks into two cases, depending on whether or not \( \beta \in \mathcal{P}^\vee \). If \( \beta \in \mathcal{P}^\vee \), then we have the relation \( p(\alpha, \beta, \alpha) = t_i e_\alpha \) in \( A(V^\vee) \). The left-hand side \( p(\alpha, \beta, \alpha) \) maps to \( 1_{\alpha\beta} \ast 1_{\beta\alpha} \in B(V) \), which corresponds to the class \( [X_{\alpha\beta}] \in H^2(X_{\alpha\alpha}) \). This is the same class to which \( t_i \) maps via the isomorphism of Equation 2, thus the relation holds in \( B(V) \). On the other hand, if \( \beta \in \mathcal{F}^\vee \setminus \mathcal{P}^\vee = B \setminus \mathcal{P} \), then we have the relation \( t_i e_\alpha = 0 \) in \( A(V^\vee) \). In this case \( H_1 \cap \Delta_\alpha = \emptyset \), so \( \phi(t_i e_\alpha) = 0 \) in \( B(V) \), as well.

The fact that \( B(V) \) is generated by the classes \( 1_{\alpha\beta} \) for \( (\alpha, \beta) \in \mathcal{P} \times \mathcal{P} \) follows from Equations 1 and 2 and the fact that for all \( \alpha \), \( H^2(X_{\alpha\alpha}) \) is spanned by the set \( \{ u_i \mid i \in I_\alpha \} \). Thus we have defined a surjective graded ring homomorphism
\[
\phi : A(V^\vee) \rightarrow B(V).
\]

To show that \( \phi \) is injective, we show that each block \( e_\alpha A(V^\vee)e_\beta \) has dimension no larger than the total dimension of the ring \( H^*(X_{\alpha\beta}) \) (or the ring \( R_{\alpha\beta} \) from Remark 4.8 in the non-rational case). By Proposition 3.6 and Corollary 3.7, we have a surjective map
\[
\chi : \mathbb{R}[u_i]_{i \in I} \otimes_{\text{Sym} V} \mathbb{R} = \text{Sym} \mathbb{R}^I / V = \text{Sym}(V^\perp)^* \rightarrow e_\alpha A(V^\vee)e_\beta
\]
given by \( \chi(f) = p \cdot f \), where \( p \) is any taut path from \( \alpha \) to \( \beta \). It will be enough to show that the monomials \( u_S \) of Equation (3) are in the kernel of \( \chi \).
The condition \( H_S \cap \Delta_\alpha \cap \Delta_\beta = \emptyset \) can be rephrased as \( H_{S'} \cap \Delta_\alpha = \emptyset \), where \( S' = S \cup \{ i \in I \mid \alpha_i \neq \beta_i \} \). This is equivalent to saying that the projection \( \bar{\alpha} \) of \( \alpha \) to \( \{ \pm 1 \}^{I-S'} \) gives an infeasible sign vector for \( V_{S'} \). By Theorem 2.3 and Lemma 2.4, this is equivalent to saying that \( \bar{\alpha} \) is unbounded for the Gale dual arrangement \( (V^\vee)_{S'} \) (note that \( \bar{\alpha} \) cannot be infeasible for \( (V^\vee)_{S'} \), since \( \Delta^\vee_\alpha \supset \Delta^\vee_\alpha \) and \( \alpha \in \mathcal{P} = \mathcal{P}^\vee \)). The vanishing of \( \chi(u_S) \) in \( A(V^\vee) \) then follows from Corollary 3.8. \( \square \)

4.5. The center. We now state and prove part (2) of Theorem (B) from the Introduction, which gives a cohomological interpretation of the center of \( B \).

**Theorem 4.11.** If \( \mathcal{V} \) is rational, we have natural graded isomorphisms

\[
Z(B) \cong H^* (X) \cong H^* (\mathcal{M}_\mathcal{H}).
\]

**Proof.** Let \( z \in Z(B) \). Since \( z \) commutes with the idempotents \( 1_{\alpha\alpha} \) for all \( \alpha \in \mathcal{P} \), we must have \( z = \sum_{\alpha \in \mathcal{P}} z_\alpha \), where \( z_\alpha \in H^* (X_{\alpha\alpha}) = H^* (X_\alpha) \). Also, since \( z \) commutes with every element \( 1_{\alpha\beta} \), we see that \( z_\alpha \) and \( z_\beta \) will restrict to the same element of \( H^* (X_{\alpha\beta}) \) for all \( \alpha, \beta \in \mathcal{P} \). On the other hand, since the elements \( 1_{\alpha\beta} \) generate \( B \) as a ring, these two conditions completely characterize \( Z(B) \). That is, we have

\[
(4) \quad Z(B) \cong \left\{ z \in \bigoplus_{\alpha \in \mathcal{P}} H^* (X_\alpha) \ \bigg| \text{ for all } \alpha, \beta \in \mathcal{P}, (z_\alpha - z_\beta)|_{X_{\alpha\beta}} = 0 \right\}.
\]

Thus we want to show that the pullback

\[
\pi^* : H^* (X) \to H^* (\tilde{X}) \cong \bigoplus_{\alpha \in \mathcal{P}} H^* (X_\alpha)
\]

is injective, with image equal to the right hand side of Equation (4).

Fix a total ordering \( \alpha^1, \ldots, \alpha^r \) of \( \mathcal{P} \) which is a refinement of the partial order described in Section 2.6. For \( 1 \leq j, k \leq r \) let

\[
X_k = X_{\alpha^k},
\]

\[
X_{jk} = X_{\alpha^j/\alpha^k},
\]

\[
C_k = X_k \setminus \bigcup_{j<k} X_{jk},
\]

\footnote{We use superscripts to avoid confusion with the notation \( \alpha_i \) from the proof of Theorem 3.10, which will resurface in Section 5.}
and
\[ X_{\leq k} := \bigcup_{i=1}^{k} C_i = \bigcup_{i=1}^{k} X_i \subset X. \]

Then \( X_{\leq k} \) is the set of all points \( x \in X \) for which the limit \( \lim_{\lambda \to \infty} \lambda \cdot x \) under the \( \mathbb{C}^\times \) action defined by \( \xi \) is a \( T \)-fixed point corresponding to a 0-flat \( H_b \) where \( b \in \text{Bas} \) and \( \mu(b) \leq k \). This implies that \( C_i \) is a rational homology cell for all \( i \). We will prove by induction on \( k \) that the chain complex
\[
(5) \quad H^*(X_{\leq k}) \to \bigoplus_{1 \leq i \leq k} H^*(X_i) \to \bigoplus_{1 \leq i < j \leq k} H^*(X_{ij})
\]
has no cohomology in the first two terms, where the first map is given by restriction, and the second map is given by restriction of \( H^*(X_i) \to H^*(X_{ij}) \) if \( i < j \), or by the negative of the restriction map if \( i > j \). The case \( k = 1 \) is trivial, since \( X_{\leq 1} = X_1 \). The case \( k = r \) gives our theorem.

Let \( C_i^0, X_{\leq k}^0, X_k^0, \) and \( X_{jk}^0 \) denote the intersections of \( C_i, X_{\leq k}, X_k, \) and \( X_{jk} \) with the core of \( \mathcal{M}_{i\ell} \) (see Remark 4.2), that is with the union of all of the compact \( X_i \). Then the inclusions
\[
C_i^0 \hookrightarrow C_i, \quad X_{\leq k}^0 \hookrightarrow X_{\leq k}, \quad X_k^0 \hookrightarrow X_k, \quad \text{and} \quad X_{jk}^0 \hookrightarrow X_{jk}
\]
are all homotopy equivalences, and the \( C_i^0 \) decompose \( X_{\leq k}^0 \) into cells. It follows that the long exact sequence of the pair \((X_{\leq k}, X_{\leq k-1})\) splits into short exact sequences
\[
0 \to H^*(X_{\leq k}, X_{\leq k-1}) \to H^*(X_{\leq k}) \to H^*(X_{\leq k-1}) \to 0.
\]
This then implies that the chain complex (5) surjects onto the same complex with \( k \) replaced by \( k - 1 \), with kernel the complex
\[
H^*(X_{\leq k}^0, X_{\leq k-1}^0) \to H^*(X_k^0) \to \bigoplus_{1 \leq j < k} H^*(X_{jk}^0).
\]
So to complete our induction it will be enough to show that this complex has no cohomology in the first two places. This follows because the compact toric variety \( X_k^0 \) is paved by the \( T \)-invariant cells \( C_i \cap X_k^0 \) for \( i \leq k \) (note that some of these cells may be empty), where the largest cell is \( C_k^0 = X_{\leq k}^0 \setminus X_{\leq k-1}^0 \), and each is contained in some \( X_j^0 \cap X_k^0 \) for \( j < k \). \( \square \)

Remark 4.12. By Remarks 4.5 and 4.8, we can formulate Theorem 4.11 even if \( V \) is not rational. We have not done this because our proof uses the geometry of the
variety $X$, but we believe that it would be possible to give a purely combinatorial proof.

5. The representation category

We begin with a general discussion of highest weight categories, quasi-hereditary algebras, self-dual projectives, and Koszul algebras. With the background in place, we analyze our algebras $A(V)$ and $B(V)$ in light of these definitions.

5.1. Highest weight categories. Let $C$ be an abelian, Artinian category enriched over $\mathbb{R}$ with simple objects $\{L_{\alpha} \mid \alpha \in I\}$, projective covers $\{P_{\alpha} \mid \alpha \in I\}$, and injective hulls $\{I_{\alpha} \mid \alpha \in I\}$. Let $\leq$ be a partial order on the index set $I$.

Definition 5.1. We call $C$ highest weight with respect to this partial order if there is a collection of objects $\{V_{\alpha} \mid \alpha \in I\}$ and epimorphisms $P_{\alpha} \xrightarrow{\Pi_{\alpha}} V_{\alpha} \xrightarrow{\pi_{\alpha}} L_{\alpha}$ such that for each $\alpha \in I$, the following conditions hold:

1. The object $\ker \pi_{\alpha}$ has a filtration such that each successive quotient is isomorphic to $L_{\beta}$ for some $\beta < \alpha$.
2. The object $\ker \Pi_{\alpha}$ has a filtration such that each successive quotient is isomorphic to $V_{\gamma}$ for some $\gamma > \alpha$.

The objects $V_{\alpha}$ are called standard objects. Classic examples of highest weight categories in representation theory include the various integral blocks of parabolic category $\mathcal{O}$ [FM99, 5.1].

Suppose that $C$ is highest weight with respect to a given partial order on $I$. To simplify the discussion, we will assume that the endomorphism algebras of every simple object in $C$ is just the scalar ring $\mathbb{R}$; this will hold for the categories we consider. For all $\alpha \in I$, let $C_{\neq \alpha}$ be the subcategory of objects whose composition series contain no simple objects $L_{\beta}$ with $\beta > \alpha$.

Proposition 5.2. The standard object $V_{\alpha}$ of Definition 5.1 is isomorphic to the projective cover of $L_{\alpha}$ in the subcategory $C_{\neq \alpha}$ of $C$.

Proof. We must show that

$$\text{Ext}^i (V_{\alpha}, L_{\beta}) \cong \begin{cases} \mathbb{R} & \text{if } \alpha = \beta \text{ and } i = 0, \\ 0 & \text{if } \alpha = \beta \text{ and } i > 0, \\ 0 & \text{if } \alpha \not\leq \beta. \end{cases}$$
By definition, we have

$$\text{Ext}^i(P_{\alpha}, L_{\beta}) = \begin{cases} \mathbb{R} & \text{if } \alpha = \beta \text{ and } i = 0, \\ 0 & \text{otherwise}. \end{cases}$$

If $\alpha \nless \beta$, then $\ker \Pi_{\alpha}$ has a filtration by $V_{\gamma}$ with $\gamma > \alpha \nless \beta$, and therefore with $\gamma \nless \beta$. Thus $\text{Ext}^i(\ker \Pi_{\alpha}, L_{\beta}) = 0$ for all $i$. Then by the long exact sequence for $\text{Ext}$, we have $\text{Ext}^i(V_{\alpha}, L_{\beta}) \cong \text{Ext}^i(P_{\alpha}, L_{\beta})$ whenever $\alpha \nless \beta$. □

Dually, we define the costandard object $\Lambda_{\alpha}$ to be the the injective hull of $L_{\alpha}$ in the same subcategory $C_{\not\geq \alpha}$.

**Definition 5.3.** An object of $C$ is called **tilting** if it admits a filtration by standard modules and one by costandard modules. An equivalent condition is that $T$ is tilt- ing if and only if $\text{Ext}^i(T, \Lambda_{\alpha}) = 0 = \text{Ext}^i(V_{\alpha}, T)$ for all $i > 0$ and $\alpha \in I$. (The first condition is equivalent to the existence of a standard filtration, and the second to the existence of a costandard filtration.) For each $\alpha \in I$, there is a unique indecomposable tilting module $T_{\alpha}$ with $V_{\alpha}$ as its largest standard submodule and $\Lambda_{\alpha}$ as its largest costandard quotient [Rin91].

We now have six important sets of objects of $C$, all indexed by the set $I$:

- the simples $\{L_{\alpha}\}$
- the indecomposable projectives $\{P_{\alpha}\}$
- The indecomposable injectives $\{I_{\alpha}\}$
- the standard objects $\{V_{\alpha}\}$
- the costandard objects $\{\Lambda_{\alpha}\}$
- the tilting objects $\{T_{\alpha}\}$.

Each of these six sets forms a basis for the Grothendieck group $K(C)$, and thus each is a minimal set of generators of the bounded derived category $D^b(C)$. In particular, any exact functor from $D^b(C)$ to any other triangulated category is determined by the images of these objects and the morphisms between them and their shifts.

Let $\{M_{\alpha} \mid \alpha \in I\}$ and $\{N_{\alpha} \mid \alpha \in I\}$ be two sets of objects that form bases for $K(C)$. We say that the second set is **left dual** to the first set (and that the first set is **right dual** to the second) if

$$\text{Ext}^i(N_{\alpha}, M_{\beta}) \cong \begin{cases} \mathbb{R} & \text{if } \alpha = \beta \text{ and } i = 0, \\ 0 & \text{otherwise}. \end{cases}$$
It is an easy exercise to check that if a dual set to \( \{ M_\alpha \} \) exists, then it is unique up to isomorphism. Note that dual sets descend to dual bases for \( K(\mathcal{C}) \) under the Euler form
\[
\langle [M], [N] \rangle := \sum_{i=0}^{\infty} (-1)^i \dim \operatorname{Ext}^i(M, N).
\]

**Proposition 5.4.** The sets \( \{ P_\alpha \} \) and \( \{ I_\alpha \} \) are left and right (respectively) dual to \( \{ L_\alpha \} \), and the set \( \{ \Lambda_\alpha \} \) is right dual to \( \{ V_\alpha \} \).

**Proof.** The first statement follows immediately from the definition of projective covers and injective hulls. For the second statement, let \( \alpha, \beta \in \mathcal{I} \), and suppose first that \( \alpha \not< \beta \). If \( \alpha \) is maximal in \( \mathcal{I} \), then \( V_\alpha = P_\alpha \) and we get that \( \operatorname{Ext}^i(V_\alpha, \Lambda_\beta) \) is \( \mathbb{R} \) if \( i = 0 \) and \( \alpha = \beta \) and is 0 otherwise, as required. If \( \alpha \) is not maximal, we can assume by induction that the statement holds for all larger \( \alpha \). Apply the cohomological functor \( \operatorname{Ext}^*(-, \Lambda_\beta) \) to the short exact sequence
\[
0 \to \operatorname{Ker} \to P_\alpha \to V_\alpha \to 0;
\]
since \( \operatorname{Ker} \) has a filtration by standards \( V_{\alpha'} \) with \( \alpha' > \alpha \), the result follows by induction on the resulting long exact sequence.

On the other hand, if \( \alpha < \beta \) then we apply the dual argument using the exact sequence \( 0 \to \Lambda_\beta \to I_\beta \to \operatorname{Coker} \to 0 \), where \( \operatorname{Coker} \) has a filtration by costandards \( \Lambda_{\beta'} \) with \( \beta' > b \). \( \square \)

### 5.2. Quasi-hereditary algebras.
We now study those algebras whose module categories are highest weight.

**Definition 5.5.** An algebra is **quasi-hereditary** if its category \( \mathcal{C}(E) \) of finitely generated right modules is highest weight with respect to some partial ordering of its simple modules.

Let \( E \) be a finite-dimensional, quasi-hereditary \( \mathbb{R} \)-algebra with respect to a fixed partial order on the indexing set \( \mathcal{I} \) of its simple modules. Let

\[
\begin{align*}
P_s &= \bigoplus_{\alpha \in \mathcal{I}} P_\alpha, & I_s &= \bigoplus_{\alpha \in \mathcal{I}} I_\alpha, & T_s &= \bigoplus_{\alpha \in \mathcal{I}} T_\alpha
\end{align*}
\]

be the sums of the indecomposable projectives, injectives, and tilting modules, respectively. Let \( D^b(E) = D^b(\mathcal{C}(E)) \) be the bounded derived category of finitely generated right \( E \)-modules.
**Definition 5.6.** We say that $E$ is basic if the simple module $L_\alpha$ is one dimensional for all $\alpha \in I$. This is equivalent to requiring that the canonical homomorphism

$$E \to \text{End}(P_\ast) \cong \text{End}(I_\ast)^{op}$$

is an isomorphism.

**Definition 5.7.** The endomorphism algebra $R(E) := \text{End}(T_\ast)$ is called the Ringel dual of $E$. It has simple modules indexed by $I$, and it is quasi-hereditary with respect to the partial order on $I$ opposite to the given one. If $E$ is basic, then the canonical homomorphism $E \to R(R(E))$ is an isomorphism [Rin91, Theorems 6,7]. The functor $\mathcal{R} = \text{RHom}^\ast(-,T_\ast)$ from $D^b(E)$ to $D^b(R(E))$ is called the Ringel duality functor.

**Proposition 5.8.** Suppose that $E$ is basic. Up to automorphisms of $E$ and $R(E)$, $\mathcal{R}$ is the unique contravariant equivalence that satisfies any of the following conditions:

1. $\mathcal{R}$ sends tilting modules to projective modules,
2. $\mathcal{R}$ sends projective modules to tilting modules,
3. $\mathcal{R}$ sends standard modules to standard modules.

**Proof.** We first prove statement (1). The Ringel duality functor is an equivalence because $D^b(E)$ is generated by $T_\ast$. Since $\mathcal{R}(T_\ast)$ is equal to $R(E)$ as a right module over itself, it is clear that $\mathcal{R}$ takes tilting modules to projective modules. Suppose that $\mathcal{R}'$ is another such equivalence. Since the indecomposable tilting modules $\{T_\alpha\}$ generate $K(E)$ and $\mathcal{R}'$ induces an isomorphism on Grothendieck groups, $\mathcal{R}'$ must take the tilting modules to the complete set of indecomposable projective $R(E)$-modules. Thus $\mathcal{R}'(T_\ast)$ is isomorphic to the direct sum of all indecomposable projective $R(E)$-modules, which is isomorphic to $R(E)$ as a right module over itself. Since any exact functor is determined by its values on sends a generator and on the endomorphisms of that generator, $\mathcal{R}'$ can only differ from $\mathcal{R}$ in its isomorphism between $\text{End}(T_\ast)$ and $R(E)$. This is precisely the uniqueness statement we have claimed for (1).

Statement (2) follows by applying statement (1) to the adjoint functor.

As for Statement (3), it was shown in [Rin91, Theorem 6] that $\mathcal{R}$ takes standard modules to standard modules. Suppose that $\mathcal{R}'$ is another such equivalence. For any $\alpha \in I$, the projective module $P_\alpha$ has a standard filtration, therefore so does $\mathcal{R}'(P_\alpha)$. Furthermore, we have $\text{Ext}^i(\mathcal{R}'(V_\beta),\mathcal{R}'(P_\alpha)) = \text{Ext}^i(P_\alpha,V_\beta) = 0$ for all $i > 0$.
and \( \beta \in \mathcal{I} \), thus \( \mathcal{R}'(P_\alpha) \) has a costandard filtration as well, and is therefore tilting. Then part (2) tells us that \( \mathcal{R}' \) is the Ringel duality functor. \( \square \)

5.3. **Self-dual projectives and the double centralizer property.** Suppose that our algebra \( E \) is basic and quasi-hereditary, and that it is endowed with an anti-involution \( \psi \), inducing an equivalence of categories \( \mathcal{C}(E) \simeq \mathcal{C}(E^{\text{op}}) \). We have another such equivalence given by taking the dual of the underlying vector space, and these two equivalences compose to a contravariant auto-involution \( d \) of \( \mathcal{C}(E) \).

We will assume for simplicity that \( \psi \) fixes all idempotents of \( E \). The case where it gives a non-trivial involution on idempotents is also interesting, but requires a bit more care in the statements below, and will not be relevant to this paper. The following proposition follows easily from the fact that any contravariant equivalence takes projectives to injectives.

**Proposition 5.9.** For all \( \alpha \in \mathcal{I} \),

\[
 dL_\alpha \cong L_\alpha, \quad dP_\alpha \cong I_\alpha, \quad dV_\alpha \cong \Lambda_\alpha, \quad \text{and} \quad dT_\alpha \cong T_\alpha.
\]

**Remark 5.10.** Proposition 5.9 has two important consequences. First, since \( d \) preserves simples, it acts trivially on the Grothendieck group of \( \mathcal{C}(E) \). In particular, we have \([V_\alpha] = [\Lambda_\alpha]\), so by Proposition 5.4, the classes \([V_\alpha]\) are an orthonormal basis of the Grothendieck group.

Second, the isomorphism \( T_\alpha \cong dT_\alpha \) induces an anti-automorphism of \( R(E) \) that fixes idempotents, and thus a duality functor on the Ringel dual category \( \mathcal{C}(R(E)) \).

**Proposition 5.11.** For all \( \alpha \in \mathcal{I} \), the following are equivalent:

1. The projective module \( P_\alpha \) is injective.
2. The projective module \( P_\alpha \) is tilting.
3. The projective module \( P_\alpha \) is self-dual, that is, \( d(P_\alpha) = P_\alpha \).

Furthermore, these conditions imply that the simple module \( L_\alpha \) is contained in the socle of some standard module \( V_\beta \).

**Proof.** (1) \( \Rightarrow \) (2): Projectives all carry standard filtrations, and injectives carry co-standard ones. Thus, any module which is both must be tilting.
(2) ⇒ (3): All tilting modules are self-dual, since indecomposable tiltings are determined by the highest simple module appearing in their composition series, which is unchanged by duality.

(3) ⇒ (1): The dual of a projective is injective, so any self-dual projective is also injective.

Suppose that the projective module \( P_\alpha \) is self-dual. Since \( P_\alpha \) is indecomposable, it is the injective hull of its socle. Since \( P_\alpha \) is self-dual, its socle is isomorphic to its cosocle \( L_\alpha \). Since \( P_\alpha \) has a standard filtration, it has at least one standard module \( V_\beta \) as a submodule. Since socle is a left exact functor and \( V_\beta \) is finite-dimensional, the socle of \( V_\beta \) is a non-trivial submodule of the socle \( L_\alpha \) of \( P_\alpha \). Since \( L_\alpha \) is simple, the socle of \( V_\beta \) must be isomorphic to \( L_\alpha \). □

Let \( \mathcal{I}_d = \{ \alpha \in \mathcal{I} \mid d(P_\alpha) \cong P_\alpha \} \). Let

\[
P = \bigoplus_{\alpha \in \mathcal{I}_d} P_\alpha \subset P_\ast
\]

be the direct sum of all of the self-dual projective right \( E \)-modules, and consider its endomorphism algebra

\[
S := \text{End}(P) \subset \text{End}(P_\ast) \cong E.
\]

**Definition 5.12.** An algebra is said to be symmetric if it is isomorphic to its vector space dual as a bimodule over itself. It is immediate from the definition that \( S \) is symmetric.

For Theorem 5.13 we make one additional assumption, which is that the last implication of Proposition 5.11 can be reversed. More precisely, we assume that if the simple module \( L_\alpha \) includes into the socle of some standard module, then \( \alpha \in \mathcal{I}_d \).

**Theorem 5.13.** With the above hypothesis, the functor from right \( E \)-modules to right \( S \)-modules taking a module \( M \) to \( \text{Hom}_E(P, M) \) is fully faithful on projectives.

**Proof.** Fix an index \( \alpha \in \mathcal{I}_d \), and let \( L \) be the socle of the standard module \( V_\alpha \). Then \( L \) is a direct sum of simple modules, and the assumption above implies that the injective hull of \( L \) is also projective; we denote this hull by \( P \). Since \( P \) is an injective module, the inclusion \( L \hookrightarrow P \) extends to \( V_\alpha \), and since the map is injective on the socle \( L \), it must be injective on all of \( V_\alpha \). Thus \( P \) is the injective hull of \( V_\alpha \). An application of [MS, 2.6] gives the desired result. □
Remark 5.14. The property attributed to the $S$-$E$-bimodule $P$ in Theorem 5.13 is known as the **double centralizer property**. See [MS] for a more detailed treatment of this phenomenon.

5.4. **Koszul algebras.** To discuss the notion of Koszulity, we must begin to work with graded algebras and graded modules. Let $E = \bigoplus_{k \geq 0} E_k$ be a graded $\mathbb{R}$-algebra, and let $R = E_0$.

**Definition 5.15.** A complex

$$\ldots \rightarrow P_k \rightarrow P_{k-1} \rightarrow \ldots \rightarrow P_1 \rightarrow P_0$$

of graded projective right $E$-modules is called **linear** if $P_k$ is generated in degree $k$.

**Definition 5.16.** The algebra $E$ is called **Koszul** if each every simple right $E$-module admits a linear projective resolution.

The notion of Koszulity gives us a second interpretation of quadratic duality.

**Theorem 5.17.** [BGS96, 2.3.3, 2.9.1, 2.10.1] If $E$ is Koszul, then $E$ is the quotient of the tensor algebra $T_R E_1$ by a quadratic ideal, and the quadratic dual $E^!$ is isomorphic to $\text{Ext}_E(R, R)^\text{op}$. Furthermore, $E^!$ is also Koszul.

**Remark 5.18.** In this case $E^!$ is also known as the **Koszul dual** of $E$.

Let $D(E)$ be the bounded derived category of graded right $E$-modules.

**Theorem 5.19.** [BGS96, 1.2.6] If $E$ is Koszul, we have an equivalence of derived categories

$$D(E) \cong D(E^!).$$

**Remark 5.20.** There are two other triangulated categories naturally associated to $E$, namely the bounded derived category $D_{\text{ug}}(E)$ of ungraded $E$-modules and the bounded derived category $D_{\text{dg}}(E)$ of dg-$E$-modules, where $E$ is thought of a dg-algebra with trivial differential. Given a complex of graded $E$-modules, one can either forget the grading on the modules to obtain a complex of ungraded modules, or add the module grading to the grading on the complex to get a dg-module over $E$. This gives us forgetful functors

$$\varepsilon_u : D(E) \rightarrow D_{\text{ug}}(E) \text{ and } \varepsilon_d : D(E) \rightarrow D_{\text{dg}}(E).$$
both of which are **degradings** (in the sense of [BGS96]) with respect to different shift functors on \( D(E) \). Using the theory of derived Morita equivalence (for example, see [Sch04, Proposition 3.20]), one can construct an equivalence of categories from \( D_{ug}(E) \) to \( D_{dg}(E^!) \) that commutes with the functors \( \varepsilon_u \) and \( \varepsilon_d \).

We conclude this section with a discussion of Koszulity for quasi-hereditary algebras. Suppose that our graded algebra \( E \) is quasi-hereditary. For all \( \alpha \in \mathcal{I} \), there exists an idempotent \( e_\alpha \in \mathbb{R} \) such that \( P_\alpha = e_\alpha E \), thus each projective module \( P_\alpha \) inherits a natural grading. Let us assume that the grading of \( E \) is compatible with the quasi-hereditary structure. In other words, we suppose that for all \( \alpha \in \mathcal{I} \), the standard module \( V_\alpha \) admits a grading that is compatible with the map \( \Pi_\alpha : P_\alpha \to V_\alpha \) of Definition 5.1. It is not hard to check that each of \( L_\alpha, V_\alpha, A_\alpha, P_\alpha, \) and \( I_\alpha \) inherits a grading as a quotient of \( E \), and that \( T_\alpha \) admits a unique grading that is compatible with the inclusion of \( V_\alpha \). Thus \( R(E) = \text{End}(T_\alpha) \) inherits a grading as well, and this grading is compatible with the quasi-hereditary structure [Zhu04].

**Theorem 5.21.** [ÁDL03, Theorem 1] Let \( E \) be a finite-dimensional graded algebra with a graded anti-automorphism that preserves idempotents. If \( E \) is graded quasi-hereditary and each standard module admits a linear projective resolution, then \( E \) is Koszul.

5.5. **The algebra** \( A(\mathcal{V}) \). Let \( \mathcal{V} = (V, \eta, \xi) \) be a polarized arrangement, and let \( A = A(\mathcal{V}) \) be the associated quiver algebra. \( A \) has a canonical anti-automorphism taking \( p(\alpha, \beta) \) to \( p(\beta, \alpha) \) for all \( \alpha \leftrightarrow \beta \) in \( \mathcal{P} \). Under the identification \( A(\mathcal{V}) \cong B(\mathcal{V}^\vee) \), this corresponds to swapping the left and right factors of \( \tilde{\mathcal{X}}^\vee \times_{\mathcal{X}^\vee} \tilde{\mathcal{X}}^\vee \). This anti-automorphism fixes the idempotents, and thus gives rise to a contravariant involution \( \mathcal{d} \) of \( \mathcal{C}(A) \) as in Section 5.3.

For all \( \alpha \in \mathcal{P} \), let

\[
L_\alpha = A/\langle e_\beta \mid \beta \neq \alpha \rangle
\]

be the one-dimensional simple right \( A \)-module supported at the node \( \alpha \), and let \( P_\alpha = e_\alpha A \) denote the projective cover of \( L_\alpha \). Since \( L_\alpha \) is one dimensional for each \( \alpha \), \( A \) is basic. Let \( a = \mu^{-1}(\alpha) \) be the basis corresponding to the sign vector \( \alpha \), and let

\[
K_{>\alpha} = \sum_{i \in a} p(\alpha, \alpha_i) \cdot A \subset P_\alpha
\]

be the right-submodule of \( P_\alpha \) generated by paths that begin at the node \( \alpha \) and move to a node that is higher in the partial order given in Section 2.6. (Recall that \( \alpha_i \) is the
unique sign vector such that \( \alpha \leftrightarrow \alpha_i \). Let

\[ V_\alpha = P_\alpha / K_{> \alpha}, \]

and let

\[ P_\alpha \to V_\alpha \to L_\alpha \]

be the natural projections.

**Lemma 5.22.** The module \( V_\alpha \) has a vector space basis consisting of a taut path from \( \alpha \) to each element of \( \mathcal{F} \cap B_a \).

**Proof.** Corollary 3.7 implies that this collection of paths is linearly independent. Any taut path which terminates outside of \( \mathcal{F} \cap B_a \) must cross a hyperplane \( H_i \) for some \( i \in a \), and by Corollary 3.8 it can be replaced by a path which crosses this hyperplane first, thus it lies in \( K_{> \alpha} \). It will therefore suffice to show that any path which is not taut will also have trivial image in \( V_\alpha \). By Proposition 3.6, this is equivalent to showing that the positive degree part of \( \text{Sym} V \) acts trivially on \( V_\alpha \), which follows from the fact that \( V \) is spanned by \( \{ t_i | i \in a \} \). \( \square \)

When \( \mathcal{V} \) is rational, the modules \( P_\alpha, V_\alpha, L_\alpha \) acquire natural geometric interpretations via the isomorphism \( A \cong B(\mathcal{V}^\vee) \) of Theorem 4.10. Let \( \mathcal{X}^\vee \) be the relative core of the hypertoric variety \( \mathcal{M}_\mathcal{H}^\vee \) associated to \( \mathcal{V}^\vee \). For each \( \alpha \in \mathcal{P} = \mathcal{P}^\vee \), we have a component \( X_\alpha^\vee \subseteq \mathcal{X}^\vee \). Let \( y_\alpha \in X_\alpha^\vee \) be an arbitrary element of the dense toric stratum (in other words, an element whose image under the moment map lies in the interior of the polyhedron \( \Delta_\alpha^\vee \)), and let \( x_\alpha = \lim_{\lambda \to \infty} \lambda \cdot y_\alpha \in X_\alpha \) be the toric fixed point whose image under the moment map is the \( \xi^\vee \)-maximum point of \( \Delta_\alpha^\vee \).

**Proposition 5.23.** If \( \mathcal{V} \) is rational, then we have module isomorphisms

\[ P_\alpha \cong H^*(X_\alpha \times_{\mathcal{X}^\vee} \tilde{\mathcal{X}}^\vee), \quad V_\alpha \cong H^*(\{ x_\alpha \} \times_{\mathcal{X}^\vee} \tilde{\mathcal{X}}^\vee), \quad \text{and} \quad L_\alpha \cong H^*(\{ y_\alpha \} \times_{\mathcal{X}^\vee} \tilde{\mathcal{X}}^\vee), \]

where \( A(\mathcal{V}) \cong B(\mathcal{V}^\vee) = H^*(\tilde{\mathcal{X}}^\vee \times_{\mathcal{X}^\vee} \tilde{\mathcal{X}}^\vee) \) acts on the right by convolution.

**Proof.** The first isomorphism is immediate from the definitions.

Restriction to the point \( x_\alpha \) defines a \( B(\mathcal{V}^\vee) \)-module surjection from

\[ P_\alpha \cong H^*(X_\alpha^\vee \times_{\mathcal{X}^\vee} \tilde{\mathcal{X}}^\vee) \to H^*(\{ x_\alpha \} \times_{\mathcal{X}^\vee} \tilde{\mathcal{X}}^\vee). \]

Note that \( x_\alpha \in X_\beta^\vee \) if and only if \( \alpha(i) = \beta(i) \) for all \( i \notin a \), or in other words, if and only if \( \beta \in \mathcal{F}_a \cap \mathcal{B} = \mathcal{B}^\vee_a \cap \mathcal{F}^\vee \). The second isomorphism now follows from Lemma
5.22, using the fact that a taut path from $\alpha$ to $\beta$ in the algebra $A(V)$ gives rise to the unit class $1_{\alpha\beta} \in H^0(X_{\alpha\beta}) \subset B(V)$. The third isomorphism follows from the fact that $H^*(\{y_\alpha\} \times_{\check{X}} \check{X}) \cong H^*\{y_\alpha\}$ is one-dimensional, $e_\alpha$ acts by the identity, and $e_\beta$ acts by zero for all $\beta \neq \alpha$. □

**Theorem 5.24.** The algebra $A$ is quasi-hereditary with respect to the partial order on $P$ given in Section 2.6, with the modules \{\(V_\alpha\)\} as the standard modules. This structure is compatible with the grading on $A$.

**Proof.** We must show that the modules \{\(V_\alpha\mid \alpha \in I\}\) satisfy the conditions of Definition 5.1. Condition (1) follows from Lemmas 5.22 and 2.8.

For condition (2), we define a filtration of $K_{>\alpha} = \ker \Pi_\alpha$ as follows. For $\gamma \in P$, let $P_{\gamma}^{\alpha} \subset P_\alpha$ be the submodule generated by paths that pass through the node $\gamma$, and for any $\gamma \in P$ let

$$P_{\alpha}^{\geq \gamma} = \sum_{\delta \geq \gamma} P_{\alpha}^\delta, \quad P_{\alpha}^{> \gamma} = \sum_{\delta > \gamma} P_{\alpha}^\delta.$$ 

Note that $P_{\alpha}^{\geq \alpha} = P_\alpha$ and $P_{\alpha}^{> \alpha} = K_{>\alpha}$. Then $P_{\alpha}^\gamma \subset K_{>\alpha}$ for all $\gamma > \alpha$, and these submodules form a filtration with successive quotients

$$M_{\alpha}^\gamma := P_{\alpha}^{\geq \gamma}/P_{\alpha}^{> \gamma}.$$ 

Let $g = \mu^{-1}(\gamma) \in \text{Bas}$. If $\alpha$ is not in the negative cone $B_{g'}$, then there exists $i \in g$ such that $\alpha(i) \neq \gamma(i)$. It follows from Corollary 3.7 that $P_{\alpha}^\gamma = P_{\alpha}^{> \gamma}$, hence $M_{\alpha}^\gamma = 0$.

If $\alpha \in B_{g'}$, then composition with a taut path $p$ from $\alpha$ to $\gamma$ defines a map $P_{\gamma} \rightarrow P_{\alpha}^{\geq \gamma}$ which induces a map $V_{\gamma} \rightarrow M_{\alpha}^\gamma$. We will show that this induced map is an isomorphism. First, note that $M_{\alpha}^\gamma$ is spanned by the classes of paths which pass through $\gamma$. Using Proposition 3.6, such a path is equivalent to one which begins with a taut path from $\alpha$ to $\gamma$, and by Corollary 3.7 implies that this taut path can be taken to be $p$, so our map is surjective.

To see that it is injective, it will be enough to show that $\dim_{\mathbb{R}} P_\alpha$ is bounded below by

$$\sum_{\alpha \in B_g} \dim_{\mathbb{R}} V_{\mu(g)} = |\{(\delta, g) \in P \times \text{Bas} \mid \alpha, \delta \in B_g\}|,$$

or, adding over all $\alpha$, that

$$\dim_{\mathbb{R}} A \geq |\{(\alpha, \delta, g) \in P \times P \times \text{Bas} \mid \alpha, \delta \in B_g\}|.$$
Assume for the moment that $\mathcal{V}$ is rational. By Theorem 4.10 we have

$$\dim_\mathbb{R} A = \dim_\mathbb{R} H^*(\tilde{X}^\vee \times_X \tilde{X}^\vee) = \sum_{\alpha, \delta \in \mathcal{P}^\vee} \dim_\mathbb{R} H^*(X^\vee_{\alpha} \cap X^\vee_{\delta}) = |\{ (\alpha, \delta, b) \in \mathcal{P}^\vee \times \mathcal{P}^\vee \times \text{Bas}^\vee | H^\vee_{\alpha} \subset \Delta^\vee_{\alpha} \cap \Delta^\vee_{\delta}\}|.$$ 

Here the last equality uses the fact that the variety $X^\vee_{\alpha} \cap X^\vee_{\delta}$ is a semi-projective toric variety, so the total dimension of its cohomology is equal to the number of torus fixed points, which is the same as the number of vertices $H^\vee_{\delta}$, $b \in \text{Bas}$ contained in the moment polytope $\Delta^\vee_{\alpha} \cap \Delta^\vee_{\delta}$.

Now our result follows from the fact that for any basis $b \in \text{Bas}$,

$$H^\vee_{\delta} \subset \Delta^\vee_{\alpha} \iff \alpha \in \mathcal{F}^\vee_{\delta} \iff \alpha \in \mathcal{B}^\vee_{b}.$$ 

If $\mathcal{V}$ is not rational, the same argument works, replacing the cohomology ring of $X^\vee_{\alpha\delta}$ by the appropriate combinatorial ring $R^\vee_{\alpha\delta}$ as in Remark 4.8. 

**Theorem 5.25.** Let $\mathcal{V}$ be a polarized arrangement. The algebras $A(\mathcal{V})$ and $B(\mathcal{V})$ are Koszul, and Koszul dual to each other.

**Proof.** By Theorems 3.10, 4.10, and 5.17, it is enough to prove that $A = A(\mathcal{V})$ is Koszul. By Theorem 5.21, it is enough to show that each standard module $V^\alpha_a$ has a linear projective resolution.

Let $a = \mu^{-1}(\alpha)$ be the basis associated with the sign vector $\alpha$. For any subset $S \subset a$, let $\alpha_S$ be the sign vector that differs from $\alpha$ in exactly the indices in $S$. Thus, for example, $\alpha_{\emptyset} = \alpha$, and $\alpha_{\{i\}} = \alpha_i$ for all $i \in a$. (Note that the sign vectors that arise this way are exactly those in the set $\mathcal{F}_a$.) If $S = S' \cup \{i\} \subset a$, then we have a map $\varphi_{S,i} : P_{\alpha_S} \to P_{\alpha_{S'}}$ given by left multiplication by the element $p(\alpha_{S'}, \alpha_S)$. We adopt the convention that $P_{\alpha_S} = 0$ if $\alpha_S \notin \mathcal{P}$ and $\varphi_{S,i} = 0$ if $i \notin S$.

Let

$$\Pi_{\alpha} = \bigoplus_{S \subset a} P_{\alpha_S}$$

be the sum of all of the projective modules associated to the sign vectors $\alpha_S$. This module is multi-graded by the abelian group $\mathbb{Z}^a = \mathbb{Z}\{\epsilon_i | i \in a\}$, with the summand
$P_{\alpha S}$ sitting in multi-degree $\epsilon_S := \sum_{i \in S} \epsilon_i$. For each $i \in a$, we define a differential
\[
\partial_i = \sum_{i \in S \subset a} \varphi_{S,i}
\]
of degree $-\epsilon_i$. These differentials commute because of the relation (A2), and thus define a multi-complex structure on $\Pi_\alpha$. The total complex $\Pi^\alpha$ of this multi-complex is linear and projective; we claim that it is a resolution of the standard module $V_\alpha$. It is clear from the definition that $H^0(\Pi^\alpha) \cong V_\alpha$, thus we need only show that our complex is exact in positive degrees.

We will use two important facts about filtered chain complexes and multi-complexes. Both are manifest from the theory of spectral sequences, but could also easily be proven by hand by any interested reader.

(*) If any one of the differentials in a multi-complex is exact, then the total complex is exact as well.

(**) If a chain complex $C^\alpha$ has a filtration such that the associated graded $\tilde{C}^\alpha$ is exact at an index $i$, then $C^\alpha$ is also exact at $i$.

As in the proof of Theorem 5.24, we may filter each projective module $P_{\alpha S}$ by submodules of the form $P^\beta_{\alpha S}$ for $\beta \geq \alpha_S$, which consists of paths from $\alpha_S$ that pass through the node $\beta$. We extend this filtration to all $\beta$ by defining $P^\beta_{\alpha S}$ to be the sum of $P^\beta_{\alpha S}$ over all $\beta' \in \mathcal{P}$ for which $\beta' \geq \alpha_S$ and $\beta' \geq \beta$. It is easy to see that this filtration is compatible with the differentials, hence we obtain an associated graded multi-complex
\[
\tilde{\Pi}^\alpha := \bigoplus_{\beta} (\Pi^\alpha)^\beta / (\Pi^\alpha)^{>\beta} = \bigoplus_{\beta, S} M^\beta_{\alpha S}.
\]

Take $\beta \in \mathcal{P}$, and let $b = \mu^{-1}(\beta)$. We showed in the proof of Theorem 5.24 that $M^\beta_{\alpha S}$ is nonzero if and only if $\alpha_S \in B_b$, in which case it is isomorphic to $V_\beta$. If $\beta = \alpha$, then we have a nonzero summand only when $S = \emptyset$, so that summand sits in total degree zero. For $\beta \neq \alpha$, choose an element $i \in a \cap b^c$. This ensures that if $S = S' \cup \{i\}$, then $\alpha_{S'} \in B_b$ if and only if $\alpha_S \in B_b$. For such a pair $S$ and $S'$, we have
\[
M^\beta_{\alpha S} \cong V_\beta \cong M^\beta_{\alpha_{S'}},
\]
and $\tilde{\partial}^\beta_{i}$ is the isomorphism given by left-composition with $p(\alpha_{S'}, \alpha_S)$. Thus $\tilde{\partial}^\beta_{i}$ is exact in nonzero degree. By (*) we can conclude that the total complex $\tilde{\Pi}^\alpha$ is exact in nonzero degree, and thus by (**) so is $\Pi^\alpha$.

We next determine which projective $A$-modules are self-dual.
Theorem 5.26. For all $\alpha \in \mathcal{P}$, the following are equivalent:

1. The projective $P_\alpha$ is injective.
2. The projective $P_\alpha$ is tilting.
3. The projective $P_\alpha$ is self-dual, that is, $d(P_\alpha) = P_\alpha$.
4. The simple $L_\alpha$ is contained in the socle of some standard module $V_\beta$.
5. The cone $\Sigma_\alpha \subset V$ has non-trivial interior.
6. The chamber $\Delta^\vee_\alpha \subset V^\perp - \xi$ is compact.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (4) were proved in Proposition 5.11.

(4) $\Rightarrow$ (5): Let $b = \mu^{-1}(\beta) \in \text{Bas}$. By Lemma 5.22, $V_\beta$ is spanned as a vector space by taut paths $p_\gamma$ from $\beta$ to nodes $\gamma \in F \cap B_b$. The socle of $V_\beta$ is spanned by those $p_\gamma$ for which $\gamma$ is as far away from $\beta$ as possible; i.e. any hyperplane $H_i, i \notin b$ which meets $\Delta_\gamma$ must separate $\Delta_\gamma$ and $\Delta_\beta$. This implies that any ray starting at the point $H_b$ and passing through an interior point $q$ of $\Delta_\gamma$ will not leave this chamber once it enters. It follows that the direction vector of this ray lies in $\Sigma_\gamma$. Since this holds for any $q$, $\Sigma_\alpha$ has nonempty interior.

(5) $\Rightarrow$ (6): The fact that $\Sigma_\alpha$ has non-empty interior implies that $\alpha$ is feasible for the polarized arrangement $(V, \eta', \xi)$ for any $\eta' \in \mathbb{R}^I/V$. Dually, $\alpha$ is bounded for $(V^\perp, -\xi, -\eta')$ for any covector $\eta'$, and thus $\Delta^\vee_\alpha$ is compact.

(6) $\Rightarrow$ (3): If $V$ is rational, then Proposition 5.23 tells us that $P_\alpha$ is isomorphic to the vector space $H^*(X^\vee_\alpha \times \tilde{X}^\vee)$. Since $\Delta^\vee_\alpha$ is compact, so is $X^\vee_\alpha$, and the Poincaré pairing provides an isomorphism between $P_\alpha$ and its vector space dual $d(P_\alpha)$. In the non-rational case, the same argument goes through using the fact that the rings $R_{\alpha\beta}$ of Remark 4.8 are Gorenstein [Sta83, II.5.2].

Remark 5.27. The equivalence (1) $\Leftrightarrow$ (6) is part (3) of Theorem (B) from the Introduction, keeping in mind that $A = A(V) \cong B(V^\vee)$. If $V$ is rational, then the set of $\alpha \in \mathcal{P}$ for which $\Delta^\vee_\alpha$ is compact indexes the components of the core of the hypertoric variety $\mathcal{M}_{H^\vee}$ (see Remark 4.2), which may also be described as the set of all irreducible projective Lagrangian subvarieties of $\mathcal{M}_{H^\vee}$.

Remark 5.28. Theorems 5.24, 5.25, and 5.26 are all analogous to theorems that arise in the study of parabolic category $\mathcal{O}$ and other important categories in representation theory [MS].
6. Derived Equivalences

The purpose of this section is to show that the dependence of \( A(V) \) on the parameters \( \xi \) and \( \eta \) is relatively minor. Indeed, suppose that

\[
V_1 = (V, \eta_1, \xi_1) \quad \text{and} \quad V_2 = (V, \eta_2, \xi_2)
\]

are polarized arrangements with the same underlying linear subspace \( V \subset \mathbb{R}^I \). Thus \( V_1 \) and \( V_2 \) are related by translations of the hyperplanes and a change of affine-linear functional on the affine space in which the hyperplanes live. The associated quiver algebras \( A_1 = A(V_1) \) and \( A_2 = A(V_2) \) are not necessarily isomorphic, nor even Morita equivalent. They are, however, derived Morita equivalent, as stated in Theorem (C) of the Introduction and proved in Theorem 6.13 of this section. That is, the triangulated category \( D(V) \) defined in Section 5.4 is an invariant of the subspace \( V \subset \mathbb{R}^I \). The corresponding results for \( D_{ug}(V) \) and \( D_{dg}(V) \) can be obtained by similar reasoning.

6.1. Definition of the functors. We begin by restricting our attention to the special case in which \( \xi_1 = \xi_2 = \xi \) for some \( \xi \in V^* \). On the dual side, this means that \( \eta_1^\vee = \eta_2^\vee = \eta^\vee = -\xi \), and therefore that \( \mathcal{H}_1^\vee = \mathcal{H}_2^\vee = \mathcal{H}^\vee \). Our first task will be to define an \( A_1 - A_2 \) bimodule \( N \). We will give two equivalent descriptions of \( N \), one on the A-side and one on the B-side, exploiting the isomorphism \( A(V_j) \cong B(V_j^\vee) \) of Theorem 4.10. We begin with the B-side description, as it is the easier of the two to motivate. Let \( A_j = A(V_j) \) and \( B_j^\vee = B(V_j^\vee) \). For simplicity, we will assume that each \( V_j \) is rational.

Recall that the relative core

\[
X_j^\vee := \bigcup_{\alpha \in \mathcal{P}_j} X_{\alpha}^\vee
\]

is defined as a union of toric varieties indexed by the chambers of \( \mathcal{H}^\vee \) which are bounded above with respect to the affine linear function \( \xi_j^\vee \). This sits inside of the extended core (Remark 4.2)

\[
X_{\text{ext}}^\vee := \bigcup_{\alpha \in \mathcal{F}^\vee} X_{\alpha}^\vee,
\]

in which we use all of the chambers of \( \mathcal{H}^\vee \); the extended core depends only on \( \mathcal{H}^\vee \) and is therefore the same for \( V_1^\vee \) and \( V_2^\vee \). We then define \( N \) to be the graded vector space

\[
H^*(\tilde{X}_1^\vee \times X_{\text{ext}}^\vee \tilde{X}_2^\vee) = \bigoplus_{(\alpha, \beta) \in \mathcal{P}_1 \times \mathcal{P}_2} H^*(X_{\alpha, \beta}^\vee)[-d_{\alpha, \beta}],
\]
which has a natural left $B'_{1'}$-action and right $B'_{2'}$-action via convolution.

To formulate this definition on the A-side, rather than considering all feasible sign vectors we must consider all bounded sign vectors. Let $A_{\text{ext}}(V)$ be the algebra defined by the same relations as $A(V)$, but without the feasibility restrictions. That is, we begin with a quiver $Q_{\text{ext}}$ whose nodes are indexed by the set $\{\pm 1\}^I$ of all sign vectors, and let $A_{\text{ext}}(V)$ be the quotient of $P(Q_{\text{ext}}) \otimes_{\mathbb{R}} \text{Sym} V^*$ by the following relations:

A_{\text{ext}1}: If $\alpha \in \{\pm 1\}^I \setminus \mathcal{B}$, then $e_\alpha = 0$.

A_{\text{ext}2}: If four distinct elements $\alpha, \beta, \gamma, \delta \in \{\pm 1\}^I$ satisfy $\alpha \leftrightarrow \beta \leftrightarrow \gamma \leftrightarrow \delta \leftrightarrow \alpha$, then $p(\alpha, \beta, \gamma) = p(\alpha, \delta, \gamma)$.

A_{\text{ext}3}: If $\alpha, \beta \in \{\pm 1\}^I$ and $\alpha \leftrightarrow \beta$, then $p(\alpha, \beta, \alpha) = t_i e_\alpha$.

We note that since $\mathcal{B}_1 = \mathcal{B}_2$, we have $A_{\text{ext}}(V_1) = A_{\text{ext}}(V_2) = A_{\text{ext}}$. Let

$$e_{\eta_j} = \sum_{\alpha \in P_j} e_\alpha \in A_{\text{ext}}.$$

Then $A_j$ is isomorphic to the subalgebra $e_{\eta_j} A_{\text{ext}} e_{\eta_j}$ of $A_{\text{ext}}$. Consider the graded vector space

$$N := e_{\eta_1} A_{\text{ext}} e_{\eta_2},$$

which is a left $A_1$-module and a right $A_2$-module in the obvious way.

The following proposition is an easy extension of Theorem 4.10; its proof will be left to the reader.

**Proposition 6.1.** The quiver algebra $A_{\text{ext}}$ is isomorphic to the extended convolution algebra $H^* (\tilde{X}_{\text{ext}}^\vee \times \tilde{X}_{\text{ext}}^\vee \times \tilde{X}_{\text{ext}}^\vee)$. Furthermore, the isomorphisms $A_j \cong B'_j$ induce a graded bimodule isomorphism between $N$ and $H^* (\tilde{X}_1^\vee \times \tilde{X}_{\text{ext}}^\vee \times \tilde{X}_2^\vee)$.

We define a functor $\Phi : D(V_1) \to D(V_2)$ by the formula

$$\Phi(M) := M \otimes_{A_1} N.$$

For $\alpha \in P_j$, let $P^j_\alpha$ and $V^j_\alpha$ denote the corresponding projective module and standard module for $A_j$.

**Proposition 6.2.** If $\alpha \in P_1 \cap P_2$, then $\Phi(P^1_\alpha) = P^2_\alpha$. 
Proof. The natural map
\[ \Gamma : P^2_\alpha = e_\alpha A_2 \to e_\alpha A_1 \otimes A_1 e_{\eta_1} A_{\text{ext}} e_{\eta_2} = P^1_\alpha \otimes A_1 N = \Phi(P^1_\alpha) \]
taking \( e_\alpha \) to \( e_\alpha \otimes e_{\eta_1} e_{\eta_2} \), is an isomorphism, where surjectivity follows from an ana-
logue of Proposition 3.7 for the module \( N \). □

Remark 6.3. Note that by Proposition 6.2 and the equivalence \((3) \iff (6)\) of Theorem
5.26, \( \Phi \) takes self-dual projectives to self-dual projectives.

Consider a basis \( b \in \text{Bas}(V_1) = \text{Bas}(V_2) \), and recall that we have bijections
\[ P_1 \xleftarrow{\mu_1} \text{Bas}(V_1) = \text{Bas}(V_2) \xrightarrow{\mu_2} P_2. \]
Let \( \nu : P_1 \to P_2 \) denote the composition. Recall also that the sets \( B_b \subset B \), defined in
Section 2.6, do not depend on \( \eta \).

Lemma 6.4. For any \( \alpha \in P_1 \), the \( A_2 \)-module \( \Phi(P^1_\alpha) \) has a filtration with standard sub-
quotients. If \( \alpha \in B_b \) then the standard module \( V^2_{\mu_2(b)} \) appears with multiplicity 1 in the
associated graded, and otherwise it does not appear.

Proof. As in the proof of Proposition 6.2, we have \( \Phi(P^1_\alpha) = e_\alpha A_1 \otimes A_1 e_{\eta_1} A_{\text{ext}} e_{\eta_2} \), thus
we may represent an element of \( \Phi(P^1_\alpha) \) by a path in \( B \) that begins at \( \alpha \) and ends at
an element of \( P_2 = B \cap F_2 \). For \( \beta \in P_2 \), let \( \Phi(P^1_\alpha)_{\beta} \) be the submodule generated by
those paths \( p \) such that \( \beta \) is the maximal element of \( P_2 \) through which \( p \) passes, and let
\[ \Phi(P^1_\alpha)_{> \beta} = \bigcup_{\gamma > \beta} \Phi(P^1_\alpha)_{\gamma} \quad \text{and} \quad \Phi(P^1_\alpha)_{\geq \beta} = \bigcup_{\gamma \geq \beta} \Phi(P^1_\alpha)_{\gamma}. \]
We then obtain a filtration
\[ \Phi(P^1_\alpha) = \bigcup_{\beta} \Phi(P^1_\alpha)_{\geq \beta}. \]
Suppose that \( \beta = \mu_2(b) \); we claim that the quotient \( \Phi(P^1_\alpha)_{\geq \beta} / \Phi(P^1_\alpha)_{> \beta} \) is isomorphic
to \( V^2_{\beta} \) if \( \alpha \in B_b \), and is trivial otherwise.

If \( \alpha \in B_b \), then we have a map
\[ V^2_{\beta} \to \Phi(P^1_\alpha)_{\geq \beta} / \Phi(P^1_\alpha)_{> \beta} \]
given by precomposition with any taut path from \( \alpha \) to \( \beta \), and an adaptation of the
proof of Theorem 5.24 shows that it is an isomorphism. If \( \alpha \notin B_b \), then there exists
\( i \in b \) such that \( \alpha_i \neq \beta(i) \), and any path from \( \alpha \) to \( \beta \) will be equivalent to one that
passes through \( \beta_i > \beta \). Thus in this case the quotient is trivial. □
Proposition 6.5. For all $\alpha \in \mathcal{P}_1$, we have $[\Phi(V^1_\alpha)] = [V^2_{\nu(\alpha)}]$ in the Grothendieck group of (ungraded) right $A_2$-modules. Thus $\Phi$ induces an isomorphism of Grothendieck groups.

Proof. For all $b \in \text{Bas}$, we have

$$\sum_{\alpha \in B_b} [\Phi(V^1_\alpha)] = [\Phi(P^1_{\mu_1(b)})] = \sum_{\alpha \in B_b} [V^2_{\nu(\alpha)}],$$

where the first equality follows from the proof of Theorem 5.24 and the second follows from Lemma 6.4. The first statement of the theorem then follows from induction on $b$. The second statement follows from the fact that the Grothendieck group of modules over a quasi-hereditary algebra is freely generated by the classes of the standard modules. \hfill \Box

Remark 6.6. We emphasize that $\Phi(V^1_\beta)$ and $V^2_{\nu(\beta)}$ are not isomorphic as modules; Proposition 6.5 says only that they have the same class in the Grothendieck group. In fact, the next proposition provides an explicit description of $\Phi(V^1_\beta)$ as a module.

Proposition 6.7. $\Phi(V^1_\alpha)$ is the quotient of $\Phi(P^1_\alpha)$ by the submodule generated by all paths which cross the hyperplane $H_i$ for some $i \in \mu^{-1}_1(\alpha)$. In particular, $\text{Tor}^A_1(V^1_\alpha, N) = 0$ for all $k > 0$.

Proof. It is clear that if we take a projective resolution of $V^1_\alpha$ and tensor it with $N$, the degree zero cohomology of the resulting complex will be this quotient. Thus we need only show that the complex is exact in positive degree, that is, that it is a resolution of $V^1_\alpha \otimes N$. The proof of this fact is identical to the proof of Lemma 6.4. \hfill \Box

Corollary 6.8. If a right $A_1$-module $M$ admits a filtration by standard modules, then $\text{Tor}^A_1(M, N) = 0$ for all $k > 0$, and thus $\Phi(M) = M \otimes_{A_1} N$.

Remark 6.9. Though we will not need this fact, it is interesting to note that $\Phi$ takes the exceptional collection $\{V^1_\alpha\}$ to the mutation of $\{V^2_{\nu(\alpha)}\}$ with respect to a linear refinement of our partial order. (See [Bez06] for definitions of exceptional collections and mutations.) We leave the proof as an exercise to the reader.
6.2. Ringel duality and Serre functors. In this section we pass to an even further
special case; we still require that $\xi_1 = \xi_2$, and we will now assume in addition that
$\eta_1 = -\eta_2$. Rather than referring to $V_1$ and $V_2$, we will write
$$V = (V, \eta, \xi), \quad \tilde{V} = (V, -\eta, \xi),$$
and we will refer to $\tilde{V}$ as the reverse of $V$. Let
$$\Phi^- : D(V) \to D(\tilde{V}) \quad \text{and} \quad \Phi^+ : D(\tilde{V}) \to D(V)$$
be the functors constructed above.

**Theorem 6.10.** The algebras $A$ and $\tilde{A}$ are Ringel dual, and the Ringel duality functor is
$$d \circ \Phi^- = \Phi^- \circ d.$$ In particular, $\Phi^-$ sends projectives to tiltings, tiltings to injectives, and standards to costandards.

**Proof.** Using the B-side description of the functor $\Phi^-$, we find that, for any $\alpha \in P$,
$$\Phi^-(P_\alpha) = \bigoplus_{\beta \in P} H^*(X_{\alpha\beta})[-d_{\alpha\beta}].$$
The polyhedron $\Delta_{\alpha\beta}$ is always compact, thus $H^*(X_{\alpha\beta})$ is Gorenstein and $\Phi^-(P_\alpha)$ is self-dual. We showed in Lemma 6.4 that $\Phi^-(P_\alpha)$ admits a filtration with standard subquotients, with $\tilde{V}_{\nu(\alpha)}$ as its largest standard submodule, from which we can con-
clude that $\Phi^-(P_\alpha)$ is isomorphic to $\tilde{T}_{\nu(\alpha)}$. Thus $d \circ \Phi^-$ is a contravariant functor that
sends projective modules to tilting modules; by Proposition 5.8, it will now be suffi-
cient to show that $\Phi^-$ is an equivalence.

For all $\alpha, \beta \in P$, the functor $\Phi^-$ induces a map $\text{Hom}(P_\alpha, P_\beta) \to \text{Hom}(\tilde{T}_{\nu(\alpha)}, \tilde{T}_{\nu(\beta)})$. We will show that this map is an isomorphism by first showing it to be injective and then comparing dimensions. By the double centralizer property (Remark 5.14), there exists a self-dual projective $P_{\alpha'}$ and a map $P_{\alpha'} \to P_\alpha$ such that composition with this map defines an injection from $\text{Hom}(P_\alpha, P_\beta)$ to $\text{Hom}(P_{\alpha'}, P_\beta)$. On the other hand, the injective hull of $P_\beta$ is the same as the injective hull of its socle. Since $P_\beta$
has a standard filtration, each simple summand of this socle lies in the socle of some
standard module. Then the implication $(4) \Rightarrow (3)$ of Theorem 5.26 tells us that the
injective hull of $P_\beta$ is isomorphic to some self-dual projective $P_{\beta'}$.

Now consider the commutative diagram below, in which the vertical arrow on
the left is injective. To prove injectivity of the top horizontal arrow, it is enough to
show injectivity of the bottom horizontal arrow, which follows from Proposition 6.2.
Next we need to prove that the two Hom-spaces on the top of the diagram have the same dimension. Since standards and costandards are dual sequences (Proposition 5.4), we have

$$\text{Ext}^i(T_\alpha, T_\beta) = 0$$ for all \(i > 0\).

By Lemma 6.4, we have the decomposition

$$[\bar{T}_\nu(\alpha)] = [\Phi^-(P_\alpha)] = \sum_{\alpha \in B_h} [V_{\mu(b)}]$$

in the Grothendieck group of \(\bar{A}\)-modules. From this statement and Proposition 6.5, we may deduce that

$$[P_\alpha] = \sum_{\alpha \in B_h} [V_{\mu_1(b)}]$$

in the Grothendieck group of \(A\)-modules. The standard classes form an orthonormal basis with respect to the Euler form (Remark 5.10), thus

$$\dim \text{Hom}(\bar{T}_\nu(\alpha), \bar{T}_\nu(\beta)) = \langle [\bar{T}_\nu(\alpha)], [\bar{T}_\nu(\beta)] \rangle$$

$$= \#\{b \in \text{Bas} \mid \alpha, \beta \in B_h\}$$

$$= \langle [P_\alpha], [P_\beta] \rangle$$

$$= \dim \text{Hom}(P_\alpha, \bar{P}_\beta).$$

Thus \(\Phi^-\) is an equivalence of categories.

By Propositions 5.8 and 5.9, it is now sufficient to show that \(R(A)\) is isomorphic to \(\bar{A}\). To this end, consider the equivalence \(\Phi^+\) from \(\bar{A}\) modules to \(A\) modules, which takes \(\bar{P}_\nu(\alpha)\) to \(T_\alpha\) for all \(\alpha \in \mathcal{P}\). From this we find that

$$R(A) = \text{End}_A(T_\alpha) \cong \text{End}_{\bar{A}}(\bar{P}_\nu(\alpha)) = \bar{A}.$$

The last statement follows from Proposition 5.9. \(\square\)

The functors \(\Phi^\pm\) are not mutually inverse (we will see this explicitly in Theorem 6.11), but their composition is interesting and natural from a categorical perspective. For any graded algebra \(E\), an auto-equivalence \(S : D(E) \to D(E)\) is called
a Serre functor\(^7\) if we have isomorphisms of vector spaces
\[
\text{Hom}(M, SM') \cong \text{Hom}(M', M)^*
\]
that are natural in both \(M\) and \(M'\).

By the 5-lemma, to check that a functor is Serre, we need only show it is exact, and satisfies the Serre property on homomorphisms between objects in a generating set of the category. If \(E\) is finite-dimensional and has finite global dimension, then \(D(E)\) is generated by \(E\) as a module under right multiplication, so an exact functor \(S : D(E) \to D(E)\) is Serre if and only if
\[
SE \cong \text{Hom}(E, S(E)) \cong \text{Hom}(E, E)^* \cong E^*.
\]
Since every right \(E\)-module has a free resolution, any Serre functor is equivalent to the derived tensor product with \(S(E)\), hence \(- \otimes_E E^*\) is the unique Serre functor on \(D(E)\). It follows that \(E\) is symmetric in the sense of Definition 5.12 if and only if its Serre functor is trivial.

**Theorem 6.11.** The endofunctor \(\Phi^+ \circ \Phi^-\) is a Serre functor of \(D(V)\).

**Proof.** We use the characterization of [MS, 3.4]: \(S\) is a Serre functor if and only if
1. \(S\) sends projectives to injectives, and
2. \(S\) agrees with the Serre functor of \(S\) (Equation (6) of Section 5.3) on the subcategory of projective-injective modules.

Condition (1) follows immediately from Theorem 6.10, since \(\Phi^-\) sends projectives to tilttings, which \(\Phi^+\) (by symmetry) sends to injectives. Since \(S\) is symmetric (Definition 5.12), its Serre functor is trivial, and condition (2) says simply that \(S\) must act trivially on projective-injective modules. This follows from Theorem 5.26 and Proposition 6.2. \(\square\)

### 6.3. Composing functors

We now return to the situation of Section 6.1, in which we have two polarized arrangements
\[
\nu_1 = (V, \eta_1, \xi) \quad \text{and} \quad \nu_2 = (V, \eta_2, \xi).
\]

\(^7\)This terminology is of course motivated by Serre duality on a projective variety.
To this mix we add a third polarized arrangement \( V_3 = (V, \eta_3, \xi) \), and study the composition of the two functors

\[
D(V_1) \xrightarrow{\Phi_{12}} D(V_2) \xrightarrow{\Phi_{23}} D(V_3).
\]

Since \( \Phi_{12} \) and \( \Phi_{23} \) are the derived functors of tensor product with a bimodule, their composition is the derived functor of the derived tensor product of these bimodules. It is an easy exercise to check that the right \( A_2 \)-module \( N_{12} \) admits a standard filtration, hence Corollary 6.8 tells us that the higher levels of the derived tensor product of \( N_{12} \) and \( N_{23} \) vanish. So for any right \( A_1 \)-module \( M \), we have

\[
\Phi_{23} \circ \Phi_{12}(M) = (M \otimes_{A_1} N_{12} \otimes_{A_2} N_{23} = M \otimes_{A_1} (N_{12} \otimes_{A_2} N_{23}).
\]

There a natural map \( N_{12} \otimes_{A_2} N_{23} \to N_{13} \) given by composition of paths, which induces a natural transformation \( \Phi_{23} \circ \Phi_{12} \to \Phi_{13} \). Furthermore, Proposition 6.5 tells us that \( \Phi_{23} \circ \Phi_{12} \) and \( \Phi_{13} \) induce the same map on Grothendieck groups. In particular, this implies that the bimodules \( N_{12} \otimes_{A_2} N_{23} \) and \( N_{13} \) have the same dimension.

Suppose that \( \eta_3 = -\eta_1 \), so that \( V_3 = \bar{V}_1 \),

\[
\Phi_{23} = \Phi_{21}: D(V_2) \to D(\bar{V}_1), \quad \text{and} \quad \Phi_{13} = \Phi^- : D(V_1) \to D(\bar{V}_1).
\]

Lemma 6.12 says that, in this case, the natural transformation from \( \Phi_{23} \circ \Phi_{12} \) to \( \Phi_{13} \) is an isomorphism.

**Lemma 6.12.** \( \Phi^- \cong \Phi_{21} \circ \Phi_{12} \)

**Proof.** We would like to show that the natural map \( N_{12} \otimes_{A_2} N_{21} \to N_{11} \) is an isomorphism. We have already observed that the source and target have the same dimension, so it is enough to show that the map is surjective. In other words, we must show that for any \( \alpha \in P_1 \) and \( \beta \in \bar{P}_1 \), every element of \( e_\alpha A_{\text{ext}} e_\beta \) may be represented by a path in \( Q_{\text{ext}} \) that passes through a node in \( P_2 \).

The existence of a non-zero element of \( e_\alpha A_{\text{ext}} e_\beta \) is equivalent to both sign vectors remaining bounded if the set \( S \) of hyperplanes separating them is deleted. Thus we may assume that \( \alpha|_{I \setminus S} = \beta|_{I \setminus S} \) is bounded feasible for both \( (V_1)_S \) and its reversal \( (\bar{V}_1)_S = (V_1)_S \). But this implies that the same sign vector is bounded feasible for \( (V_2)_S \), thus there must exist a sign vector \( \gamma \in P_2 \) such that \( \gamma|_{I \setminus S} = \alpha|_{I \setminus S} = \beta|_{I \setminus S} \). Then by Corollary 3.8, our element can be written as a sum of paths passing through the node \( \gamma \). \( \square \)
This allows us to prove the main theorem of Section 6.

**Theorem 6.13.** The categories $D(V_1)$ and $D(V_2)$ are equivalent.

**Proof.** We first note that by Theorems 5.19 and 5.25, we have equivalences

$$D(V_1) \simeq D(V_1^\vee) \quad \text{and} \quad D(V_2) \cong D(V_2^\vee).$$

Since $\eta_1^\vee = -\xi_j$, replacing the parameter $\xi_1$ with $\xi_2$ can be interpreted on the Gale dual side as replacing the parameter $\eta_1^\vee$ with $\eta_2^\vee$. Thus we may reduce Theorem 6.13 to the case where $\xi_1$ and $\xi_2$ coincide.

By Lemma 6.12, $\Phi_{21} \circ \Phi_{12} = \Phi^-$, which we know from Theorem 6.10 is an equivalence of derived categories. Thus $\Phi_{12}$ is faithful and $\Phi_{21}$ is full and essentially surjective. By symmetry, $\Phi_{12}$ is also full and essentially surjective, thus an equivalence. □

### 7. Canonical Deformations

Let $V$ be a polarized arrangement, and let $A$ and $B$ be the associated algebras. Both $A$ and $B$ have canonical deformations, which we describe as follows.

**Definition 7.1.** Let $A' = A'(V)$ be the quotient of $P(Q) \otimes \text{Sym}(R^f)^*$ by the two-sided ideal generated by the following relations:

1. $A1$: $e_\alpha = 0$ for $\alpha \in F \setminus P$.
2. $A2$: If four distinct elements $\alpha, \beta, \gamma, \delta \in F$ satisfy $\alpha \leftrightarrow \beta \leftrightarrow \gamma \leftrightarrow \delta \leftrightarrow \alpha$ then $p(\alpha, \beta, \gamma) = p(\alpha, \delta, \gamma)$.
3. $A3$: If $\alpha, \beta \in F$ and $\alpha \uparrow \beta$, then $p(\alpha, \beta, \alpha) = v_i e_\alpha$, where $v_i$ is the $i$th basis vector of $(R^f)^*$.

It is clear from this definition that

$$A \cong A' \otimes_{\text{Sym}(R^f)} \text{Sym} V^* \cong A' \otimes_{\text{Sym} V^*} R.$$

**Remark 7.2.** It is via the deformation $A'$ that we can best understand the relationship between $A$ and the Deligne groupoid, to which we referred in Remark 3.4. Inside of the Deligne groupoid sits a semi-groupoid generated by paths in this quiver, which we may call the **Deligne semi-groupoid**. The algebra $A'$ is isomorphic to the quotient by $\{e_\alpha \mid \alpha \in F \setminus P\}$ of the semi-groupoid algebra of the Deligne semi-groupoid.
Definition 7.3. Let

\[ B' = B'(\mathcal{V}) = H^*_T(\tilde{\mathcal{X}} \times \mathcal{X}) \]

where \( T = V^*/\text{Im}(Z_I)^* \) is the torus with Lie algebra \( V^* \) which acts naturally on \( \pi : \tilde{\mathcal{X}} \to \mathcal{X} \). Then the natural convolution product on \( B' \) makes it into a \( \text{Sym} V \) algebra, and we have a natural algebra isomorphism

\[ B \cong B' \otimes_{\text{Sym} V} \mathbb{R} \]

Furthermore, it is not hard to adapt Theorem 4.11 to the equivariant context to produce a natural \( \text{Sym} V \)-algebra isomorphism

\[ Z(B') \cong H^*_T(\mathcal{X}) \]

As before, all of these statements can be understood even when \( \mathcal{V} \) is not rational, using the fact that the (equivariant) cohomology rings of \( \mathcal{X} \) and \( \tilde{\mathcal{X}} \times \mathcal{X} \) have purely combinatorial descriptions in terms of \( \mathcal{H} \).

The purpose of this section is to show that these two deformations are both instances of a completely canonical deformation of any quadratic algebra. As in Section 3.3, Let \( R = \mathbb{R}\{e_\alpha \mid \alpha \in I\} \) be a ring spanned by finitely many pairwise orthogonal idempotents, and let \( M \) be an \( R \)-bimodule, and \( W = \bigoplus_{\alpha,\beta} W_{\alpha\beta} \) an \( R \)-bimodule equipped with an inclusion

\[ \iota : W \hookrightarrow M \otimes_R M \]

For shorthand, we will write

\[ r_w := \iota(w) \quad \text{for any } w \in W. \]

Let

\[ E = T_R(M)/\langle r_w \mid w \in W \rangle \]

be the associated quadratic algebra. Observe that \( W \) is naturally dual to \( E^!_{2\alpha} \), the degree 2 part of the quadratic dual of \( E \). Let \( Z(E^!) \) denote the center of \( E^! \), and let \( L \subseteq W \) be the linear forms that cut \( Z(E^!)^2 \) out of \( E^!_{2\alpha} \).

Definition 7.4. Let

\[ \hat{E} = T_R(M) \otimes_R \text{Sym} W/\langle r_w - e_\alpha w_\beta \mid w \in W_{\alpha\beta} \rangle \]

and

\[ \bar{E} = \hat{E} \otimes_{\text{Sym} W} \text{Sym} W/L. \]
The algebra $\hat{E}$ is a deformation of $E$ over the base $\text{Spec} \text{Sym} W \cong E^!_2$, and $\tilde{E}$ is the restriction of this deformation to the linear subspace $Z(E'_1)_2$ of $E^!_2$. Both algebras are naturally graded, with the generators of $W$ lying in degree 2. We will refer to $\tilde{E}$ as the canonical deformation of $E$.

**Remark 7.5.** Note that $e_\beta w e_\alpha$ may also be written as $\delta_{\alpha, \beta} w e_\alpha$. In particular, if we think of $T_R(M)$ as the path algebra of a quiver with vertex set $J$ and $e_\beta T_R(M) e_\alpha$ as the space of paths from node $\alpha$ to node $\beta$, then our deformations do not affect any relations among paths that start and end at different vertices.

**Remark 7.6.** We will further explore the properties of this deformation in a future paper. In particular, we will prove that if $E$ is Koszul, then $\tilde{E}$ is flat over $Z(E'_1)_2$.

Consider the case of the quiver algebra $A$. In this case
\[ W/L \cong Z(A^!_1)^*_2 \cong Z(B^!_1)^*_2 \cong V^\perp, \]
where the second isomorphism follows from Theorem 3.10 and the third from Theorem 4.11. We then have the following result, whose proof we will delay for a moment.

**Theorem 7.7.** There is a natural $\text{Sym} V^\perp$-algebra isomorphism $\tilde{A} \to A'$.

Next consider the convolution algebra $B$. In this case
\[ W/L \cong Z(B^!_1)^*_2 \cong V, \]
where the third isomorphism may be deduced from Theorems 3.10, 4.4, 4.10, and 4.11. Then we have the following analogue of Theorem 7.7.

**Theorem 7.8.** There is a natural $\text{Sym} V$-algebra isomorphism $\tilde{B} \to B'$.

**Remark 7.9.** It is striking that the algebra $\tilde{B}$ can be recovered from the algebra $B$ by a general construction. Recall that $B$ is defined as the ordinary cohomology of the variety $\widetilde{X} \times_T \widetilde{X}$, while $\tilde{B}$ is defined as the $T$-equivariant cohomology. This means that the algebra $B$ somehow "knows about" the action of $T$ on $\widetilde{X} \times_T \widetilde{X}$.

**Proof of 7.7:** In the case of the algebra $A$, we have
\[ R = \mathbb{R}\{e_\alpha \mid \alpha \in P\} \quad \text{and} \quad M = \mathbb{R}\{p(\alpha, \beta) \mid \alpha \leftrightarrow \beta\}. \]
For distinct nodes $\alpha \neq \gamma$, the image of $W_{\alpha \gamma}$ in $T_R(M)$ is generated by expressions of the form of item $A'2$ in Definition 7.1. At a single node, we explained in the proof of Proposition 3.10 that $W_{\alpha \alpha}$ may be identified with the intersection $V_{\alpha \alpha} \cap R^I \subset R^J$, which projects isomorphically to a subspace of $R^{J_\alpha}$. The map
\[ \iota_{\alpha} : W_{\alpha \alpha} \hookrightarrow e_\alpha T_R(M) e_\alpha \]
is given by taking the $j^{th}$ basis vector of $R^{J_\alpha}$ to $p(\alpha, \alpha_j, \alpha)$ for each $j \in J_\alpha$. Finally, we have
\[ W/L \cong V^\perp \]
via the map that takes $W_{\alpha \gamma}$ to zero for $\alpha \neq \gamma$, and takes $W_{\alpha \alpha}$ to $V^\perp$ by the natural inclusion.

By Remark 7.5, $\tilde{A}$ is defined by relations of two forms: $r_w$ for $w \in W_{\alpha \gamma}$ ($\alpha \neq \gamma$), and $r_w - w e_{\alpha}$ for $w \in W_{\alpha \alpha}$. The first type of relations are those of type $A'2$ in Definition 7.1, and the second type are those of type $A'3$.

**Proof of 7.8:** In Theorem 4.10, we exhibited an isomorphism
\[ A(V^\vee) \rightarrow B(V), \]
and it is easy to check that our map lifts to a homomorphism
\[ A'(V^\vee) \rightarrow B'(V) \]
of $\text{Sym} V$-algebras. Since the source and target are free $\text{Sym} V$-algebras generated in the same degrees, and the map becomes an isomorphism over $0 \in V^*$, it must itself be an isomorphism. The result then follows from Theorem 7.7 applied to $V^\vee$. □

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