Arithmetic and topology of hypertoric varieties

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Abstract. A hypertoric variety is a quaternionic analogue of a toric variety. Just as the topology of toric varieties is closely related to the combinatorics of polytopes, the topology of hypertoric varieties interacts richly with the combinatorics of hyperplane arrangements and matroids. Using finite field methods, we obtain combinatorial descriptions of the Betti numbers of hypertoric varieties, both for ordinary cohomology in the smooth case and intersection cohomology in the singular case. We also introduce a conjectural ring structure on the intersection cohomology of a hypertoric variety.

Let T^k be an algebraic torus acting linearly and effectively on an affine space $\mathbb{A}^n$, by which we mean a vector space over an unspecified field, or even over the integers. Though much of our paper is devoted to the finite field case, for the purposes of the introduction one may simply think of a complex vector space. A character α of T^k defines a lift of the action to the trivial line bundle on $\mathbb{A}^n$, and the corresponding geometric invariant theory (GIT) quotient $\mathfrak{X} = \mathbb{A}^n /\!\!/_{\alpha} T^k$ is a toric variety.[3] A hypertoric variety is a symplectic quotient

$$\mathfrak{M} = T^*\mathbb{A}^n /\!\!/\!\!/\!\!/_{(\alpha,0)} T^k = \mu^{-1}(0) /\!\!/_{\alpha} T^k, \tag{1}$$

where $\mu : T^*\mathbb{A}^n \to (\mathfrak{t}^k)^*$ is the algebraic moment map for the T^k action on $T^*\mathbb{A}^n$. Over the complex numbers, this construction may be interpreted as a hyperkähler quotient [BD, §3], or equivalently as a real symplectic quotient of $\mu^{-1}(0)$ by the compact form of T^k. For this reason, $\mathfrak{M}$ may be thought of as a 'quaternionic' or hyperkähler analogue of $\mathfrak{X}$. In this paper, however, we will focus on the algebro-geometric construction, which lets us work over arbitrary fields.

The data of T^k acting on $\mathbb{A}^n$, along with the character α, can be conveniently encoded by an arrangement $\mathcal{A}$ of cooriented hyperplanes in an affine space of dimension $d = n - k$. The topology of the corresponding complex toric variety $\mathfrak{X}(\mathcal{A})_{\mathbb{C}}$ is deeply related to the combinatorics of the polytope cut out by $\mathcal{A}$ over the real numbers [S1, S2]. The hypertoric variety $\mathfrak{M}(\mathcal{A})$ is sensitive to a different side of the combinatorial data. As a topological space, the complex variety $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$ does *not* depend on the coorientations of the hyperplanes [HP, 2.2], and hence has little relationship to the polytope that controls $\mathfrak{X}(\mathcal{A})$. Instead, the

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[3]See [P2] for an elementary definition of GIT quotients, and an exposition of toric varieties from this perspective. The reader with a more differential bent can think of $\mathfrak{X}$ as a real symplectic quotient of $\mathbb{C}^n$ by the compact torus sitting inside of T^k. This is not to be confused with the algebraic (or complex) symplectic quotient of Equation (1).

topology of $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$ interacts richly with the combinatorics of the matroid associated to $\mathcal{A}$, as explained in [Ha]. We now describe the sort of combinatorial structures that arise in this setting.

Let Δ be a simplicial complex of dimension $d-1$ on the ground set $\{1,\ldots,n\}$. The ***f*-vector** of Δ is the $(d+1)$-tuple $(f_0,\ldots,f_d)$, where f_i is the number of faces of Δ of cardinality i (and therefore of dimension $i-1$). The ***h*-vector** $(h_0,\ldots,h_d)$ and ***h*-polynomial** $h_\Delta(q)$ of Δ are defined by the equations

$$h_\Delta(q) = \sum_{i=0}^{d} h_i\, q^i = \sum_{i=0}^{d} f_i\, q^i (1-q)^{d-i}.$$

To each simplicial complex Δ, we associate its **Stanley-Reisner ring** $\mathcal{SR}(\Delta)$, which is defined to be the the quotient of $\mathbb{C}[e_1,\ldots,e_n]$ by the ideal generated by the monomials $\prod_{i\in S} e_i$ for all non-faces S of Δ. The complex Δ is called **Cohen-Macaulay** if there exists a d-dimensional subspace $L \subseteq \mathcal{SR}(\Delta)_1$ such that $\mathcal{SR}(\Delta)$ is a free module over the polynomial ring $\operatorname{Sym} L$. Such a subspace is called a **linear system of parameters**. If Δ is Cohen-Macaulay and L is a linear system of parameters for Δ, then $\mathcal{SR}_0(\Delta) := \mathcal{SR}(\Delta) \otimes_{\operatorname{Sym} L} \mathbb{C}$ has Hilbert series equal to $h_\Delta(q)$ [S3, 5.9].

Let $\mathcal{A} = \{H_1,\ldots,H_n\}$ be a collection of labeled hyperplanes in a vector space V, and let $a_i \in V^*$ be a nonzero normal vector to H_i for all i. The **matroid complex** $\Delta_{\mathcal{A}}$ associated to $\mathcal{A}$ is the collection of sets $S \subseteq \{1,\ldots,n\}$ such that $\{a_i \mid i \in S\}$ is linearly independent. A **circuit** of $\Delta_{\mathcal{A}}$ is a minimal dependent set. Let σ be an ordering of the set $\{1,\ldots,n\}$. A ***σ*-broken circuit** of $\Delta_{\mathcal{A}}$ is a set $C \smallsetminus \{i\}$, where C is a circuit, and i is the σ-minimal element of C. The ***σ*-broken circuit complex** $\mathrm{bc}_\sigma\Delta_{\mathcal{A}}$ is defined to be the collection of subsets of $\{1,\ldots,n\}$ that do not contain a σ-broken circuit. The two complexes $\Delta_{\mathcal{A}}$ and $\mathrm{bc}_\sigma\Delta_{\mathcal{A}}$ are both Cohen-Macaulay (in fact shellable [Bj, §7.3 & §7.4]); their h-polynomials will be denoted $h_{\mathcal{A}}(q)$ and $h^{br}_{\mathcal{A}}(q)$, respectively. As the notation suggests, the polynomial $h^{br}_{\mathcal{A}}(q)$ is independent of our choice of ordering σ [Bj, §7.4].

Let $\mathcal{A}$ be a central hyperplane arrangement, and $\tilde{\mathcal{A}}$ a simplification of $\mathcal{A}$. By this we mean that all of the hyperplanes in $\mathcal{A}$ pass through the origin, and $\tilde{\mathcal{A}}$ is obtained by translating those hyperplanes away from the origin in such a way so that all nonempty intersections are generic. Then $\mathfrak{M}(\mathcal{A})$ is an affine cone, and $\mathfrak{M}(\tilde{\mathcal{A}})$ is an orbifold resolution of $\mathfrak{M}(\mathcal{A})$. Our goal is to study the topology of the complex varieties $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$ and $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$, relating them to the combinatorics of the arrangement $\mathcal{A}$. To achieve this goal, we count points on the corresponding varieties over finite fields.

Our approach to counting points on $\mathfrak{M}(\tilde{\mathcal{A}})$ is motivated by a paper of Crawley-Boevey and Van den Bergh [CBVdB], who work in the context of representations of quivers. In Section 3 we use an exact sequence that appeared first in [CB] to obtain a combinatorial formula for the number of $\mathbb{F}_q$ points of $\mathfrak{M}(\tilde{\mathcal{A}})$. Then the Weil conjectures allow us to translate this formula into a description of the Poincaré polynomial of $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$ (Theorem 3.5).

Theorem. *The Poincaré polynomial of* $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$ *coincides with the h-polynomial of* $\Delta_{\mathcal{A}}$.

This theorem has been proven by different means in [BD, 6.7] and [HS, 1.2]. One noteworthy aspect of our approach is that it sheds light on a mysterious theorem of Buchstaber and Panov [BP, §8], who produce a seemingly unrelated space with the same Poincaré polynomial (see Remark 3.6).

In the case of the singular variety $\mathfrak{M}(\mathcal{A})$, we follow the example of Kazhdan and Lusztig [KL, §4], who study the singularities of Schubert varieties. These singularities are measured by local intersection cohomology Poincaré polynomials, and Kazhdan and Lusztig obtain a recursive formula for these polynomials using Deligne's extension of the Weil conjectures. In our paper, we extend the argument in [KL, 4.2] to apply to more general classes of varieties. Roughly speaking, we consider a collection of stratified affine cones with polynomial point count, which is closed under taking closures of strata, and normal cones to strata. (For details, see Theorem 4.1.) In Section 2 we give such a stratification of $\mathfrak{M}(\mathcal{A})$, and in Section 4 we obtain the following new result (Theorem 4.3).

Theorem. *The intersection cohomology Poincaré polynomial of* $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$ *coincides with the* h*-polynomial of* $\mathrm{bc}_\sigma\Delta_{\mathcal{A}}$.

Section 5 is devoted to the comparison of the two h-polynomials using the decomposition theorem of [BBD, 6.2.5]. The map from $\mathfrak{M}(\tilde{\mathcal{A}})$ to $\mathfrak{M}(\mathcal{A})$ is semismall (Corollary 2.7), hence the decomposition theorem expresses the cohomology of $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$ in terms of the intersection cohomology of the strata of $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$ and the cohomology of the fibers of the resolution (Equation (13)). By our previous results, we thus obtain a combinatorial formula relating the h-numbers of a matroid complex to those of its broken circuit complex. This formula turns out to be a special case of the Kook-Reiner-Stanton convolution formula, which is proven from a strictly combinatorial perspective in [KRS, 1].

We note that this suggests yet another avenue leading to the computation of the Betti numbers of $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$. Knowing only the intersection Betti numbers of $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$, we could have computed these numbers using the KRS formula and the recursion that we obtained from the decomposition theorem. This approach is one that generalizes naturally to other settings. For example, Nakajima's quiver varieties form a class of stratified affine varieties which is closed under taking closures of strata and normal cones to strata [Na, §3]. These varieties have semismall resolutions whose Betti numbers are relevant to the representation theory of Kač-Moody algebras, and are the subject of an outstanding conjecture of Lusztig [Lu, 8]. If a polynomial point count for the singular varieties could be obtained, then the decomposition theorem would provide recursive formulas for the Betti numbers of the smooth ones.

Section 6 deals with the problem of ring structures. Hausel and Sturmfels show that the cohomology ring of $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$ is isomorphic to $\mathcal{SR}_0(\Delta_{\mathcal{A}})$, which strengthens Theorem 3.5. Intersection cohomology is in general only a group, so we have no analogous theorem to prove for $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$. When $\mathcal{A}$ is a unimodular arrangement, however, we define a ring $R_0(\mathcal{A})$ which is isomorphic to the intersection cohomology of $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$ as a graded vector space. This ring does not depend on a choice of ordering σ of the set $\{1, \ldots, n\}$, but it degenerates flatly to $\mathcal{SR}_0(\mathrm{bc}_\sigma\Delta_{\mathcal{A}})$ for any σ (Theorem 6.2). We conjecture that this isomorphism is natural, and that the multiplicative structure can be interpreted in terms of the geometry of $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$ (Conjecture 6.4).

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1 Hypertoric varieties

Let T^n and T^d be split algebraic tori defined over $\mathbb{Z}$, with Lie algebras $\mathfrak{t}^n$ and $\mathfrak{t}^d$. Let $\{x_1, \ldots, x_n\}$ be a basis for $\mathfrak{t}^n = \mathrm{Lie}(T^n)$, and let $\{e_1, \ldots, e_n\}$ be the dual basis for the dual lattice $(\mathfrak{t}^n)^*$. Suppose given n nonzero integer vectors $\{a_1, \ldots, a_n\} \subseteq \mathfrak{t}^d$ such that the map $\mathfrak{t}^n \to \mathfrak{t}^d$ taking x_i to a_i has rank d, and let $\mathfrak{t}^k$ be the kernel of this map. Then we have an exact sequence

$$0 \longrightarrow \mathfrak{t}^k \xrightarrow{\iota} \mathfrak{t}^n \longrightarrow \mathfrak{t}^d \longrightarrow 0, \tag{2}$$

which exponentiates to an exact sequence of groups

$$0 \longrightarrow T^k \longrightarrow T^n \longrightarrow T^d \longrightarrow 0, \tag{3}$$

where $T^k = \ker\bigl(T^n \to T^d\bigr)$. Thus T^k is an algebraic group with Lie algebra $\mathfrak{t}^k$, which is connected if and only if the vectors $\{a_i\}$ span the lattice $\mathfrak{t}^d$ over the integers. Every algebraic subgroup of T^n arises in this way.

Consider the cotangent bundle $T^*\mathbb{A}^n \cong \mathbb{A}^n \times (\mathbb{A}^n)^*$ along with its natural algebraic symplectic form

$$\omega = \sum dz_i \wedge dw_i,$$

where z and w are coordinates on $\mathbb{A}^n$ and $(\mathbb{A}^n)^*$, respectively. The restriction to T^k of the standard action of T^n on $T^*\mathbb{A}^n$ is hamiltonian, with moment map

$$\mu(z, w) = \iota^* \sum_{i=1}^{n} (z_i w_i)\, e_i.$$

Suppose given an integral element $\alpha \in (\mathfrak{t}^k)^*$. This descends via the exponential map to a character of T^k, which defines a lift of the action of T^k to the trivial bundle on $T^*\mathbb{A}^n$. The symplectic quotient

$$\mathfrak{M} = T^*\mathbb{A}^n /\!\!/\!\!/\!\!/_{\alpha} T^k = \mu^{-1}(0) /\!\!/_{\alpha} T^k$$

is called a **hypertoric variety**. Here the second quotient is a projective GIT quotient[4]

$$\mu^{-1}(0) /\!\!/_{\alpha} T^k := \mathrm{Proj} \bigoplus_{m=0}^{\infty} \Bigl\{ f \in \mathcal{O}_{\mu^{-1}(0)} \Bigm| \nu^*(f) = \alpha^m \otimes f \Bigr\},$$

[4]For a careful treatment of geometric invariant theory over the integers, see Appendix B of [CBVdB].

where

$$\nu^* : \mathcal{O}_{\mu^{-1}(0)} \to \mathcal{O}_{T^k \times \mu^{-1}(0)} \cong \mathcal{O}_{T^k} \otimes \mathcal{O}_{\mu^{-1}(0)}$$

is the map on functions induced by the action map $\nu : T^k \times \mu^{-1}(0) \to \mu^{-1}(0)$. If α is omitted from the subscript, it will be understood to be equal to zero. The hypertoric variety $\mathfrak{M}$ is a symplectic variety of dimension $2d$, and admits an effective hyperhamiltonian action of the torus $T^d = T^n/T^k$, with moment map

$$\Phi[z,w] = \sum_{i=1}^{n} (z_i w_i)\, e_i \in \ker(\iota^*) = (\mathfrak{t}^d)^*.$$

Here $[z,w]$ is used to denote the image in $\mathfrak{M}$ of a pair (z,w) with closed T^k-orbit in $\mu^{-1}(0)$.

Remark 1.1. *The word 'hypertoric' comes from the fact that the complex variety $\mathfrak{M}_{\mathbb{C}}$ may be constructed as a hyperkähler quotient of $T^*\mathbb{C}^n$ by the compact real form of T^k, thus making it a 'hyperkähler analogue' of the toric variety $\mathfrak{X} = \mathbb{C}^n /\!\!/_{\alpha} T^k$. This was the original approach of Bielawski and Dancer* [BD]*, who used the name 'toric hyperkähler manifolds'. For more on the general theory of hyperkähler analogues of Kähler quotients, see* [P1]*.*

It is convenient to encode the data that were used to construct $\mathfrak{M}$ in terms of an arrangement of affine hyperplanes in $(\mathfrak{t}^d)^*$, with some additional structure. A **weighted, cooriented, affine hyperplane** $H \subseteq (\mathfrak{t}^d)^*$ is a hyperplane along with a choice of nonzero integer normal vector $a \in \mathfrak{t}^d$. Here "affine" means that H need not pass through the origin, and "weighted" means that a is not required to be primitive. Let $r = (r_1, \ldots, r_n) \in (\mathfrak{t}^n)^*$ be a lift of α along ι^*, and let

$$H_i = \{v \in (\mathfrak{t}^d)^* \mid v \cdot a_i + r_i = 0\}$$

be the weighted, cooriented, affine hyperplane with normal vector $a_i \in \mathfrak{t}^d$. We will denote the arrangement $\{H_1, \ldots, H_n\}$ by $\mathcal{A}$, and the associated hypertoric variety by $\mathfrak{M}(\mathcal{A})$. Choosing a different lift r' of α corresponds geometrically to translating $\mathcal{A}$ inside of $(\mathfrak{t}^d)^*$. The rank of the lattice spanned by the vectors $\{a_i\}$ is called the **rank** of $\mathcal{A}$; in our case, we have already made the assumption that $\mathcal{A}$ has rank d. Observe that $\mathcal{A}$ is defined over the integers, and therefore may be realized over any field. Intuitively, it is useful to think of $\mathcal{A}$ as an arrangement of real hyperplanes, but in Section 4 we will need to consider the complement of $\mathcal{A}$ over a finite field.

Remark 1.2. *We note that we allow repetitions of hyperplanes in our arrangement ($\mathcal{A}$ may be a multi-set), and that a repeated occurrence of a particular hyperplane is* not *the same as a single occurrence of that hyperplane with weight 2. On the other hand, little is lost by restricting one's attention to arrangements of distinct hyperplanes of weight one.*

We call the arrangement $\mathcal{A}$ **simple** if every subset of m hyperplanes with nonempty intersection intersects in codimension m. We call $\mathcal{A}$ **unimodular** if every collection of d linearly independent vectors $\{a_{i_1}, \ldots, a_{i_d}\}$ spans $\mathfrak{t}^d$ over the integers. An arrangement which is both simple and unimodular is called **smooth**.

Theorem 1.3. [BD, 3.2 & 3.3] *The hypertoric variety $\mathfrak{M}(\mathcal{A})$ has at worst orbifold (finite quotient) singularities if and only if $\mathcal{A}$ is simple, and is smooth if and only if $\mathcal{A}$ is smooth.*

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a **central** arrangement, meaning that $r_i = 0$ for all i. Let $\tilde{\mathcal{A}} = \{\tilde{H}_1, \ldots, \tilde{H}_n\}$ be a **simplification** of $\mathcal{A}$, by which we mean an arrangement defined by the same vectors $\{a_i\} \subset \mathfrak{t}^d$, but with a different choice of $r \in (\mathfrak{t}^n)^*$, such that $\tilde{\mathcal{A}}$ is simple. This corresponds to translating each of the hyperplanes in $\mathcal{A}$ away from the origin by some generic amount. We then have

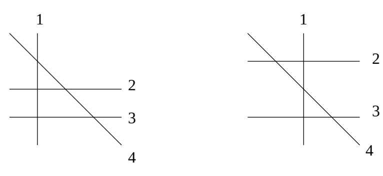
$$\mathfrak{M}(\mathcal{A}) \,=\, \operatorname{Proj} \bigoplus_{m=0}^{\infty} \mathcal{O}_{\mu^{-1}(0)}^{T^k} \,=\, \operatorname{Spec} \mathcal{O}_{\mu^{-1}(0)}^{T^k} \,=\, \operatorname{Spec} \mathcal{O}_{\mathfrak{M}(\tilde{\mathcal{A}})},$$

hence there is a surjective, projective map $\pi : \mathfrak{M}(\tilde{\mathcal{A}}) \to \mathfrak{M}(\mathcal{A})$. Geometrically, π may be understood to be the map induced by the T^k-equivariant inclusion of $\mu^{-1}(0)^{\alpha-\mathrm{st}}$ into $\mu^{-1}(0)$, where $\mu^{-1}(0)^{\alpha-\mathrm{st}}$ is the stable locus for the linearization of the T^k action given by $\alpha = \iota^*(r)$. The central fiber $\mathcal{L}(\tilde{\mathcal{A}}) = \pi^{-1}(0)$ is called the **core** of $\mathfrak{M}(\tilde{\mathcal{A}})$.

Theorem 1.4. [BD, §6],[HS, 6.4] *The core $\mathcal{L}(\tilde{\mathcal{A}})$ is isomorphic to a union of toric varieties with moment polytopes given by the bounded complex of $\tilde{\mathcal{A}}$. Over the complex numbers, $\mathcal{L}(\tilde{\mathcal{A}})_{\mathbb{C}}$ is a $T^d_{\mathbb{R}}$-equivariant deformation retract of $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$, where $T^d_{\mathbb{R}} \cong U(1)^d$ is the compact real form of $T^d_{\mathbb{C}}$.*

It follows that the dimension of the core is at most d, with equality if and only $\mathcal{A}$ is coloop-free (see Remark 2.3).

Example 1.5. *The two arrangements pictured below are each simplifications of a central arrangement of four hyperplanes in $\mathbb{R}^2$, in which the second and third hyperplane coincide. All hyperplanes are taken with weight 1, and coorientations may be chosen arbitrarily.*



Consider the complex hypertoric varieties associated to these two arrangements. Both are obtained as symplectic quotients of $T^*\mathbb{C}^4$ by the same T^2 action, but with different choices of character. Both varieties are resolutions of the affine variety given by the associated central arrangement. The hypertoric variety associated to the left-hand arrangement has a core consisting of a projective plane glued to a Hirzebruch surface along a projective line. The hypertoric variety associated to the right-hand arrangement has a core consisting of two projective planes glued together at a point. As manifolds, they are diffeomorphic, as are any two complex hypertoric varieties corresponding to different simplifications of the same central arrangement [HP, 2.1].

2 The stratification

Let $\mathcal{A}$ be a rank d central arrangement of n weighted, cooriented hyperplanes in $(\mathfrak{t}^d)^*$. Our goal for this section is to define and analyze a stratification of the singular affine variety $\mathfrak{M}(\mathcal{A})$. This stratification will be a refinement of the Sjamaar-Lerman stratification, introduced for real symplectic quotients in [SL], and adapted to the algebraic setting in [Na, §3]. Our refinement will prove to be more natural from a combinatorial perspective (see Remark 2.3).

Given any subset $S \subseteq \{1, \ldots, n\}$, let $H_S = \cap_{i \in S} H_i$. A **flat** of $\mathcal{A}$ is a subset $F \subseteq \{1, \ldots, n\}$ such that $F = \{i \mid H_i \supseteq H_F\}$. We let $L(\mathcal{A})$ denote the lattice of flats for the arrangement $\mathcal{A}$. For any flat F, we define the **restriction**

$$\mathcal{A}^F := \{H_i \cap H_F \mid i \notin F\},$$

an arrangement of $|F^c|$ hyperplanes in the affine space H_F, and the **localization**

$$\mathcal{A}_F := \{H_i / H_F \mid i \in F\},$$

an arrangement of $|F|$ hyperplanes in the affine space $(\mathfrak{t}^d)^*/H_F$. The lattice $L(\mathcal{A}^F)$ is isomorphic to the sublattice of $L(\mathcal{A})$ consisting of those flats which contain F; likewise, $L(\mathcal{A}_F)$ may be identified with the sublattice of $L(\mathcal{A})$ consisting of flats contained in F. We define the **rank** of a flat $\operatorname{rk} F = \operatorname{rk} \mathcal{A}_F$, and the **corank** $\operatorname{crk} F = \operatorname{rk} \mathcal{A}^F = \operatorname{rk} \mathcal{A} - \operatorname{rk} F$. Given a simplification $\tilde{\mathcal{A}}$ of $\mathcal{A}$, there is a natural simplification $\tilde{\mathcal{A}}_F$ of the localization.

We now fix notation regarding the various tori associated to the localization and restriction of $\mathcal{A}$ at F. The $\mathcal{A}_F$ analogue of the exact sequence (2) is

$$0 \to \hat{\mathfrak{t}} \to \mathfrak{t}^F \to \mathfrak{t}^{\operatorname{rk} F} \to 0,$$

where $\mathfrak{t}^F$ is the coordinate subtorus of $\mathfrak{t}^n$ supported on F, $\mathfrak{t}^{\operatorname{rk} F} \cong H_F$ is the image of $\mathfrak{t}^F$ in $\mathfrak{t}^d$, and $\hat{\mathfrak{t}} = \mathfrak{t}^k \cap \mathfrak{t}^F$. Similarly, the restriction $\mathcal{A}^F$ corresponds to an exact sequence

$$0 \to \mathfrak{t} \to \mathfrak{t}^{F^c} \to \mathfrak{t}^{\operatorname{crk} F} \to 0,$$

where $\mathfrak{t}^{F^c}$ is the coordinate subtorus of $\mathfrak{t}^n$ supported on F^c, $\mathfrak{t}^{\operatorname{crk} F} \cong \mathfrak{t}^d / H_F$, and $\mathfrak{t} = \mathfrak{t}^k / \hat{\mathfrak{t}}$. The tori $\hat{T}, T^F, T^{\operatorname{rk} F}$ and $T, T^{F^c}, T^{\operatorname{crk} F}$ are defined analogously, as in the exact sequence (3). Let $T^*\mathbb{A}^F$ and $T^*\mathbb{A}^{F^c}$ be the cotangent bundles of the corresponding coordinate subspaces of A^n. Then the hypertoric varieties $\mathfrak{M}(\mathcal{A}_F)$ and $\mathfrak{M}(\mathcal{A}^F)$ are obtained as symplectic quotients of $T^*\mathbb{A}^F$ and $T^*\mathbb{A}^{F^c}$ by $\hat{T}$ and T, respectively.

Proposition 2.1. *Let F be a flat of $\mathcal{A}$. The subvariety of $\mathfrak{M}(\mathcal{A})$ given by the equations $z_i = w_i = 0$ for all $i \in F$ is isomorphic to $\mathfrak{M}(\mathcal{A}^F)$.*

Proof: The inclusion of $T^*\mathbb{A}^{F^c}$ into $T^*\mathbb{A}^n$ is T^k-equivariant, where the action of T^k on $T^*\mathbb{A}^{F^c}$ factors through T. The Lie coalgebra $\mathfrak{t}^*$ of T includes into $(\mathfrak{t}^k)^*$, and the T^k-moment map on $T^*\mathbb{A}^n$ restricts to the T-moment map on $T^*\mathbb{A}^{F^c}$, as shown in the following diagram.

$$\begin{tikzcd} T^*\mathbb{A}^n \arrow[r, "\mu"] & (\mathfrak{t}^k)^* \\ T^*\mathbb{A}^{F^c} \arrow[u] \arrow[r, "\mu^F"] & \mathfrak{t}^* \arrow[u] \end{tikzcd}$$

Thus we have

$$\mathfrak{M}(\mathcal{A}^F) = T^*\mathbb{A}^{F^c} /\!\!/\!\!/\!\!/ T = \left(\mu^F\right)^{-1}(0) /\!\!/ T \cong \left(T^*\mathbb{A}^{F^c} \cap \mu^{-1}(0)\right) /\!\!/ T^k,$$

which is cut out of $\mathfrak{M}(\mathcal{A}) = \mu^{-1}(0)/T^k$ by the equations $z_i = w_i = 0$ for all $i \in F$. □

Remark 2.2. *The subvariety of Proposition 2.1 may also be described as $\Phi^{-1}(H_F)$, where $\Phi : \mathfrak{M}(\mathcal{A}) \to (\mathfrak{t}^d)^*$ is the moment map for the T^d-action described in Section 1. We then have a commutative diagram*

$$\begin{tikzcd} \mathfrak{M}(\mathcal{A}) \arrow[r, "\Phi"] & (\mathfrak{t}^d)^* \\ \mathfrak{M}(\mathcal{A}^F) \arrow[u] \arrow[r, "\Phi^F"] & H_F \arrow[u] \end{tikzcd},$$

where Φ^F is the moment map for the action of $T^{\operatorname{crk} F}$ on $\mathfrak{M}(\mathcal{A}^F)$.

Let S, Y, and Z be schemes over an arbitary ring, with $Z \subseteq Y$ a locally closed subscheme Z, and $s \in S$ a basepoint. We will say that S is a **normal slice** to Z in Y if there exists a collection of étale open cover $\{U_\alpha\}$ of a neighborhood of Z, such that each U_α admits a dominant étale map to $S \times \mathbb{A}^{\dim Z}$, with $U_\alpha \cap Z$ dominating $\{s\} \times \mathbb{A}^{\dim Z}$. Since étale maps are locally invertible in the analytic category, this implies that our definition of normal slices for complex schemes agree with the usual notion. In other words, there is an analytic neighborhood of Z in Y which is locally biholomorphic to a neighborhood of $Z \times \{s\}$ in $Z \times S$.

A **stratification** of a scheme Y over $\mathbb{Z}$ is a partition of Y into smooth, locally closed subschemes Y_β indexed by a finite poset B, along with normal slices S_β, with the property that for all $\beta \in B$,

$$\overline{Y_\beta} = \bigsqcup_{\beta' \leq \beta} Y_{\beta'}.$$

To define a stratification of $\mathfrak{M}(\mathcal{A})$, we begin by putting

$$\mathring{\mathfrak{M}}(\mathcal{A}) = \left\{[z, w] \in \mathfrak{M}(\mathcal{A}) \;\middle|\; \text{there does not exist } i \text{ such that } z_i = w_i = 0\right\}.$$

The identification of $\mathfrak{M}(\mathcal{A}^F)$ with a subvariety of $\mathfrak{M}(\mathcal{A})$ as in Proposition 2.1 induces the identification

$$\mathring{\mathfrak{M}}(\mathcal{A}^F) = \left\{[z, w] \in \mathfrak{M}(\mathcal{A}) \;\middle|\; z_i = w_i = 0 \text{ if and only if } i \in F\right\}. \tag{4}$$

By [BD, 3.1], any point $[z,w] \in \mathfrak{M}(\mathcal{A})$ has the property that the set $\{i \mid z_i = w_i = 0\}$ is a flat, hence we have a decomposition

$$\mathfrak{M}(\mathcal{A}) = \bigsqcup_F \mathring{\mathfrak{M}}(\mathcal{A}^F). \tag{5}$$

One interpretation of our decomposition is that points are grouped according to the stabilizers in T^n of their lifts to $\mu^{-1}(0)$. This is therefore a refinement of the Sjamaar-Lerman stratification [SL, §2], which groups points by their stabilizers in the subtorus T^k. It follows from the defining property of a moment map that $\mathring{\mathfrak{M}}(\mathcal{A})$ is smooth, and therefore that each piece of the decomposition is smooth. To see that our decomposition is a stratification we must produce normal slices to the strata, which we will do in Lemma 2.4. The largest stratum $\mathring{\mathfrak{M}}(\mathcal{A})$ will be referred to as the **generic stratum** of $\mathfrak{M}(\mathcal{A})$.

Remark 2.3. *If $[z,w] \in \mathring{\mathfrak{M}}(\mathcal{A}^F)$, then the stabilizer of $(z,w) \in \mu^{-1}(0)$ is equal to $\hat{T} = T^k \cap T^F$. An element $i \in F$ is called a* **coloop** *of F if a_i may not be expressed as a linear combination of $\{a_j \mid j \in F \smallsetminus \{i\}\}$. If F and G are two flats, then $T^k \cap T^F = T^k \cap T^G$ if and only if F and G agree after deleting all coloops. Hence the Sjamaar-Lerman stratification of $\mathfrak{M}(\mathcal{A})$ is naturally indexed by coloop-free flats, rather than all flats.*

Lemma 2.4. *The variety $\mathfrak{M}(\mathcal{A}_F)$ is a normal slice to $\mathring{\mathfrak{M}}(\mathcal{A}^F) \subseteq \mathfrak{M}(\mathcal{A})$, thus the decomposition of Equation* (5) *is a stratification.*

Proof: Let S be a subset of F^c such that the coordinate vectors in $(\mathfrak{t}^{F^c})^*$ indexed by S descend to a basis of $\mathfrak{t}^*$, and let

$$W_S = \Big\{[z,w] \in \mathfrak{M}(\mathcal{A}) \;\Big|\; z_i \neq 0 \text{ for all } i \in S\Big\}. \tag{6}$$

Any element of W_S may be represented by an element $(z,w) \in \mu^{-1}(0)$ such that $z_i = 1$ for all $i \in S$, and any two such representations differ by a element of the subtorus $\hat{T} \subseteq T^k$. Next observe that the coordinate projection of $T^*\mathbb{A}^n$ onto $T^*\mathbb{A}^F$ takes $\mu^{-1}(0)$ to the zero set of the moment map for the action of $\hat{T}$ on $T^*\mathbb{A}^F$. We may therefore define a map $p_F : W_S \to \mathfrak{M}(\mathcal{A}_F)$ by taking an element of W_S, representing it in the form described above, and projecting to the F coordinates. This map is smooth on the locus $\mathring{\mathfrak{M}}(\mathcal{A}^F) \cap W_S$.

Suppose that $y \in W_S$. By [BLR, 2.2.14], there is a neighborhood U of y in W_S and a smooth map $\eta : U \to \mathbb{A}^{\operatorname{crk} F}$ such that the restriction of p_F to $\eta^{-1}(0)$ is étale. Let

$$\vartheta = \eta \times p_F : U \to \mathbb{A}^{\operatorname{crk} F} \times \mathfrak{M}(\mathcal{A}_F).$$

Then the derivative of ϑ at y is a surjection, hence ϑ is smooth at y. Since its source and target have the same dimension, it must be étale.

If y is not in S, then we may modify the definition of W_S by changing some of the z_i to w_i in Equation (6), and adjust the definition of the map p_F accordingly. Then y will be contained in the new set W_S, and the proof will go through as before. $\square$

We next prove a result similar to Lemma 2.4 by working purely in the analytic category. The advantage of Lemma 2.5 is that we obtain a statement that is compatible with the affinization map π, which will be useful in Section 5.

Lemma 2.5. *For all $y \in \mathring{\mathfrak{M}}(\mathcal{A}^F)$ there is an analytic neighborhood $\mathfrak{U}$ of $y \in \mathfrak{M}(\mathcal{A})_{\mathbb{C}}$ and a map $\varphi : \mathfrak{U} \to \mathring{\mathfrak{M}}(\mathcal{A}^F)_{\mathbb{C}} \times \mathfrak{M}(\mathcal{A}_F)_{\mathbb{C}}$ such that $\varphi(y) = (y, 0)$, and φ is a diffeomorphism onto its image. Furthermore, there is a map $\tilde{\varphi} : \pi^{-1}(\mathfrak{U}) \to \mathring{\mathfrak{M}}(\mathcal{A}^F)_{\mathbb{C}} \times \mathfrak{M}(\tilde{\mathcal{A}}_F)_{\mathbb{C}}$ which covers φ, and is also a diffeomorphism onto its image.*

$$\begin{tikzcd}
\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}} \arrow[d, "\pi"] & \pi^{-1}(\mathfrak{U}) \arrow[l] \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"] & \mathring{\mathfrak{M}}(\mathcal{A}^F)_{\mathbb{C}} \times \mathfrak{M}(\tilde{\mathcal{A}}_F)_{\mathbb{C}} \arrow[d, "\mathrm{id}\times\pi_F"] \\
\mathfrak{M}(\mathcal{A})_{\mathbb{C}} & \mathfrak{U} \arrow[l] \arrow[r, "\varphi"] & \mathring{\mathfrak{M}}(\mathcal{A}^F)_{\mathbb{C}} \times \mathfrak{M}(\mathcal{A}_F)_{\mathbb{C}}
\end{tikzcd}$$

Proof: Let $\bar{y} \in T^*\mathbb{C}^n$ be a representative of y. Since y is contained in the stratum $\mathring{\mathfrak{M}}(\mathcal{A}^F)_{\mathbb{C}}$, we may assume that the i^{th} coordinates of $\bar{y}$ are zero for all $i \in F$. Let $V = T_{\bar{y}}\big(T^k_{\mathbb{C}} \cdot \bar{y}\big)$ be the tangent space to the orbit of $T^k_{\mathbb{C}}$ through $\bar{y}$. Then $V \subseteq T^*\mathbb{C}^n$ is isotropic with respect to the symplectic form ω, and the inclusion of $T^*\mathbb{C}^F$ into $T^*\mathbb{C}^n$ induces a $\hat{T}_{\mathbb{C}}$-equivariant inclusion of $T^*\mathbb{C}^F$ into the quotient V^ω/V, where V^ω is the symplectic perpendicular space to V inside of $T^*\mathbb{C}^n$. The torus $\hat{T}_{\mathbb{C}}$ acts trivially on the quotient of V^ω/V by $T^*\mathbb{C}^F$, which may be identified with the tangent space $T_y\mathring{\mathfrak{M}}(\mathcal{A}^F)_{\mathbb{C}}$. The lemma then follows from the discussion in [Na, §3.2 & 3.3]. $\square$

Corollary 2.6. *The restriction of π to $\pi^{-1}\Big(\mathring{\mathfrak{M}}(\mathcal{A}^F)_{\mathbb{C}}\Big)$ is a locally trivial topological fiber bundle over the stratum $\mathring{\mathfrak{M}}(\mathcal{A}^F)_{\mathbb{C}}$, with fiber isomorphic to the core $\mathcal{L}(\tilde{\mathcal{A}}_F)_{\mathbb{C}} \subseteq \mathfrak{M}(\tilde{\mathcal{A}}_F)_{\mathbb{C}}$.*

If $Y = \sqcup_{\beta\in B} Y_\beta$ is a stratified space and $f : X \to Y$ is a map, then f is called **semismall** if for all $y_\beta \in Y_\beta$, the dimension of $f^{-1}(y_\beta)$ is at most half of the codimension of Y_β in Y. This seemingly arbitrary condition can be motivated by the observation that

$$f \text{ semismall} \;\Leftrightarrow\; \dim X = \dim Y = \dim\Big(Y \times_X Y\Big).$$

Corollary 2.7. *The map π is semismall.*

Proof: $2\dim \mathcal{L}(\tilde{\mathcal{A}}_F) \leq \dim \mathfrak{M}(\tilde{\mathcal{A}}_F) = \operatorname{codim} \mathring{\mathfrak{M}}(\mathcal{A}^F)$, with equality if and only if F is coloop-free. $\square$

Remark 2.8. *The decomposition defined by Equations* (4) *and* (5) *make sense for noncentral arrangements as well as central ones. For most of the paper we will use this decomposition only in the central case, but in Section 6 we will consider the generic stratum of an arbitrary hypertoric variety.*

3 The Betti numbers of $\mathfrak{M}(\tilde{\mathcal{A}})$

Let X be a variety defined over the integers, and let q be a prime power. By an $\mathbb{F}_q$ **point** of X, we mean a closed point of the variety $X_{\mathbb{F}_q} = X \otimes_{\mathbb{Z}} \mathbb{F}_q$. We say that X has **polynomial point count** if there exists a polynomial $\nu_X(q)$ such that, when q is a power of a sufficiently large prime, $\nu_X(q)$ is equal to the number of $\mathbb{F}_q$ points of X. For the remainder of the paper, when we refer to the number of $\mathbb{F}_q$ points on a given variety, we will always implicitly assume that q is a power of a sufficiently large prime. Suppose that X has at worst orbifold singularities, so that the cohomology of $X_{\mathbb{C}}$ is Poincaré dual to the compactly supported cohomology. The Betti numbers for the compactly supported cohomology of $X_{\mathbb{C}}$ agree with the Betti numbers for compactly supported ℓ-adic étale cohomology of $X_{\bar{\mathbb{F}}_p}$ for large enough primes p [BBD, 6.1.9]. If the ℓ-adic étale cohomology of $X_{\bar{\mathbb{F}}_p}$ is pure, we may use the Lefschetz fixed point theorem in ℓ-adic étale cohomology to relate the Betti numbers of $X_{\mathbb{C}}$ to the number of $\mathbb{F}_q$-points on X (see for example [CBVdB, A.1]).

Theorem 3.1. *Suppose that X has polynomial point count and at worst orbifold singularities, and that the ℓ-adic étale cohomology of the variety $X_{\bar{\mathbb{F}}_q}$ is pure. Then $X_{\mathbb{C}}$ has Poincaré polynomial $P_X(q) = q^{\dim X} \cdot \nu_X(q^{-1})$, where q has degree 2. (In particular, the odd cohomology of X vanishes.)*

The purpose of this section is to apply Theorem 3.1 to compute the Poincaré polynomial $P_{\tilde{A}}(q)$ of $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$. The fact that the ℓ-adic étale cohomology of $\mathfrak{M}(\tilde{\mathcal{A}})_{\bar{\mathbb{F}}_q}$ is pure follows from [CBVdB, A.2],[5] using the $\mathbb{G}_m$-action studied in [HP].

Recall that $\mathfrak{M}(\tilde{\mathcal{A}})$ is defined as the GIT quotient $\mu^{-1}(0) /\!\!/_{\alpha} T^k$, where μ is the moment map for the action of T^k on $T^*\mathbb{A}^n$. For any $\lambda \in (\mathfrak{t}^k)^*$, let

$$\mathfrak{M}_\lambda(\tilde{\mathcal{A}}) = \mu^{-1}(\lambda) /\!\!/_{\alpha} T^k.$$

If λ is a regular value of μ, then T^k will act locally freely on $\mu^{-1}(\lambda)$, meaning that the stabilizer in T^k of any point in $\mu^{-1}(\lambda)$ is finite over any field.[6] This in turn implies that the GIT quotient of $\mu^{-1}(\lambda)$ by T^k over any algebraically closed field will be an honest geometric quotient. Fix a regular value λ. By an argument completely analogous to that of Nakajima's appendix to [CBVdB], the varieties $\mathfrak{M}(\tilde{\mathcal{A}})$ and $\mathfrak{M}_\lambda(\tilde{\mathcal{A}})$ have the same point count. Thus Theorem 3.1 tells us that we can compute $P_{\tilde{A}}(q)$ by counting points on $\mathfrak{M}_\lambda(\tilde{\mathcal{A}})$ over finite fields.

Lemma 3.2. *Let X be a variety defined over $\mathbb{F}_q$, let T be a split torus of rank k acting on X, and let T' be a (possibly disconnected) rank ℓ subgroup of T which acts locally freely. Then the number of $\mathbb{F}_q$ points of X is equal to $(q-1)^\ell$ times the number of $\mathbb{F}_q$ points of X/T'.*

[5] The authors state this theorem only for smooth varieties, but their argument clearly extends to the orbifold case.

[6] When we speak of a finite subgroup of a torus that is defined over a finite field, we always mean that the subgroup remains finite after passing to the algebraic closure.

Proof: By Hilbert's Theorem 90, every T-orbit in X which defines an $\mathbb{F}_q$ point of X/T contains an $\mathbb{F}_q$ point of X. This tells us that we may count points on X T-orbit by T-orbit, and thereby reduce to the case where T acts transitively. In this case X is isomorphic to a split torus, and T' acts on X via a homomorphism with finite kernel. Thus X/T' is a split torus with $\operatorname{rk} X/T' = \operatorname{rk} X - \ell$. This completes the proof. $\square$

Corollary 3.3. *The number of $\mathbb{F}_q$ points of $\mu^{-1}(\lambda)$ is equal to $(q-1)^k$ times the number of $\mathbb{F}_q$ points of $\mathfrak{M}_\lambda(\tilde{\mathcal{A}})$.*

Proposition 3.4. *The variety $\mathfrak{M}_\lambda(\tilde{\mathcal{A}})$ has polynomial point count, with*

$$\nu_{\mathfrak{M}_\lambda(\tilde{\mathcal{A}})}(q) = q^{2d} \cdot h_{\mathcal{A}}(q^{-1}).$$

Proof: For any element $z \in \mathbb{A}^n$, we have an exact sequence[7]

$$0 \to \{w \mid \mu(z,w) = 0\} \to T_z^*\mathbb{A}^n \overset{\mu(z,-)}{\longrightarrow} (\mathfrak{t}^k)^* \to \operatorname{stab}(z)^* \to 0, \tag{7}$$

where $\operatorname{stab}(z)^* = (\mathfrak{t}^k)^*/\operatorname{stab}(z)^\perp$ is the Lie coalgebra of the stabilizer of z in T^k. Consider the map $\phi : \mu^{-1}(\lambda) \to \mathbb{A}^n$ given by projection onto the first coordinate. By exactness of (7) at $(\mathfrak{t}^k)^*$, we have

$$\operatorname{Im}\phi = \{z \mid \lambda \cdot \operatorname{stab}(z) = 0\} = \{z \mid \operatorname{stab}(z) = 0\},$$

where the last equality follows from the fact that λ is a regular value. Furthermore, we see that for $z \in \operatorname{Im}\phi$, $\phi^{-1}(z)$ is a torsor for the d-dimensional vector space $\{w \mid \mu(z,w) = 0\}$. Hence the number of $\mathbb{F}_q$ points of $\mu^{-1}(\lambda)$ is equal to q^d times the number of $\mathbb{F}_q$ points of $\mathbb{A}^n$ at which T^k is acting locally freely.

A point $z \in \mathbb{A}^n$ is acted upon locally freely by T^k if and only if $\{i \mid z_i = 0\} \in \Delta_{\mathcal{A}}$. Hence the total number of such points over $\mathbb{F}_q$ is equal to

$$\begin{aligned}
\sum_{S \in \Delta_{\mathcal{A}}} (q-1)^{n-|S|} &= \sum_{i=0}^{d} f_i(\Delta_{\mathcal{A}}) \cdot (q-1)^{n-i} \\
&= (q-1)^k \sum_{i=0}^{d} f_i(\Delta_{\mathcal{A}}) \cdot (q-1)^{d-i} \\
&= (q-1)^k \cdot q^d \cdot \sum_{i=0}^{d} f_i(\Delta_{\mathcal{A}}) \cdot q^{-i}(1-q^{-1})^{d-i} \\
&= (q-1)^k \cdot q^d \cdot h_{\mathcal{A}}(q^{-1}).
\end{aligned}$$

To find the number of $\mathbb{F}_q$ points of $\mathfrak{M}_\lambda(\tilde{\mathcal{A}})$ we multiply by q^d and divide by $(q-1)^k$, and thus obtain the desired result. $\square$

Theorem 3.1 and Proposition 3.4, along with the observation that $\nu_{\mathfrak{M}(\tilde{\mathcal{A}})}(q) = \nu_{\mathfrak{M}_\lambda(\tilde{\mathcal{A}})}(q)$, combine to give us the Poincaré polynomial of $\mathfrak{M}(\tilde{\mathcal{A}})$.

[7] The analogous exact sequence in the context of representations of quivers first appeared in [CB, 3.3], and was used to count points on quiver varieties over finite fields in [CBVdB, §2.2].

Theorem 3.5. *The Poincaré polynomial of $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$ coincides with the h-polynomial of the matroid complex associated to $\mathcal{A}$, that is $P_{\tilde{\mathcal{A}}}(q) = h_{\mathcal{A}}(q)$.*

Remark 3.6. *Implicit in the work of Buchstaber and Panov* [BP, §8] *is a calculation of the cohomology ring of the nonseparated complex variety $W/T^k_{\mathbb{C}}$, where $W \subseteq \mathbb{C}^n$ is the locus of points at which $T^k_{\mathbb{C}}$ acts locally freely. Their description of this ring coincides with the description of $H^*(\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}})$ that we will give in Theorem 6.1, due originally to* [Ko, HS]. *We now have an explanation of why these rings are the same: $\mathfrak{M}(\tilde{\mathcal{A}})_{\mathbb{C}}$ is homeomorphic to $\mathfrak{M}_\lambda(\tilde{\mathcal{A}})_{\mathbb{C}}$, which, by the proof of Proposition 3.4, is an affine space bundle over $W/T^k_{\mathbb{C}}$.*

4 The Betti numbers of $\mathfrak{M}(\mathcal{A})$

Our aim in this section is to prove an analogue of Theorem 3.5 for the intersection cohomology of the singular variety $\mathfrak{M}(\mathcal{A})$. Intersection cohomology was defined for complex varieties in [GM1, GM2]. The sheaf theoretic definition naturally extends to an ℓ-adic étale version for varieties in positive characteristic, which was studied extensively in [BBD].

Let Y be a variety of dimension m, defined over the integers, with a stratification

$$Y = \bigsqcup_{\beta \in B} Y_\beta.$$

Let us suppose further that for all β, the normal slice S_β to the stratum Y_β is an affine cone, meaning that it is equipped with an action of the multiplicative group $\mathbb{G}_m$ having the basepoint s as its unique fixed point, and that s is an attracting fixed point. Let $IH^*(Y)$ denote the global ℓ-adic étale intersection cohomology of $Y_{\bar{\mathbb{F}}_p}$ for p a large prime, and $IH^*_\beta(Y)$ the local ℓ-adic étale intersection cohomology at any point in the stratum $(Y_\beta)_{\bar{\mathbb{F}}_p}$. Since local intersection cohomology is preserved by any étale map, and the global intersection cohomology of a cone is the same as the local intersection cohomology at the vertex by [KL, §3], we have natural isomorphisms

$$IH^*_\beta(Y) \cong IH^*_s(S_\beta) \cong IH^*(S_\beta) \tag{8}$$

for all $\beta \in B$.

In this case, let

$$P_Y(q) = \sum_{i=0}^{m-1} \dim IH^{2i}(Y) \cdot q^i$$

be the even degree intersection cohomology Poincaré polynomial of Y, and let

$$P^\beta_Y(q) = \sum_i \dim IH^{2i}_\beta(Y) \cdot q^i$$

be the corresponding Poincaré polynomial for the local intersection cohomology at a point in Y_β. (In the cases of interest to us, odd degree global and local cohomology will always vanish.) Provided that p is chosen large enough, these polynomials agree with the Poincaré polynomials for global and local topological intersection cohomology of the complex analytic space $Y_{\mathbb{C}}$ by [BBD, 6.1.9].

Let $\mathcal{T}$ be a class of stratified schemes over $\mathbb{Z}$ satisfying the following two conditions.

(1) For each stratum Y_β of $Y \in \mathcal{T}$, the normal slice S_β to Y_β in Y is isomorphic to an element of $\mathcal{T}$.

(2) For each $Y \in \mathcal{T}$, the group $IH^*(Y)$ is pure.

The following analogue of Theorem 3.1 is a generalization of the main result of [KL, §4], in which the place of $\mathcal{T}$ was taken by the class consisting of the intersections of Schubert varieties and opposite Schubert cells.

Theorem 4.1. *Suppose that every element of $\mathcal{T}$ has polynomial point count. Then all global and local intersection cohomology groups of elements of $\mathcal{T}$ vanish in odd degree, and for all $Y \in \mathcal{T}$, we have*

$$q^m \cdot P_Y(q^{-1}) = \sum_{\beta \in B} P_Y^\beta(q) \cdot \nu_{Y_\beta}(q). \tag{9}$$

Proof: Let $\mathrm{Fr}_*^s : IH^*(Y_{\bar{\mathbb{F}}_p}) \to IH^*(Y_{\bar{\mathbb{F}}_p})$ be the map induced by the s^{th} power of the Frobenius automorphism $\mathrm{Fr} : Y_{\bar{\mathbb{F}}_p} \to Y_{\bar{\mathbb{F}}_p}$. Purity of IH^* implies that the eigenvalues of Fr_*^s on IH^i all have absolute value $p^{is/2}$. The polynomial point count hypothesis implies that each eigenvalue α of Fr_* must satisfy $\alpha^s = f(p^s)$ for some polynomial f. This is only possible when $f(x) = x^{i/2}$ and i is even. Thus odd cohomology vanishes, and the eigenvalues of Fr_*^s on IH^{2i} is p^{si}. Since IH_β^* is isomorphic to the global intersection cohomology of the normal slice S_β, and $IH^*(S_\beta)$ is pure by conditions (1) and (2), the odd cohomology vanishes and eigenvalues of Fr_*^s are all p^{is} for IH_β^* as well. Thus

$$P_Y(p^s) = \mathrm{Tr}\left(\mathrm{Fr}_*^s, IH^*\right) \quad \text{and} \quad P_Y^\beta(p^s) = \mathrm{Tr}\left(\mathrm{Fr}_*^s, IH_\beta^*\right). \tag{10}$$

By Poincaré duality and the Lefschetz formula [KW, II.7.3 & III.12.1(4)], we have

$$\begin{aligned} p^{ms} \cdot \mathrm{Tr}\left(\mathrm{Fr}_*^{-s}, IH^*\right) &= \mathrm{Tr}\left(\mathrm{Fr}_*^s, IH_c^*\right) \\ &= \sum_{\mathrm{Fr}^s(y)=y} \mathrm{Tr}\left(\mathrm{Fr}_*^s, IH_y^*\right) \\ &= \sum_{\beta \in B} \nu_{Y_\beta}(p^s) \cdot \mathrm{Tr}\left(\mathrm{Fr}_*^s, IH_\beta^*\right). \end{aligned} \tag{11}$$

Equation (9) follows immediately from substitution of (10) into (11). □

Let $\mathcal{A}$ be a central hyperplane arrangement as in Section 2, and let $P_{\mathcal{A}}(q) = P_{\mathfrak{M}(\mathcal{A})}(q)$. Our goal is to show that $\mathfrak{M}(\mathcal{A})$ has polynomial point count, and to use Theorem 4.1 to compute its intersection cohomology Poincaré polynomial. Let

$$M(\mathcal{A}) = (\mathfrak{t}^d)^* \smallsetminus \bigcup_{i=1}^n H_i$$

be the complement of $\mathcal{A}$ in $(\mathfrak{t}^d)^*$. Then $M(\mathcal{A})$ has polynomial point count, and the polynomial $\chi_{\mathcal{A}}(q) := \nu_{M(\mathcal{A})}(q)$ is known as the **characteristic polynomial** of $\mathcal{A}$ [At, 2.2]. Let $r(\mathcal{A})$ denote the number of components of the real manifold $M(\mathcal{A})_{\mathbb{R}}$.

Proposition 4.2. *The hypertoric variety $\mathring{\mathfrak{M}}(\mathcal{A})$ has polynomial point count, with*

$$\nu_{\mathring{\mathfrak{M}}(\mathcal{A})}(q) = (q-1)^d \cdot \sum_F \chi_{\mathcal{A}^F}(q) \cdot r(\mathcal{A}_F).$$

Proof: Consider the decomposition

$$(\mathfrak{t}^d)^* = \bigsqcup_F M(\mathcal{A}^F)$$

into complements of restrictions of the arrangement $\mathcal{A}$ to various flats. We will count points of $\mathring{\mathfrak{M}}(\mathcal{A})$ on the individual fibers of the moment map $\Phi : \mathfrak{M}(\mathcal{A}) \to (\mathfrak{t}^d)^*$, and then add up the contributions of each fiber. The fiber

$$\Phi^{-1}(0) = \big\{[z,w] \mid z_i w_i = 0 \text{ for all } i\big\}$$

is called the **extended core** of $\mathfrak{M}(\mathcal{A})$, and we will denote it $\mathcal{L}_{ext}(\mathcal{A})$. The extended core is T^d-equivariantly isomorphic to a union of affine toric varieties, with moment polytopes equal to the closures of the components of $M(\mathcal{A})$ [HP, §2]. An element $[z,w] \in \mathcal{L}_{ext}(\mathcal{A})$ lies on the toric divisor corresponding to the hyperplane H_i if and only if $z_i = w_i = 0$ [BD, §3.1], hence the generic points on $\mathcal{L}_{ext}(\mathcal{A})$ consist precisely of the free T^d orbits. There is one such orbit for every component of $M(\mathcal{A})_{\mathbb{R}}$, hence the number of generic $\mathbb{F}_q$ points in the fiber $\Phi^{-1}(0)$ is equal to $(q-1)^d \cdot r(\mathcal{A})$.

Fix a flat F, and let $T^{\operatorname{rk} F}$ be the image of T^F in T^d, as in Section 2. Then $T^{\operatorname{rk} F}$ acts on $\mathfrak{M}(\mathcal{A}_F)$, and the projection $p_F : \mathfrak{M}(\mathcal{A}) \to \mathfrak{M}(\mathcal{A}_F)$ is $T^{\operatorname{rk} F}$-equivariant. Choose a point $x \in M(\mathcal{A}^F) \subseteq (\mathfrak{t}^d)^*$. Recall from the proof of Lemma 2.4 that we have a map p_F from an open subset of $\mathfrak{M}(\mathcal{A})$ to $\mathfrak{M}(\mathcal{A}_F)$. This open subset includes $\Phi^{-1}(x)$, and the restriction of p_F maps $\Phi^{-1}(x)$ surjectively onto $\mathcal{L}_{ext}(\mathcal{A}_F)$. The generic points in $\Phi^{-1}(x)$ are precisely those points which map to generic points of $\mathfrak{M}(\mathcal{A}_F)$. Choose a subset $S \subseteq \{1, \ldots, n\}$ of size $\operatorname{crk} F$ such that $\{a_i \mid i \in F \cup S\}$ is a spanning set for $\mathfrak{t}^d$, and let T' be the image in T^d of the coordinate torus T^S. Then T' acts freely on the generic fibers of p_F, and by dimension count, this action is transitive as well. Hence

$$\nu_{\Phi^{-1}(x)}(q) = \nu_{T'}(q) \cdot \nu_{T^{\operatorname{rk} F}}(q) \cdot r(\mathcal{A}_F) = (q-1)^d \cdot r(\mathcal{A}_F).$$

Summing over all $x \in M(\mathcal{A}^F)$ contributes a factor of $\chi_{\mathcal{A}^F}(q)$, and summing over all flats F yields the desired formula. □

Since every stratum of $\mathfrak{M}(\mathcal{A})$ is itself the generic stratum of some hypertoric variety, Proposition 4.2 implies that $\mathfrak{M}(\mathcal{A})$ has polynomial point count, and furthermore gives us a combinatorial formula for counting points on each stratum. Let $\mathcal{T}$ be the class of all hypertoric varieties corresponding to central arrangements. By Lemma 2.4, $\mathcal{T}$ satisfies condition (1). To see that the intersection cohomology groups of hypertoric varieties are pure, we use the decomposition theorem of [BBD, 6.2.5], which will be discussed further in Section 5. This theorem implies that the projective map of Corollary 2.7 induces an injection of $IH^*(\mathfrak{M}(\mathcal{A}))$ into the ℓ-adic étale cohomology group of $\mathfrak{M}(\tilde{\mathcal{A}})_{\bar{\mathbb{F}}_p}$, which we observed was pure in Section 3. This injection is equivariant with respect to the Frobenius action, therefore $IH^*(\mathfrak{M}(\mathcal{A}))$ is pure as well. Let $P_{\mathcal{A}}(q) = P_{\mathfrak{M}(\mathcal{A})}(q)$. Combining Theorem 4.1 with Proposition 4.2, Lemma 2.4, and the isomorphism (8) produces the following equation:

$$q^{2d} \cdot P_{\mathcal{A}}(q^{-1}) = \sum_F P_{\mathcal{A}_F}(q) \cdot (q-1)^{\operatorname{crk} F} \cdot \sum_{G \supseteq F} \chi_{\mathcal{A}^G}(q) \cdot r(\mathcal{A}^F_G), \tag{12}$$

where $\mathcal{A}^F_G = (\mathcal{A}^F)_G = (\mathcal{A}_G)^F$.

Theorem 4.3. *The intersection cohomology Poincaré polynomial of $\mathfrak{M}(\mathcal{A})$ coincides with the h-polynomial of $\operatorname{bc}_\sigma \Delta_{\mathcal{A}}$, that is $P_{\mathcal{A}}(q) = h^{br}_{\mathcal{A}}(q)$.*

Proof: The polynomial $P_{\mathcal{A}}(q)$ is completely determined by Equation (12) and the fact that $\deg P_{\mathcal{A}}(q) \leq d-1$. It therefore suffices to prove the recursion

$$q^{2d} \cdot h^{br}_{\mathcal{A}}(q^{-1}) = \sum_F h^{br}_{\mathcal{A}_F}(q) \cdot (q-1)^{\operatorname{crk} F} \cdot \sum_{G \supseteq F} \chi_{\mathcal{A}^G}(q) \cdot r(\mathcal{A}^F_G).$$

We proceed by expressing every piece of the equation in terms of the Möbius function[8]

$$\mu : L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{Z}.$$

The function μ is defined by the recursion

$$\mu(F,G) = 0 \text{ unless } F \subseteq G, \quad \text{and if } F \subseteq G, \text{ then } \sum_{F \subseteq H \subseteq G} \mu(H,G) = \delta(F,G),$$

where δ is the Kronecker delta function. Let $\mu(F) = \mu(\emptyset, F)$ for all flats $F \in L(\mathcal{A})$. We may express all relevant polynomials in terms of the Möbius function as follows [Bj, §7.4][9]:

$$\chi_{\mathcal{A}}(q) = \sum_F \mu(F) q^{\operatorname{crk} F}, \quad r(\mathcal{A}) = (-1)^{\operatorname{rk} \mathcal{A}} \chi_{\mathcal{A}}(-1) = \sum_F (-1)^{\operatorname{rk} F} \mu(F),$$

$$\text{and} \quad h^{br}_{\mathcal{A}}(q) = (-q)^{\operatorname{rk} \mathcal{A}} \chi_{\mathcal{A}}(1-q^{-1}) = (-1)^{\operatorname{rk} \mathcal{A}} \sum_F \mu(F) q^{\operatorname{rk} F} (q-1)^{\operatorname{crk} F}.$$

[8]The Möbius function should not be confused with the moment map, for which we have also used the symbol μ. The Möbius function will not appear in this paper outside of the proof of Theorem 4.3.

[9]Note that Björner's definition of the h-polynomial differs from ours in that the order of the coefficients is reversed.

It follows that

$$\begin{aligned}
&\sum_F h^{br}_{\mathcal{A}_F}(q)\cdot(q-1)^{\operatorname{crk}F}\cdot\sum_{G\supseteq F}\chi_{\mathcal{A}^G}(q)\cdot r(\mathcal{A}^F_G)\\
=&\sum_{H\subseteq F\subseteq J\subseteq G\subseteq I}(-1)^{\operatorname{rk}F}\mu(H)q^{\operatorname{rk}H}(q-1)^{\operatorname{rk}F-\operatorname{rk}H}\cdot(q-1)^{\operatorname{crk}F}\cdot\mu(G,I)q^{\operatorname{crk}I}\cdot\mu(F,J)(-1)^{\operatorname{rk}J-\operatorname{rk}F}\\
=&\sum_{H\subseteq F\subseteq J\subseteq G\subseteq I}\mu(H)q^{\operatorname{rk}H}(q-1)^{\operatorname{crk}H}\cdot\mu(G,I)q^{\operatorname{crk}I}\cdot\mu(F,J)(-1)^{\operatorname{rk}J}.
\end{aligned}$$

We now apply the recursive definition of μ twice, once to the sum over F and once to the sum over G, to obtain a sum over a single variable. This yields

$$\begin{aligned}
\sum_F(-1)^{\operatorname{rk}F}\mu(F)q^{\operatorname{rk}F+\operatorname{crk}F}(q-1)^{\operatorname{crk}F} &= q^{\operatorname{rk}\mathcal{A}}\sum_F(-1)^{\operatorname{rk}F}\mu(F)(q-1)^{\operatorname{crk}F}\\
&= (-1)^{\operatorname{rk}\mathcal{A}}q^{2\operatorname{rk}\mathcal{A}}\sum_F\mu(F)q^{-\operatorname{rk}F}(q^{-1}-1)^{\operatorname{crk}F}\\
&= q^{2\operatorname{rk}\mathcal{A}}\cdot h^{br}_{\mathcal{A}}(q^{-1}).
\end{aligned}$$

Since $d=\operatorname{rk}\mathcal{A}$, this completes the recursion, and therefore the proof of Theorem 4.3. $\square$

5 The KRS convolution formula

In this section we use the decomposition theorem of [BBD, 6.2.5] to compare the intersection cohomology groups of $\mathfrak{M}(\mathcal{A})$ to the ordinary cohomology groups of its resolution $\mathfrak{M}(\tilde{\mathcal{A}})$. By the results of Sections 3 and 4, we know that the formula that we obtain will involve the h-numbers of matroid complexes and their broken circuit complexes. In fact, this formula turns out to be a special case of the Kook-Reiner-Stanton convolution formula, which is proven by combinatorial means in [KRS].[10]

Rather than stating the decomposition theorem for arbitrary projective maps $f:X\to Y$, we specialize to the case where X is a complex orbifold, $Y=\sqcup_{\beta\in B}Y_\beta$ is a stratified complex variety, and f is semismall. For the remainder of the paper we will always work over the complex numbers, and omit the subscript $\mathbb{C}$. Let n_β be the codimension on Y_β inside of Y.

Proposition 5.1. [BM, 4], [Gi, 5.4] *There is a direct sum decomposition*

$$H^*(X)=\bigoplus_{\beta\in B}IH^*(\overline{Y_\beta};\xi_\beta)[-n_\beta],$$

where ξ_β is the local system $R^{n_\beta}f_*\mathbb{C}_{X_\beta}$. If this local system is trivial for all β, then

$$H^*(X)=\bigoplus_{\beta\in B}IH^*(\overline{Y_\beta})\otimes H^{n_\beta}(\pi^{-1}(y_\beta)),$$

where $y_\beta\in Y_\beta$, and $H^{n_\beta}(\pi^{-1}(Y_\beta))$ is understood to lie in degree n_β.

[10] We thank Ed Swartz for this observation.

We begin by showing that in the hypertoric setting, the affinization map induces trivial local systems.

Proposition 5.2. *For any flat F of $\mathcal{A}$, the local system ξ_F on $\mathring{\mathfrak{M}}(\mathcal{A}^F)$ induced by the affinization map $\pi : \mathfrak{M}(\tilde{\mathcal{A}}) \to \mathfrak{M}(\mathcal{A})$ is trivial.*

Proof: The stratum $\mathring{\mathfrak{M}}(\mathcal{A}^F)$ admits a free action of $T^{\operatorname{crk} F} = T^d/T^{\operatorname{rk} F}$, where $T^{\operatorname{rk} F}$ is the image of the coordinate torus $T^F \subseteq T^n$, as in Section 2. Let $T_{\mathbb{R}}^{\operatorname{crk} F}$ be the compact real form of $T^{\operatorname{crk} F}$. The local system ξ_F is naturally $T_{\mathbb{R}}^{\operatorname{crk} F}$-equivariant, which means that it may be pulled back from a local system on the quotient. The quotient space $\mathring{\mathfrak{M}}(\mathcal{A}^F)/T_{\mathbb{R}}^{\operatorname{crk} F}$ is homeomorphic, via the *hyperkähler* moment map, to the space

$$H_F^{\mathbb{R}} \oplus H_F^{\mathbb{C}} \smallsetminus \bigcup_{i \in F^c} H_{F\cup\{i\}}^{\mathbb{R}} \oplus H_{F\cup\{i\}}^{\mathbb{C}} \quad \text{[BD, §3.1]}.$$

Since we are removing linear subspaces of real codimension three, the resulting space is simply connected, thus all of its local systems are trivial. □

By Corollary 2.6, Corollary 2.7, Theorem 5.1, and Proposition 5.2, we obtain the following isomorphism:

$$H^*\big(\mathfrak{M}(\tilde{\mathcal{A}})\big) \cong \bigoplus_{F \in L(\mathcal{A})} IH^*\big(\mathfrak{M}(\mathcal{A}^F)\big) \otimes H^{2\operatorname{rk} F}\big(\mathcal{L}(\tilde{\mathcal{A}}_F)\big). \tag{13}$$

Remark 5.3. *The core $\mathcal{L}(\tilde{\mathcal{A}}_F)$ has a component of dimension $\operatorname{rk} F$ if and only if F is coloop-free (Remark 2.3), hence only coloop-free flats contribute to the right hand side.*

By Theorems 3.5 and 4.3, Equation (13) translates into the following recursion.

Corollary 5.4. $h_{\mathcal{A}}(q) = \sum_{F \in L(\mathcal{A})} h_{\mathcal{A}^F}^{br}(q) \cdot h_{\operatorname{rk} F}(\mathcal{A}_F)\, q^{\operatorname{rk} F}.$

Remark 5.5. *The* **Tutte polynomial** *$T_{\mathcal{A}}(x, y)$ of $\Delta_{\mathcal{A}}$ is a bivariate polynomial invariant of matroids with several combinatorially significant specializations; see, for example,* [Bj, 7.12 & 7.15]. *In particular, we have*

$$h_{\mathcal{A}}(q) = q^{\operatorname{rk}\mathcal{A}} T_{\mathcal{A}}(q^{-1}, 1),$$

$$h_{\mathcal{A}}^{br}(q) = q^{\operatorname{rk}\mathcal{A}} T_{\mathcal{A}}(q^{-1}, 0),$$

$$\textit{and} \quad h_{\operatorname{rk}\mathcal{A}}(\Delta_{\mathcal{A}}) = T_{\mathcal{A}}(0, 1).$$

Corollary 5.4 is the specialization at $x = q^{-1}$ and $y = 1$ of the following recursion [KRS, 1]:

$$T_{\mathcal{A}}(x, y) = \sum_{F \in L(\mathcal{A})} T_{\mathcal{A}^F}(x, 0) \cdot T_{\mathcal{A}_F}(0, y).$$

6 Cohomology rings

In Sections 3 and 4, we computed the Betti numbers of $\mathfrak{M}(\tilde{\mathcal{A}})$ and $\mathfrak{M}(\mathcal{A})$. In this section, we discuss the equivariant cohomology ring $H^*_{T^d}\big(\mathfrak{M}(\tilde{\mathcal{A}});\mathbb{C}\big)$, and the equivariant intersection cohomology group $IH^*_{T^d}\big(\mathfrak{M}(\mathcal{A});\mathbb{C}\big)$. (As in Section 5, we will consider all of our varieties exclusively over the complex numbers.) In general, while ordinary cohomology is a ring, intersection cohomology groups have no naturally defined ring structure. Nonetheless, in the case of a hypertoric variety defined by a unimodular arrangement, we conjecture that that there is a natural isomorphism between $IH^*_{T^d}\big(\mathfrak{M}(\mathcal{A});\mathbb{C}\big)$ and a combinatorially defined ring $R(\mathcal{A})$, and furthermore that the multiplication in this ring has a geometric interpretation.

We begin with the ring $H^*_{T^d}\big(\mathfrak{M}(\tilde{\mathcal{A}});\mathbb{C}\big)$, which was computed independently in [Ko] and [HS]. Hausel and Sturmfels observed that this ring is isomorphic to the Stanley-Reisner ring of the matroid complex $\Delta_{\mathcal{A}}$.

Theorem 6.1. [HS, 5.1],[Ko, 3.1] *There are natural ring isomorphisms $H^*_{T^d}\big(\mathfrak{M}(\tilde{\mathcal{A}});\mathbb{C}\big) \cong \mathcal{SR}(\Delta_{\mathcal{A}})$ and $H^*\big(\mathfrak{M}(\tilde{\mathcal{A}});\mathbb{C}\big) \cong \mathcal{SR}_0(\Delta_{\mathcal{A}})$ that reduce degrees by half.*

We now proceed to the singular case. The intersection cohomology group $IH^*_{T^d}\big(\mathfrak{M}(\mathcal{A});\mathbb{C}\big)$ is a module over $H^*_{T^d}(pt)$, and, as in the smooth case, it is a free module [GKM, 14.1(1)]. We *could* therefore identify $IH^*_{T^d}\big(\mathfrak{M}(\mathcal{A});\mathbb{C}\big)$ with the Stanley-Reisner ring of the broken circuit complex $\mathrm{bc}_\sigma\Delta_{\mathcal{A}}$, since this ring is also a free module over a polynomial ring of dimension d, and has the correct Hilbert series by Theorem 4.3. We submit, however, that this identification would not be natural. One immediate objection is that the broken circuit complex depends on a choice of ordering of the set $\{1,\ldots,n\}$, while the hypertoric variety $\mathfrak{M}(\mathcal{A})$ does not. Instead we introduce the the following ring, which (if $\mathcal{A}$ is unimodular) is a deformation of the Stanley-Reisner ring of the broken circuit complex for *any* choice of ordering.

For any circuit C of $\mathcal{A}$, there exist nonzero integers $\{\lambda_i \mid i \in C\}$, unique up to simultaneous scaling, such that $\sum_{i\in C}\lambda_i a_i = 0$. Let

$$f_C := \sum_{i\in C} \operatorname{sign}(\lambda_i)\cdot \prod_{j\in C\setminus\{i\}} e_j, \tag{14}$$

where $\operatorname{sign}(\lambda) = \frac{\lambda}{|\lambda|}$. Note that the leading term of f_C with respect to an ordering σ is equal to a multiple of the σ-broken circuit monomial corresponding to the circuit C. We now define the rings

$$R(\mathcal{A}) := \mathbb{C}[e_1,\ldots,e_n]\Big/\Big\langle\, f_C \;\Big|\; C \text{ a circuit}\,\Big\rangle \quad\text{and}\quad R_0(\mathcal{A}) := R(\mathcal{A}) \otimes_{\operatorname{Sym}(\mathfrak{t}^d)^*} \mathbb{C},$$

where $\operatorname{Sym}(\mathfrak{t}^d)^*$ acts on $R(\mathcal{A})$ via the inclusion $(\mathfrak{t}^d)^* \hookrightarrow (\mathfrak{t}^n)^* = \mathbb{C}\{e_1,\ldots,e_n\}$. If $\mathcal{A}$ is unimodular, then λ_i may be chosen to be plus or minus 1 for all i. In this case, $\operatorname{sign}(\lambda_i) = \lambda_i$, and the ring $R(\mathcal{A})$ may be interpreted as the subring of rational functions on $(\mathfrak{t}^d)^*$ generated by the inverses of the linear functions $\{a_i\}$ [PS, §1].

Theorem 6.2. [PS, 4,7] *If $\mathcal{A}$ is unimodular, then the set $\{ f_C \mid C \text{ a circuit}\}$ is a universal Gröbner basis for the ideal of relations in $R(\mathcal{A})$. Thus for any ordering σ, $R(\mathcal{A})$ is a deformation of the Stanley-Reisner ring $\mathcal{SR}(\mathrm{bc}_\sigma\Delta_\mathcal{A})$. Furthermore, $R(\mathcal{A})$ is a free module over $\mathrm{Sym}(\mathbf{t}^d)^*$, hence $R_0(\mathcal{A})$ has Hilbert series $h^{br}_\mathcal{A}(q)$.*

Theorem 6.2 fails if $\mathcal{A}$ is not unimodular. If, for example, we take $\mathcal{A}$ to consist of four lines in the plane, then we will find that the Krull dimension of $R(\mathcal{A})$ is 1, and the Krull dimension of $\mathcal{SR}(\mathrm{bc}_\sigma\Delta_\mathcal{A})$ is 2. Theorems 4.3 and 6.2 together give us the following corollary.

Corollary 6.3. *If $\mathcal{A}$ is unimodular, then there exist graded vector space isomorphisms*

$$IH^*_{T^d}\big(\mathfrak{M}(\mathcal{A});\mathbb{C}\big) \cong R(\mathcal{A}) \quad \textit{and} \quad IH^*\big(\mathfrak{M}(\mathcal{A});\mathbb{C}\big) \cong R_0(\mathcal{A})$$

that reduce degrees by half.

Conjecture 6.4. *These isomorphisms are natural, and the multiplication in the ring $R(\mathcal{A})$ may be interpreted as an intersection pairing on $\mathfrak{M}(\mathcal{A})$.*

We conclude this section by providing evidence for Conjecture 6.4. Although intersection cohomology is not functorial with respect to arbitrary maps, any map between stratified spaces with the property that perverse i-chains on the source push forward to perverse i-chains on the target induces a pullback in intersection cohomology. In [GM2, §5.4], the authors define the notion of a **normally nonsingular map**, and prove that such maps have this property.

For any flat F of $\mathcal{A}$, consider the map $s_F : \mathfrak{M}(\mathcal{A}_F) \to \mathfrak{M}(\mathcal{A})$ which exhibits $\mathfrak{M}(\mathcal{A}_F)$ as a normal slice to the stratum $\mathring{\mathfrak{M}}(\mathcal{A}^F) \subseteq \mathfrak{M}(\mathcal{A})$. The stratification of $\mathfrak{M}(\mathcal{A}_F)$ is pulled back from the stratification of $\mathfrak{M}(\mathcal{A})$, hence s_F is a normally nonsingluar inclusion. It is also T^d-equivariant, where T^d acts on $\mathfrak{M}(\mathcal{A}_F)$ by first projecting onto $T^{\mathrm{rk}\,F}$. This means that if Conjecture 6.4 is correct, then we should expect to find a map of rings from $R(\mathcal{A})$ to $R(\mathcal{A}_F)$.

The ring $R(\mathcal{A})$ is defined to be a quotient of $\mathbb{C}[e_1,\ldots,e_n]$, while $R(\mathcal{A}_F)$ is a quotient of $\mathbb{C}[e_i]_{i\in F}$. Let $s(e_i) = e_i$ for all $i \in F$, and and zero otherwise. To check that s is well defined, we must examine its behavior on the element f_C for every circuit C of $\mathcal{A}$. If C is contained in F, then it is also a circuit of $\mathcal{A}_F$, and is therefore zero in $R(\mathcal{A}_F)$. If C is not contained in F, then the fact that F is a flat implies that $|C \cap F^c| \geq 2$, and therefore that $s(f_C) = 0$. Thus there is a map from $R(\mathcal{A})$ to $R(\mathcal{A}_F)$ arising naturally from the combinatorial perspective. The inclusion $\mathfrak{M}(\mathcal{A}^F) \hookrightarrow \mathfrak{M}(\mathcal{A})$ of Proposition 2.1 does not push perverse chains forward to perverse chains, therefore we do not expect to find a natural map from $R(\mathcal{A})$ to $R(\mathcal{A}^F)$, and indeed we cannot.

The inclusion of the generic stratum $\mathring{\mathfrak{M}}(\mathcal{A})$ into $\mathfrak{M}(\mathcal{A})$ is a good map, therefore there is a natural restriction homomorphism ρ from $IH^*_{T^d}\big(\mathfrak{M}(\mathcal{A});\mathbb{C}\big)$ to $H^*_{T^d}(\mathring{\mathfrak{M}}(\mathcal{A});\mathbb{C})$. We know that $IH^*_{T^d}\big(\mathfrak{M}(\mathcal{A});\mathbb{C}\big)$ is concentrated in even degree, and by the discussion that follows, so is $H^*_{T^d}(\mathring{\mathfrak{M}}(\mathcal{A});\mathbb{C})$. A spectral sequence argument using parity vanishing along the lines of [BGS, 3.4.1] shows that the relative equivariant intersection cohomology $IH^*_{T^d}\big(\mathfrak{M}(\mathcal{A}), \mathring{\mathfrak{M}}(\mathcal{A});\mathbb{C}\big)$ also has no odd cohomology, thus the long exact sequence for intersection cohomology tells us that ρ is surjective.

The T^d-equivariant cohomology of $\mathring{\mathfrak{M}}(\mathcal{A})$ is isomorphic to the $T^d_{\mathbb{R}}$-equivariant cohomology, where $T^d_{\mathbb{R}}$ is the compact real form of T^d. As we observed in the proof of Proposition 5.2, $T^d_{\mathbb{R}}$ acts freely on $\mathring{\mathfrak{M}}(\mathcal{A})$ with quotient

$$N(\mathcal{A}) := (\mathfrak{t}^d)^*_{\mathbb{R}} \oplus (\mathfrak{t}^d)^*_{\mathbb{C}} \smallsetminus \bigcup_{i=1}^{n} H_i^{\mathbb{R}} \oplus H_i^{\mathbb{C}} \quad \text{[BD, §3.1]}.$$

This space is the complement of a collection of codimension 3 real linear subspaces of the vector space $(\mathfrak{t}^d)^*_{\mathbb{R}} \oplus (\mathfrak{t}^d)^*_{\mathbb{C}}$, with intersection lattice identical to that of $\mathcal{A}$. The equivariant cohomology ring of $\mathring{\mathfrak{M}}(\mathcal{A})$ is isomorphic to the ordinary cohomology ring of $N(\mathcal{A})$, which is shown in [dLS, 5.6] to be isomorphic to $R(\mathcal{A})/\langle e_1^2, \ldots, e_n^2\rangle$, with $\deg e_i = 2$ for all i.[11] Thus $R(\mathcal{A})$ surjects naturally onto $H^*_{T^d}(\mathring{\mathfrak{M}}(\mathcal{A}); \mathbb{C})$, providing further evidence for Conjecture 6.4.

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