Math 634: Algebraic Topology I, Fall 2015
Solutions to Homework #2

Exercises from Hatcher: Chapter 1.1, Problems 2, 3, 6, 12, 16(a,b,c,d,f), 20.

2. Suppose that the path h and i from x_0 to x_1 are homotopic. It follows easily that $\bar{h}$ is homotopic to $\bar{i}$, as well. Then for any loop f based at x_1, $\beta_h[f] = [h \cdot f \cdot \bar{h}] = [i \cdot f \cdot \bar{i}] = \beta_i[f]$.

3. Suppose that $\pi_1(X, x_1)$ is abelian. Let h and i be two arbitrary paths from x_0 to x_1, and let f be a loop based at x_1. Then

$$\begin{aligned}
\beta_h[f] &= [h \cdot f \cdot \bar{h}] \\
&= [i \cdot \bar{i} \cdot h \cdot f \cdot \bar{h} \cdot i \cdot \bar{i}] \\
&= [i \cdot \bar{i} \cdot h \cdot f \cdot \overline{\bar{i} \cdot h} \cdot \bar{i}] \\
&= \beta_i\left([\bar{i} \cdot h][f][\bar{i} \cdot h]^{-1}\right) \\
&= \beta_i[f],
\end{aligned}$$

where the last equality follows from the fact that $\pi_1(X, x_1)$ is abelian.

Now suppose that $\pi_1(X, x_0)$ is nonabelian, and choose loops f and h based at x_0 such that $[h][f] \neq [f][h]$. Let c be the constant path at x_0. Then $\beta_h[f] = [h][f][h]^{-1} \neq [f] = \beta_c[f]$.

6. The key observation in this problem is the following. Let f be a path from x_0 to x_1, g a path from x_1 to x_2, and h a path from x_2 to x_0. Then $f \cdot g \cdot h$ is a loop based at x_0, and therefore defines a map from S^1 to X. Similarly, $g \cdot h \cdot f$ and $h \cdot f \cdot g$ both define maps from S^1 to X. These three maps from S^1 to X are all homotopic, since they are related to each other by precomposing by a rotation of S^1. In other words, $\Phi_{x_0}[f \cdot g \cdot h] = \Phi_{x_1}[g \cdot h \cdot f] = \Phi_{x_2}[h \cdot f \cdot g]$.

Okay, now let's apply this observation to solve the problem. Let $f : S^1 \to X$ be any map, and define $g : [0,1] \to X$ by putting $g(t) = f(e^{2\pi i t})$. Let $x_1 = g(0) = g(1)$, and let h be any path from x_0 to x_1. Then

$$\Phi_{x_0}[h \cdot g \cdot \bar{h}] = \Phi_{x_1}[g \cdot \bar{h} \cdot h] = \Phi_{x_1}[g] = [f],$$

so Φ is surjective. Now let f and h be loops based at x_0. Then $\Phi[h \cdot f \cdot \bar{h}] = \Phi[f \cdot \bar{h} \cdot h] = \Phi[f]$. This shows that $\Phi[f] = \Phi[g]$ if $[f]$ and $[g]$ are conjugate. We still need to prove the converse (which is the hardest part).

Let f and g be loops based at x_0 and suppose that $\Phi[f] = \Phi[g]$. That means that there exists a family of maps $F_t : [0,1] \to X$ such that $F_0 = f$, $F_1 = g$, and $F_t(0) = F_t(1)$ for all t (so that F_t descends to a map from S^1). Note that this is not the same as saying that $f \simeq g$, because we do not require that $F_t(0) = F_t(1) = x_0$ for all t. Define a loop h based at x_0 by putting $h(t) = F_t(0) = F_t(1)$. Then F induces a homotopy from $h \cdot g \cdot \bar{h}$ to f. You can work this out explicitly, or by invoking Lemma 1.19, with $[f]$ and $[g]$ being the images of $1 \in \mathbb{Z} \cong \pi_1(S^1)$ along the two different maps from S^1 to X.

12. We know that $\pi_1(S^1) \cong \mathbb{Z}$, and every endomorphism of $\mathbb{Z}$ is given by multiplication by some integer n. This endomorphism is induced by the map from S^1 to itself taking z to z^n.

16. We know that if A is a retract of X, then the inclusion of A into X induces an injection of $\pi_1(A)$ into $\pi_1(X)$, and the retraction induces a map from $\pi_1(X)$ to $\pi_1(A)$ whose composition with the aforementioned injection is the identity.

(a) $\pi_1(X)$ is trivial and $\pi_1(A) \cong \mathbb{Z}$.

(b) $\pi_1(X) \cong \mathbb{Z}$ and $\pi_1(A) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(c) Here $\pi_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_1(A) \cong \mathbb{Z}$, so it looks as if it would be alright. However, it is easy to see that the path which represents the generator of $\pi_1(A)$ can be contracted in X, so the homomorphism induced by the inclusion is trivial.

(d) I want to say that $\pi_1(X)$ is trivial and $\pi_1(A) = \mathbb{Z} * \mathbb{Z}$, so we win. However, we won't actually prove either of these statements until Section 1.2. Instead, let me note that if $r : X \to A$ is a retraction, then so is the composition

$$D^2 \hookrightarrow X \xrightarrow{r} A \twoheadrightarrow S^1,$$

where the first map is the inclusion of one of the disks and the last map collapses the boundary of the other disk. But we know that S^1 is not a retract of D^2.

(f) Here $\pi_1(X)$ and $\pi_1(A)$ are each isomorphic to $\mathbb{Z}$, and the map induced by the inclusion of A into X is multiplication by 2, which is indeed injective! However, there is no map from $\mathbb{Z}$ to $\mathbb{Z}$ which, when composed with multiplication by 2, gives the identity.

20. Let $h(t) = f_t(x_0)$. Since $f_1 = f_0 = \mathrm{id}_X$, $(f_1)_*$ and $(f_0)_*$ are both the identity map. Thus Lemma 1.19 says that β_h is also the identity map. But β_h is simply conjugation by $[h]$, so $[h]$ must be central.