# Parallel connections and bundles of arrangements *

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**Abstract**

Let $\mathcal{A}$ be a complex hyperplane arrangement, and let $X$ be a modular element of arbitrary rank in the intersection lattice of $\mathcal{A}$. Projection along $X$ restricts to a fiber bundle projection of the complement of $\mathcal{A}$ to the complement of the localization $\mathcal{A}_X$ of $\mathcal{A}$ at $X$. We identify the fiber as the decone of a realization of the complete principal truncation of the underlying matroid of $\mathcal{A}$ along the flat corresponding to $X$. We then generalize to this setting several properties of strictly linear fibrations, the case in which $X$ has corank one, including the triviality of the monodromy action on the cohomology of the fiber. This gives a topological realization of results of Stanley, Brylawsky, and Terao on modular factorization. We also show that (generalized) parallel connection of matroids corresponds to pullback of fiber bundles, clarifying the notion that all examples of diffeomorphisms of complements of inequivalent arrangements result from the triviality of the restriction of the Hopf bundle to the complement of a hyperplane. The modular fibration theorem also yields a new method for identifying $K(\pi,1)$ arrangements of rank greater than three. We exhibit new families of $K(\pi,1)$ arrangements, providing more evidence for the conjecture that factored arrangements of arbitrary rank are $K(\pi,1)$.

## 1 Introduction

Let $V$ be a vector space over a field $\mathbb{K}$. An *arrangement* $\mathcal{A}$ in $V$ is a finite collection of linear hyperplanes in $V$. The *complement* $M = M(\mathcal{A})$ of $\mathcal{A}$ is $V - \bigcup \mathcal{A}$. A set of hyperplanes $\mathcal{B}$ is *dependent* if the $\operatorname{codim}(\bigcap \mathcal{B}) < |\mathcal{B}|$. These dependent sets determine a matroid $G(\mathcal{A})$ with ground set $\mathcal{A}$, the *underlying matroid* of $\mathcal{A}$. Alternatively, $G(\mathcal{A})$ is the linear matroid realized by the projective point configuration $\mathcal{A}^*$ in $\mathbb{P}(V^*)$ determined by the defining linear forms for the hyperplanes of $\mathcal{A}$.

In case $\mathbb{K} = \mathbb{C}$ the complement $M(\mathcal{A})$ is a connected manifold whose topology has been studied in great detail. In this case there is a strong connection between the topological structure of $M(\mathcal{A})$ and the underlying matroid $G(\mathcal{A})$. The paradigmatic result along these lines is that the cohomology of $M(\mathcal{A})$ has a presentation depending only on $G(\mathcal{A})$, with the consequence that the Poincaré

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series of the cohomology ring of $M(\mathcal{A})$ essentially coincides with the characteristic polynomial of $G(\mathcal{A})$ [22]. It has become clear that techniques and constructions from matroid theory can have interesting and surprising implications for the topology of hyperplane complements. In this paper we interpret the matroidal notions of modular flat, principal truncation, and generalized parallel connection in this vein, in terms of bundles of complex hyperplane arrangements, their fibers, and pullbacks via inclusion maps.

Henceforth we restrict our study to complex arrangements. The *intersection lattice* $L = L(\mathcal{A})$ of $\mathcal{A}$ is the set of subspaces $X$ of $\mathbb{C}^\ell$ which are intersections of hyperplanes of $\mathcal{A}$, $X = \bigcap \mathcal{B}$ for $\mathcal{B} \subseteq \mathcal{A}$, partially ordered by reverse inclusion. The smallest element of $L$ is $O_L = \mathbb{C}^\ell$, the empty intersection, and the largest element of $L$ is $1_L = \bigcap \mathcal{A}$. For $X, Y \in L(\mathcal{A})$, the *join* $X \vee Y$ is $X \cap Y$ and the *meet* $X \wedge Y$ is $\bigcap\{H \in \mathcal{A} \mid H \supseteq X + Y\}$. The rank function $r$ of $L$ is given by $r(X) = \operatorname{codim}(X)$, and the semimodular law holds:

$$r(X \wedge Y) + r(X \vee Y) \leq r(X) + r(Y)$$

for $X, Y \in L$. Then $L$ is a geometric lattice, isomorphic to the lattice of flats of the matroid $G(\mathcal{A})$, via the identification of $X \in L$ with the flat

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supseteq X\}.$$

We will often refer to elements $X \in L(\mathcal{A})$ as flats, tacitly identifying $X$ with $\mathcal{A}_X$. For instance, "point" and "line" refer to flats of rank one and two. The *corank* of a flat $X$ is $r(1_L) - r(X)$, and "copoints" and "colines" are flats of corank one and two.

When equality holds in the formula above, $(X, Y)$ is called a *modular pair*. An element $X \in L(\mathcal{A})$ is *modular* if $(X, Y)$ is a modular pair for every $Y \in L(\mathcal{A})$. This is equivalent to the condition that $X+Y$ be an element of $L$ for every $Y \in L$. Let $\pi$ be the linear projection of $\mathbb{C}^\ell$ onto the quotient $\mathbb{C}^\ell/X$. Modularity of $X$ implies that fibers of $\pi$, the parallel translates of $X$, intersect each $Y \in L(\mathcal{A})$ in the same way, independent of position. This observation was already made by Terao in [32], who proved that $\pi|_{M(\mathcal{A})}$ is a fiber bundle projection in case $X$ has corank one. But, in fact, it is easy to show that modularity of $X$ is equivalent to $\pi$ being a map of stratified spaces, under the natural stratifications of $\mathbb{C}^\ell$ and $\mathbb{C}^\ell/X$ determined by $\mathcal{A}$ and $\mathcal{A}_X$. Being a linear projection, it is trivial to show $\pi$ restricts to a submersion on each stratum. L. Paris showed how to extend $\pi$ to a proper map of stratified spaces. Then Thom's Isotopy Lemma implies that $\pi|_{M(\mathcal{A})}$ is a fiber bundle projection for $X$ a modular flat of arbitrary rank.

This fibration result interpolates between two well-known extreme cases. In case $X$ is a modular copoint, the result was proven in [32], as already mentioned. This case gives rise to the notion of supersolvable arrangement, and its connection with fiber-type arrangements [13], a much-studied class [31, 20, 2, 5]. In case $X$ is a point, i.e., a hyperplane of $\mathcal{A}$, then $X$ is automatically modular, and the fibration is just the restriction of the defining form $\phi : \mathbb{C}^\ell \longrightarrow \mathbb{C}$ of the hyperplane $X$. This gives rise to the well-known elementary "cone-decone" construction [23]. The restriction $\phi|_{M(\mathcal{A})} : M(\mathcal{A}) \longrightarrow \mathbb{C}^*$ is in fact a trivial fibration, with fiber isomorphic to the complement in $\mathbb{C}^{\ell-1}$ of an affine arrangement, the *decone* of $\mathcal{A}$.

The general modular fibration theorem was proved by L. Paris [25]. At the same time, we were independently conducting the research reported on in this paper [28], and had arrived at the same conclusion, only to later discover an error in our treatment of the proper extension of $\pi$. We sketch the argument here, and refer the reader to [25] for a complete proof, concentrating instead on other structural results and consequences of the theorem.

In [32], Terao establishes the result for modular copoints, and proves that for general modular $X$ the fibers of $\pi$ have the same combinatorial type. But he specifically remarks that a proof of local triviality in the general case is not at hand. See Remark 2.7. The proof of Corollary 3.2 is in a sense a parametrized version of the argument of [29], where stratification techniques were first used in the theory of arrangements, several years after Terao's work.

The characteristic polynomial of a lattice was defined by G.-C. Rota. The characteristic polynomial of a matroid is the characteristic polynomial of its lattice of flats. The modular flat $X$ gives rise to a factorization of the characteristic polynomial of $G(\mathcal{A})$ over the integers, with one factor given by the characteristic polynomial of $G(\mathcal{A}_X)$. This is Stanley's modular factorization theorem [30]. Brylawski [3] identified the other factor as the characteristic polynomial of a related matroid, the *complete principal truncation* $\overline{T}_X(G)$ [36, Section 7.4] of $G = G(\mathcal{A})$ along $X$, divided by $(t-1)$. The complete principal truncation is obtained by successively adjoining generic points on the specified flat and contracting on the new points. Technically this is a matroid with multiple points; when we refer to $\overline{T}_X(G)$ we will always mean the associated simple matroid (with the same lattice of flats).

We show in Theorem 2.1 that the fiber of the bundle map $\pi|_{M(\mathcal{A})}$ is the complement of the decone of an arrangement realizing the complete principal truncation of $G(\mathcal{A})$ on the flat $X$. In addition, just as in the corank-one case [13], the monodromy of the bundle is shown to act trivially on the cohomology of the fiber (Theorem 3.5). Then the $E_2$ term in the Leray-Serre spectral sequence of $\pi|_{M(\mathcal{A})}$ is isomorphic to the tensor product of the cohomology of the base $M(\mathcal{A}_X)$ with that of the fiber. Using the identity relating characteristic polynomials and Poincaré polynomials, we obtain a topological interpretation of the Stanley-Brylawski and Terao factorization results. In fact, the factorization of the characteristic polynomial implies that the spectral sequence degenerates at the $E_2$ term, just as in the corank-one case, although we have no topological proof of this fact (Remark 3.10).

In the corank-one situation, the monodromy of the bundle gives rise to a "braid monodromy" homomorphism from $\pi_1(M)$ to the (pure) braid group on $n$ strands, where $n = |\mathcal{A} - \mathcal{A}_X|$. In the general case the analogue of this braid monodromy takes values in the fundamental group of the matroid stratum of the Grassmannian, or equivalently, the projective realization space, of the complete principal truncation $\overline{T}_X(G)$. See Remark 3.4.

The current research grew out of an attempt to clarify and generalize the construction of [7], which involved arrangements whose matroids are parallel connections. We began by studying the matroidal notion of *generalized parallel connection*. Loosely speaking, this is the free sum of two matroids along a common flat. This free sum is well-defined if and only if the flat is modular in one of the matroids. Thus we were led to the consideration of modular flats. The combinatorial study of Sections 2 and 4 formed the main part of an NSF *Research Experiences for Undergraduates* project in the summer of 1997. This work was reported on in [28], which provided the groundwork for this paper.

Given the modular fibration result, we show that generalized parallel connection, in a natural realization in terms of complex arrangements, corresponds to the pullback of fiber bundles (Theorem 4.2). The construction of [7], which yields diffeomorphisms of the complements of arrangements with non-isomorphic matroids, uses ordinary parallel connection, in which the identified flats are points. Then the diffeomorphisms of [7] are a consequence of two elementary observations, that the cone-decone construction yields a trivial bundle, and that the pullback of a trivial bundle is trivial.

When the base and fiber of a modular fibration are both aspherical, it follows that the complement $M(\mathcal{A})$ is also aspherical. In this case $\mathcal{A}$ is called a $K(\pi,1)$ arrangement. The problem of identifying $K(\pi,1)$ arrangements has been an important one in the study of complex arrangements. There are two well-known classes of $K(\pi,1)$ arrangements, the supersolvable ones, which are abundant in all ranks, and the simplicial ones, which are rare in ranks greater than three. Other techniques for identifying $K(\pi,1)$ arrangements are mostly restricted to arrangements of rank three. See [10, 14, 12] for further exposition of the $K(\pi,1)$ problem.

Corollary 3.2 provides a method for identifying $K(\pi,1)$ arrangements in ranks greater than three. In the final section we exhibit new families of such arrangements, arising from the work of P. Edelman and V. Reiner [6] and D. Bailey [1] on threshold graphs and subarrangements of the Coxeter arrangement of type $B_\ell$. By the classification result of Bailey, these new examples are all "factored" [23, Section 3.3]. So our result provides more evidence for the conjecture that factored arrangements of arbitrary rank are $K(\pi,1)$ [12]. We also give an example of an arrangement of rank four which has two different modular colines. Then Corollary 3.2 implies that a certain arrangement of rank three, the cone of one of the fibers, is not $K(\pi,1)$, an arrangement to which existing techniques do not apply.

The search for examples of high-rank $K(\pi,1)$ arrangements was motivated by a suggestion of G. Ziegler several years ago concerning counterexamples to the "homotopy type conjecture," that complex arrangements with the same underlying matroid should have homotopy-equivalent complements [23]. This idea, laid out in [28], is to find $K(\pi,1)$ arrangements of high rank, whose underlying matroids have different characteristic polynomials, but have isomorphic generic rank-three truncations. Then generic 3-dimensional sections of these arrangements will have isomorphic underlying matroids, but non-isomorphic fundamental groups. Unfortunately we are so far unable to construct such examples using the technique of this paper.

## 2 Projections and principal truncations

In this section we establish terminology and analyze the combinatorics associated with projections of hyperplane arrangements. Let $\mathcal{A}$ be a central arrangement of hyperplanes in $\mathbb{C}^\ell$. Let $L = L(\mathcal{A})$ be the intersection lattice of $\mathcal{A}$, consisting of subspaces of $\mathbb{C}^\ell$ as described in the introduction.

Let $X \in L$. Let $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supseteq X\}$, and let $\pi : \mathbb{C}^\ell \longrightarrow \mathbb{C}^\ell/X$ be the natural projection. Note that $\pi$ maps each hyperplane $H \in \mathcal{A}_X$ to a hyperplane of $\mathbb{C}^\ell/X$. Henceforth we consider the arrangement $\mathcal{A}_X$ to be an arrangement in $\mathbb{C}^\ell/X$.

We shall have occasion to study arrangements formed by the intersections of the hyperplanes of $\mathcal{A}$ with a given affine subspace $S$. This induced arrangement in $S$ is called the *restriction* of $\mathcal{A}$ to $S$, denoted $\mathcal{A}^S$.

We start by describing the combinatorial structure of the affine arrangement $\mathcal{A}_\pi$ formed by restricting $\mathcal{A}$ to a generic fiber of $\pi$. This requires some discussion of the cone-decone construction of [23], and a description of the matroid construction of principal truncation along a flat.

There is a natural correspondence between arrangements of linear hyperplanes in $\mathbb{C}^\ell$ and arrangements of affine hyperplanes in $\mathbb{C}^{\ell-1}$. The analytic operations need not concern us here; they are described in detail in [23] and [7]. One places a copy of a given affine $(\ell-1)$-arrangement $\mathcal{A}$ into $\{1\} \times \mathbb{C}^{\ell-1} \subseteq \mathbb{C}^\ell$. Then replace each of the affine subspaces in this copy of $\mathcal{B}$ by its linear span in $\mathbb{C}^\ell$ (i.e., "cone over the origin") and adjoin the "hyperplane at infinity" $\{0\} \times \mathbb{C}^{\ell-1}$, to obtain a central arrangement $c\mathcal{A}$, the *cone* of $\mathcal{A}$, in $\mathbb{C}^\ell$. The inverse operation, called "deconing," takes a central arrangement $\mathcal{A}$ to its projective image, and dehomogenizes relative to a hyperplane $H_\infty \in \mathcal{A}$ to obtain an affine $(\ell-1)$-arrangement $d\mathcal{A}$.

The intersections of hyperplanes of $d\mathcal{A}$ form a geometric semilattice [35], isomorphic to a subposet of $L(\mathcal{A})$. Specifically,

$$L(d\mathcal{A}) \cong \{Y \in L(\mathcal{A}) \mid Y \not\geq H_\infty\}.$$

The fiber arrangement $\mathcal{A}_\pi$ is an affine arrangement of dimension $\ell - r(X)$. We will show that the underlying matroid of the cone $c\mathcal{A}_\pi$ is the complete principal truncation of the matroid $G(\mathcal{A})$ along the flat $\mathcal{A}_X$.

The *principal truncation* $T_F(G)$ of a matroid $G$ along a flat $F$ is constructed by adding a generic point $p$ on the flat $F$ and then contracting $G$ on $p$ [36, Section 7.4]. The result may be a matroid with multiple points. We tacitly simplify the resulting matroid, by removing any multiple points. This does not affect the intersection lattice, characteristic polynomial, or Orlik-Solomon algebra.

This operation can be iterated. The *complete* principal truncation $\overline{T}_F(G)$ is the result of $r(F) - 1$ successive principal truncations on $F$, so that $F$ reduces to a point. Equivalently, one can add $r(F)-1$ generic points to $F$ and contract $G$ on the flat spanned by the new points. Contraction of a matroid on a point corresponds to projection of a projective point configuration from one of its points, or restriction of a hyperplane arrangement to one of its hyperplanes.

**Theorem 2.1** *Let $X \in L$ and let $\mathcal{A}_\pi$ be the affine arrangement obtained by restricting $\mathcal{A}$ to a generic fiber of $\pi : \mathbb{C}^\ell \longrightarrow \mathbb{C}^\ell/X$. Then the matroid $G(c\mathcal{A}_\pi)$ is isomorphic to the complete principal truncation $\overline{T}_X(G)$ of $G = G(\mathcal{A})$ along the flat $\mathcal{A}_X$.*

*proof:* Dualizing the description of complete principal truncation to hyperplane arrangements, we see that $\overline{T}_X(G)$ is the matroid of the arrangement $\mathcal{A}^P$ obtained by choosing a generic subspace $P$ of codimension $r(X)-1$ containing $X$, and restricting $\mathcal{A}$ to $P$. Then $P$ has dimension $\dim(X)+1$, $X \cap P$ is a hyperplane of $\mathcal{A}^P$, and an affine translate of $X \cap P$ is a generic fiber of $\pi$. It follows that $d(\mathcal{A}^P) \cong \mathcal{A}_\pi$, so $\mathcal{A}^P \cong c\mathcal{A}_\pi$. □

**Definition 2.2** A pair $(X, Y)$ forms a *modular pair* in $L$ if

$$r(X \vee Y) + r(X \wedge Y) = r(X) + r(Y).$$

An element $X \in L$ is *modular* if $(X, Y)$ is a modular pair for every $Y \in L$.

The following lemma is the key to the proof of the modular fibration theorem, and is trivial to prove.

**Lemma 2.3** *Let $X, Y \in L$. Then $(X, Y)$ is a modular pair if and only if $X + Y \in L$.* □

When $X$ is modular, the conclusion of Theorem 2.1 holds for every fiber of $\pi$ over points not in $\bigcup \mathcal{A}_X$. To prove this we need to describe the rank function $r_T$ on the lattice of flats $L(\overline{T}_X(G))$. According to [36, Proposition 7.4.9], the set $L(\overline{T}_X(G))$ can be identified with $\{Y \in L \mid X \wedge Y = 0_{L(\mathcal{A})} \text{ or } Y \geq X\}$. With this identification the rank function $r_T$ is given by

$$r_T(Y) := \begin{cases} r(Y) & \text{if } X \wedge Y = 0_{L(\mathcal{A})}, \\ r(Y) - r(X) + 1 & \text{if } Y \geq X. \end{cases}$$

**Theorem 2.4** *Suppose $X$ be a modular flat. Let $\overline{v} = v + X \in (\mathbb{C}^\ell/X) - \bigcup \mathcal{A}_X$ and let $\mathcal{A}_{\overline{v}}$ be the restriction of $\mathcal{A}$ to $\pi^{-1}(\overline{v})$. Then the intersection lattice $L(c\mathcal{A}_{\overline{v}})$ is isomorphic to $L(\overline{T}_X(M))$.*

*proof:* As in the proof of Theorem 2.1, the arrangement $c\mathcal{A}_{\overline{v}}$ can be identified with the restriction $\mathcal{A}^P$ of $\mathcal{A}$ to the linear subspace $P$ of codimension $r(X)-1$ spanned by $X$ and $v$. Then $L(c\mathcal{A}_{\overline{v}}) = \{P \cap Y \mid Y \in L(\mathcal{A})\}$. There are three cases.

Case 1. Suppose $Y \in L(\mathcal{A})$ satisfies $X \wedge Y = 0_{L(\mathcal{A})}$. By modularity of $X$, $X + Y = \mathbb{C}^\ell$. Then there exists $y \in Y$ such that $\overline{v} = \overline{y} = y + X$. Then $P \cap Y = (\mathbb{C}v + X) \cap Y = (\mathbb{C}y + X) \cap Y = \mathbb{C}y + (X \cap Y)$. Since $\overline{v} \notin \bigcup \mathcal{A}_X$, $y \notin X$, so

$$\begin{aligned}
\operatorname{codim}_P(P \cap Y) &= \operatorname{codim}_{\mathbb{C}y+X}(\mathbb{C}y + (X \cap Y)) \\
&= \operatorname{codim}_X(X \cap Y) \\
&= \operatorname{codim}_{\mathbb{C}^\ell}(X \cap Y) - \operatorname{codim}_{\mathbb{C}^\ell}(X) \\
&= r(X \vee Y) - r(X) \\
&= r(Y),
\end{aligned}$$

the last equality by modularity of $X$.

Case 2. Suppose $Y \geq X$. Then $Y \subseteq X \subseteq P$ so

$$\begin{aligned}
\operatorname{codim}_P(P \cap Y) &= \operatorname{codim}_P(Y) \\
&= \operatorname{codim}_X(Y) + \operatorname{codim}_P(X) \\
&= r(Y) - r(X) + 1.
\end{aligned}$$

Case 3. Suppose $0_{L(\mathcal{A})} < X \wedge Y < X$. Then $X \cup Y \subseteq H$ for some $H \in \mathcal{A}$. Note that $v \notin H$, since $\overline{v} \notin \bigcup \mathcal{A}_X$. It follows that $P \cap H = (\mathbb{C}v + X) \cap H = X$ since $X \subseteq H$ while $v \notin H$. Since $P \cap Y \subseteq P \cap H$ we have $P \cap Y = P \cap Y'$ for $Y' = X \vee Y \geq X$, which case is treated above.

These calculations verify that $L(c\mathcal{A}_v)$ can be identified with $L(\overline{T}_X(G))$ as described above, with the same rank function. □

**Remark 2.5** The same calculations as in case 1 above can be used to show that the converse of Theorem 2.4 also holds. That is, $X$ is modular if the lattice $L(c\mathcal{A}_{\overline{v}})$ is constant over $M(\mathcal{A}_X)$. This will be used to identify modular flats in the examples of Section 5. □

Let $M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup \mathcal{A}$ and $M(\mathcal{A}_X) = (\mathbb{C}^\ell / X) - \bigcup \mathcal{A}_X$. Note that $\pi$ maps $M(\mathcal{A})$ onto $M(\mathcal{A}_X)$.

**Corollary 2.6** *The fibers of $\pi|_{M(\mathcal{A})} : M(\mathcal{A}) \longrightarrow M(\mathcal{A}_X)$ are diffeomorphic.*

*proof:* The fiber of $\pi|_{M(\mathcal{A})}$ over $\overline{v}$ is the complement of the arrangement $\mathcal{A}_{\overline{v}}$ in $\pi^{-1}(\overline{v}) \cong \mathbb{C}^{\ell - r(X)}$. Since the base $M(\mathcal{A}_X)$ is path-connected, Theorem 2.4 implies that the arrangements $c\mathcal{A}_v$ are lattice-isotopic. Then the assertion follow from [29]. □

**Remark 2.7** Theorem 2.4 was essentially proved by Terao in [32]. Our result explicitly identifies the lattice. In case $X$ is a copoint, Corollary 2.6 follows without using Randell's lattice isotopy theorem, which had not been discovered at the time of Terao's work. In fact Corollary 2.6 and the fibration result

Corollary 3.2 of the next section confirm the suggestion stated after Proposition 2.12 of [32]. The proof of Corollary 3.2 uses the stratification technique first introduced to arrangement theory by Randell in his proof of the isotopy theorem.

# 3 Modular flats and fibrations

The arrangement $\mathcal{A}$ defines a stratification $\mathcal{S}$ of $\mathbb{C}^\ell$:

$$\mathbb{C}^\ell = \bigcup \{S_Y \mid Y \in L\}$$

whose strata $S_Y$ are given by

$$S_Y = Y - \bigcup_{Z>Y} Z.$$

Thus $S_Y$ is a connected dense open subset of the linear space $Y$. In particular, $S_Y$ is a smooth submanifold of $\mathbb{C}^\ell$. Note that the closed stratum $\overline{S_Y}$ is equal to $Y$. Also $S_Y \cap \overline{S_Z} \neq \emptyset$ if and only if $S_Y \subseteq \overline{S_Z}$ if and only if $Y \geq Z$.

This stratification satisfies Whitney's conditions (a) and (b) [16]. Indeed these conditions involve tangent and secant lines, and tangent spaces to strata, which are trivial to verify because $S_Y$, as an open subset of the linear space $Y$, has tangent space at any point equal to $Y$.

Let $X \in L$ be a modular element of rank $p$. We may identify $\mathbb{C}^\ell/X$ with $\mathbb{C}^p$. Let $\pi : \mathbb{C}^\ell \longrightarrow \mathbb{C}^p$ be the natural projection. The arrangement $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supseteq X\}$, considered as an arrangement in $\mathbb{C}^\ell/X \cong \mathbb{C}^p$, determines a stratification of $\mathbb{C}^p$ as above. Elements of $L(\mathcal{A}_X)$ have the form $\pi Y = X + Y/X$ for $Y \in L(\mathcal{A})$. Referring to Lemma 2.3, one sees that the preimage of a stratum is a union of strata, that is, that $\pi$ is a map of stratified spaces, precisely when $X$ is modular. Since $\pi$ is a linear surjection, it restricts to a submersion on each stratum. In order to apply the Thom Isotopy Lemma, it is necessary to extend $\pi$ to a proper map of stratified spaces. This step was carried out by L. Paris [25].

**Theorem 3.1** *There exists a stratified space $P_X$ containing $\mathbb{C}^\ell$ as an open dense subset, and an extension of $\pi$ to a proper stratified map $\hat{\pi} : P_X \longrightarrow \mathbb{C}^p$.* $\square$

The space $P_X$ is obtained by compactifying the fibers of $\pi$, i.e., the parallel translates of $X$, via projective completion, so that $P_X$ is diffeomorphic to $\mathbb{P}(\mathbb{C}^q) \times \mathbb{C}^p$, where $q = \ell - p = \dim(X)$. This can be viewed as a parametrized version of R. Randell's construction in his proof of the lattice isotopy theorem [29]. The stratification of $\mathbb{C}^\ell$ is extended to a stratification of $P_X$ by adjoining closed strata formed by intersecting the closures of the $S_Y$ in $P_X$ with $(\mathbb{P}(\mathbb{C}^q) - \mathbb{C}^q) \times \mathbb{C}^p$. These new strata have the form $S_Y^\infty \times \mathbb{C}^p$, for $Y \geq X$, where $S_Y^\infty = (S_Y \cap X) \cap (\mathbb{P}(\mathbb{C}^q) - \mathbb{C}^q)$. The map $\hat{\pi}$ is projection on the second factor.

Let $M(\mathcal{A})$ and $M(\mathcal{A}_X)$ denote the complements of $\bigcup \mathcal{A}$ and $\bigcup \mathcal{A}_X$ in $\mathbb{C}^\ell$ and $\mathbb{C}^p$ respectively.

**Corollary 3.2** *The map $\pi|_{M(\mathcal{A})} : M(\mathcal{A}) \longrightarrow M(\mathcal{A}_X)$ is a fiber bundle projection.*

*proof:* The complement $M(\mathcal{A})$ coincides with the open stratum $S_{0_L}$ of $\mathbb{C}^\ell \subset P_X$. So Theorem 3.1 implies that the restriction of $\widehat{\pi}$ to $M(\mathcal{A})$ is a fiber bundle projection, by the Thom Isotopy Lemma [16, 21, 34]. □

We proceed to generalize the properties of strictly linear fibrations [13], where $X$ is a modular copoint, to general modular fibrations. Henceforth let $X$ be a modular flat of $L(\mathcal{A})$, and let us denote the bundle projection $\pi|_{M(\mathcal{A})}$ by $\pi_X$.

We say $\mathcal{A}$ is a $K(\pi, 1)$ *arrangement* if $M(\mathcal{A})$ is an aspherical space.

**Corollary 3.3** *If $\mathcal{A}_X$ and the coned fiber arrangement $c\mathcal{A}_{\overline{v}}$ are $K(\pi, 1)$ arrangements, then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.*

*proof:* This follows immediately from the long exact homotopy sequence of the fibration $\pi_X$. □

**Remark 3.4** In case $X$ is a modular copoint, the monodromy of $\pi_X$ induces a homomorphism from $\pi_1(M(\mathcal{A}_X))$ to $P_n$, the pure braid group on $n = |\mathcal{A} - \mathcal{A}_X|$ strands, which we call the *braid monodromy homomorphism* after its similarity to the Moishezon construction. See [4]. For a modular flat $X$ of arbitrary rank, the pure braid group is replaced by the fundamental group of a certain subvariety of the Grassmanian, a *matroid stratum* defined as follows. If $P \in \mathcal{G}_\ell(\mathbb{C}^n)$ is a point of the Grassmannian of $\ell$-planes in $\mathbb{C}^n$, then $P$ determines a vector configuration in $\mathbb{C}^\ell$, unique up to linear change of coordinates, obtained by projecting the standard basis vectors of $\mathbb{C}^n$ onto $P$ [15]. Let $G_P$ denote the linear matroid realized by this configuration; $G_P$ is independent of the choice of basis in $P$. The *matroid stratum* of an arbitrary matroid $G$ is the subset $\Gamma(G)$ of $\mathcal{G}_\ell(\mathbb{C}^n)$ given by

$$\Gamma(G) = \{P \in \mathcal{G}_\ell(\mathbb{C}^n) \mid G_P = G\}.$$

An ordered arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ of rank $\ell$, with specified defining forms $\{\phi_1, \ldots, \phi_n\}$, determines a point $P \in \mathcal{G}_\ell(\mathbb{C}^n)$ given by the image of $(\phi_1, \ldots, \phi_n) : \mathbb{C}^\ell \longrightarrow \mathbb{C}^n$. The original arrangement $\mathcal{A}$ is isomorphic to the arrangement in $P$ formed by the intersections of $P$ with the coordinate hyperplanes in $\mathbb{C}^n$, and the point $P$ lies in $\Gamma(G(\mathcal{A}))$. See [9].

The monodromy of the stratified map $\pi$ induces a homomorphism

$$\pi_1(M(\mathcal{A}_X)) \longrightarrow \pi_1(\Gamma(\overline{T}_X(G))).$$

Indeed, a path $\{\overline{v}_t\}_{t\in[0,1]}$ in the base space $M(\mathcal{A}_X)$ determines a one-parameter family of (coned) fiber arrangements $c\mathcal{A}_{\overline{v}_t}$, equipped with ordered sets of defining forms inherited from a fixed set of defining forms for $\mathcal{A}$. By Theorem 2.4 and the construction above, this defines a path in the matroid stratum $\Gamma(\overline{T}_X(G))$. From this one easily obtains the monodromy homomorphism described above.

This construction does indeed generalize the corank one case. For in this case $\overline{T}_X(G)$ is a uniform matroid of rank two, $\Gamma(\overline{T}_X(G))$ is configuration space, and $\pi_1(\Gamma(\overline{T}_X(G)))$ is the pure braid group. $\square$

**Theorem 3.5** *The monodromy action of $\pi_1(M(\mathcal{A}_X))$ on the fiber $M(\mathcal{A}_{\overline{v}})$ is cohomologically trivial.*

*proof:* Since the fiber $M(\mathcal{A}_{\overline{v}})$ is the complement of an arrangement, the cohomology of $M(\mathcal{A}_{\overline{v}})$ is free abelian, and is generated by $H^1(M(\mathcal{A}_{\overline{v}}))$. First of all we argue that the monodromy action on $H^1(M(\mathcal{A}_{\overline{v}}))$ is trivial, by the same reasoning as in the corank-one case [13]. The group $H^1(M(\mathcal{A}_{\overline{v}}))$ has a free basis consisting of elements dual to the hyperplanes of $\mathcal{A}_{\overline{v}}$. Using this basis, it is clear that elements of $H^1(M(\mathcal{A}_{\overline{v}}))$ are uniquely determined by their linking numbers with the hyperplanes of $\mathcal{A}_{\overline{v}}$. By naturality, these linking numbers agree with linking numbers in $\mathbb{C}^\ell$ with the hyperplanes of $\mathcal{A} - \mathcal{A}_X$. Since these linking numbers take values in a discrete space, and vary continuously, they remain locally constant under translation of the fiber, and thus are globally constant under translation around a loop in the base. This proves triviality in degree one. Since $H^*(M(\mathcal{A}_{\overline{v}}))$ is generated by $H^1(M(\mathcal{A}_{\overline{v}}))$, and the monodromy action respects cup products, it follows that the monodromy acts trivially on $H^*(M(\mathcal{A}_{\overline{v}}))$. $\square$

A *rational $K(\pi,1)$ arrangement* is an arrangement whose complement has aspherical rational completion. See [8, 14, 23, 12] for the precise definition and basic properties. We point out that this property seems to bear little relationship to the notion of $K(\pi,1)$ arrangement; the terminology arises naturally in the context of simply-connected spaces.

**Corollary 3.6** *If $\mathcal{A}_X$ and $c\mathcal{A}_{\overline{v}}$ are rational $K(\pi,1)$ arrangements, then $\mathcal{A}$ is a rational $K(\pi,1)$ arrangement.*

*proof:* The argument is the same as in the corank-one case [8]. Because the monodromy action is trivial, hence nilpotent, on the cohomology of the fiber, the map $\pi_X$ induces a fibration of the rational completion of $M(\mathcal{A})$ over that of $M(\mathcal{A}_X)$, with fiber the rational completion of $M(\mathcal{A}_{\overline{v}})$. Since $M(c\mathcal{A}_{\overline{v}}) \cong \mathbb{C}^* \times M(\mathcal{A}_{\overline{v}})$, the hypothesis implies that the rational completion of $M(\mathcal{A}_{\overline{v}})$ is aspherical. The assertion then follows from the homotopy sequence of this fibration. $\square$

At this point the only known examples of rational $K(\pi,1)$ arrangements are supersolvable. If $\mathcal{A}_X$ and $c\mathcal{A}_{\overline{v}}$ are supersolvable, then $\mathcal{A}$ is also supersolvable [24]. So the preceding corollary does not provide new examples of rational $K(\pi,1)$ arrangements.

The *Poincaré series* of a topological space $M$ is

$$P(M,t) = \sum_{n\geq 0} \dim_{\mathbb{Q}} H^n(M,\mathbb{Q}).$$

For a complex arrangement $\mathcal{A}$, a famous result of Orlik and Solomon [22] relates the Poincaré series $P(M(\mathcal{A}),t)$ to the characteristic polynomial $\chi(G(\mathcal{A}),t)$ of the underlying matroid $G(\mathcal{A})$. Specifically,

$$P(M(\mathcal{A}),t) = t^r \chi(G(\mathcal{A}), -t^{-1}),$$

where $r$ is the rank of $G(\mathcal{A})$.

For a modular flat $X$, R. Stanley proved in [30] that the characteristic polynomial of the $G(\mathcal{A}_X)$ divides that of $G(\mathcal{A})$ over the integers. In [3], T. Brylawski identified the quotient as the characteristic polynomial of the complete principal truncation $\overline{T}_X(G)$, divided by $(t-1)$. The decone operation on arrangements has the effect on Poincaré polynomials of dividing by $(1+t)$. Using Theorem 2.4 and the identity relating the characteristic polynomial of $G(\mathcal{A})$ to the Poincaré polynomial of $M(\mathcal{A})$, we may restate the Stanley and Brylawski results as follows.

**Theorem 3.7** *If $X$ is a modular flat of $G$, then*

$$P(M(\mathcal{A}),t) = P(M(\mathcal{A}_X),t)P(M(\mathcal{A}_{\overline{v}}),t).$$

□

**Corollary 3.8** *The Leray-Serre spectral sequence of $\pi_X$ satisfies*

$$E_2^{p,q} \cong H^p(M(\mathcal{A}_X)) \otimes H^q(M(\mathcal{A}_{\overline{v}})),$$

*and degenerates at the $E_2$ term.* □

*proof:* The first assertion follows from the triviality of the monodromy action established in Theorem 3.5. The second is a consequence of the factorization identity among the Poincaré series. Indeed, according to [17, Theorem 11.3], the formula of Theorem 3.7 holds for a general spectral sequence $E$, with a correction term that vanishes precisely when the differential of $E_2$ is trivial. □

**Corollary 3.9** *The cohomology $H^*(M(\mathcal{A}))$ is isomorphic as a $\mathbb{Z}$-module to the tensor product $H^*(M(\mathcal{A}_X)) \otimes H^*(M(\mathcal{A}_{\overline{v}}))$.* □

**Remark 3.10** In [32] Terao established the tensor product factorization of Corollary 3.9 in terms of Orlik-Solomon algebras, using a direct combinatorial argument. This approach yields an alternate proof of the Stanley factorization theorem. The degeneracy of the spectral sequence in case $X$ is a modular copoint is given a direct proof in [13], providing a topological proof of Terao's result in this case. The proof in [13] uses the fact that the fiber $M(\mathcal{A}_{\overline{v}})$ has nonvanishing cohomology in only two different degrees, so that the spectral sequence results in a "Gysin-like" long exact sequence. The other ingredient is the construction of a section of the bundle map $\pi_X$. In case $X$ is a modular flat of arbitrary rank, a section of $\pi_X$ is constructed by L. Paris in [25]. But we see no analogue of the Gysin long exact sequence in the general case, and do not have a topological proof, independent of the Stanley and Brylawski results, of the second part of Corollary 3.8. Nevertheless, the bundle map $\pi_X$ is seen to be a topological realization of the combinatorial and algebraic factorizations arising from a modular flat. □

Motivated by the fact that supersolvable arrangements are inductively free [20], we include with this compendium of generalizations the following conjecture.

**Conjecture 3.11** *If $X$ is a modular flat and both $\mathcal{A}_X$ and $c\mathcal{A}_{\overline{v}}$ are free arrangements, then $\mathcal{A}$ is a free arrangement.*

# 4 Parallel connections

Let $G_1$ and $G_2$ be matroids on ground sets $E_1$ and $E_2$. Suppose $E_1 \cap E_2 = F$ is a flat of both $G_1$ and $G_2$, and is modular in $G_1$. The *generalized parallel connection* of $G_1$ and $G_2$ along $F$ is the matroid $P_F(G_1, G_2)$ on the ground set $E_1 \cup E_2$ whose flats are those sets $Y \subseteq E_1 \cup E_2$ for which $Y \cap E_i$ is a flat of $G_i$ for $i = 1, 2$. The modularity condition is necessary for this definition to make sense. That is, this collection of flats will form a geometric lattice for general $G_2$ if and only if $X$ is modular in $G_1$. Modularity of $F$ in $G_1$ implies that $G_2$ is modular in $P_F(G_1, G_2)$. See [36, Section 7.6] and [24] for details about this construction.

The rank of a flat $Y$ of $P_F(G_1, G_2)$ is given by

$$r(Y) = r_1(Y \cap E_1) + r_2(Y \cap E_2) - r_1(Y \cap F),$$

where $r_i$ is the rank function of $G_i$, $i = 1, 2$. In particular, the rank of $P_F(G_1, G_2)$ is equal to $r(G_1) + r(G_2) - r(F)$. The rank formula indicates that $P_F(G_1, G_2)$ is the "free" sum of $G_1$ and $G_2$ amalgamated along their common flat $F$. Indeed, $P_F(G_1, G_2)$ is a pushout of the inclusion maps $F \hookrightarrow G_i$, $i = 1, 2$ in the category of matroids and injective strong maps.

In case $F$ is a point, automatically modular in $G_1$, the matroid $P_F(G_1, G_2)$ is called a *parallel connection* of $G_1$ and $G_2$, studied in connection with complex hyperplane arrangements in [7].

Now suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are hyperplane arrangements realizing $G_1$ and $G_2$ in $\mathbb{C}^r$ and $\mathbb{C}^s$ respectively. Then there is an arrangement $\mathcal{A}$ realizing $P_F(G_1, G_2)$, provided there is a linear isomorphism between the subarrangements of $\mathcal{A}_1$ and $\mathcal{A}_2$ corresponding to the common flat $F$. To carry out the construction, let us be more precise about the realizations $\mathcal{A}_1$ and $\mathcal{A}_2$.

Suppose the flat $F$ has rank $p$ (in both $G_1$ and $G_2$). Let $X_1$ denote the corresponding element of intersection lattice $L(\mathcal{A}_1)$. Thus $X_1$ is a linear subspace of $\mathbb{C}^r$, and we may identify $(\mathcal{A}_1)_{X_1}$ with $F$. We may assume $X_1 = \mathbb{C}^{r-p} \times \{0\} \subseteq \mathbb{C}^r$. Then the defining equations of the hyperplanes in $(\mathcal{A}_1)_{X_1} \subseteq \mathcal{A}_1$ involve only the last $p$ coordinates in $\mathbb{C}^r$. Assume that the same defining forms, expressed in terms of the first $p$ coordinates of $\mathbb{C}^s$, give the defining equations for hyperplanes of $(\mathcal{A}_2)_{X_2} \subseteq \mathcal{A}_2$, where $X_2 \in L(\mathcal{A}_2)$ corresponds to the flat $F$ of $G_2$. Then we may define an arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$, with $\ell = r+s-p$, as follows. Identify $\mathbb{C}^\ell$ with $\mathbb{C}^{r-p} \times \mathbb{C}^p \times \mathbb{C}^{s-p}$. By pulling back the defining equations via projection of coordinates, the arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ naturally embed in $\mathbb{C}^r \times \mathbb{C}^{s-p} = \mathbb{C}^\ell$ and $\mathbb{C}^{r-p} \times \mathbb{C}^s = \mathbb{C}^\ell$ respectively. Then let $\mathcal{A}$ be the union of $\mathcal{A}_1 - (\mathcal{A}_1)_{X_1}$ and $\mathcal{A}_2$ in $\mathbb{C}^\ell$.

**Theorem 4.1** [36, Prop. 7.6.11] *The arrangement $\mathcal{A}$ is a realization of the generalized parallel connection $P_F(G_1, G_2)$.* □

Let $X \in L(\mathcal{A})$ correspond to the flat $F$ of $P_F(G_1, G_2)$. By modularity of $F$ in $G_1$ and of $G_2$ in $P_F(G_1, G_2)$, the results of Section 3 yield bundle maps $M(\mathcal{A}_1) \longrightarrow M((\mathcal{A}_1)_{X_1})$ and $M(\mathcal{A}) \longrightarrow M(\mathcal{A}_2)$. We consider $(\mathcal{A}_1)_{X_1}$ to be an arrangement in $\mathbb{C}^r/X_1 \cong \mathbb{C}^s/X_2 \cong \mathbb{C}^p$. Then there is a projection $M(\mathcal{A}_2) \longrightarrow M((\mathcal{A}_1)_{X_1})$. This projection is just the inclusion of $M(\mathcal{A}_2)$ into the complement of the subarrangement $\{H \in \mathcal{A}_2 \mid H \supseteq X_2\}$ of $\mathcal{A}_2$, followed by a homotopy equivalence.

**Theorem 4.2** *The fiber bundle $M(\mathcal{A}) \longrightarrow M(\mathcal{A}_2)$ is the pullback of the bundle $M(\mathcal{A}_1) \longrightarrow M((\mathcal{A}_1)_{X_1})$ along the projection $M(\mathcal{A}_2) \longrightarrow M((\mathcal{A}_1)_{X_1})$. That is,*

$$\begin{array}{ccc} M(\mathcal{A}) & \longrightarrow & M(\mathcal{A}_1) \\ \downarrow & & \downarrow \\ M(\mathcal{A}_2) & \longrightarrow & M((\mathcal{A}_1)_{X_1}) \end{array}$$

*is a pullback diagram.*

*proof:* For these special realizations, the bundle map $M(\mathcal{A}_1) \longrightarrow M((\mathcal{A}_1)_{X_1})$ is the restriction of the projection $\pi_1 : \mathbb{C}^r \longrightarrow \mathbb{C}^p$ onto the last $p$ coordinates. Similarly, the map $M(\mathcal{A}) \longrightarrow M(\mathcal{A}_2)$ is the restriction of the projection $\pi : \mathbb{C}^\ell \longrightarrow \mathbb{C}^s$ onto the last $s$ coordinates. The map $M(\mathcal{A}_2) \longrightarrow M((\mathcal{A}_1)_{X_1})$ can be identified with the restriction of the projection $\pi_2 : \mathbb{C}^s \longrightarrow C^s/X_2 \cong \mathbb{C}^p$ onto the first $p$ coordinates.

By definition, the total space of the pullback of $M(\mathcal{A}_1) \longrightarrow M((\mathcal{A}_1)_{X_1})$ along the projection $M(\mathcal{A}_2) \longrightarrow M(\mathcal{A}_X)$ is the set of pairs $(x, v) \in \mathbb{C}^r \times \mathbb{C}^s$ such that $x \in M(\mathcal{A}_1)$, $v \in M(\mathcal{A}_2)$, and $\pi_1(x) = \pi_2(v)$ in $M((\mathcal{A}_1)_{X_1}) \subseteq C^p$. But this means that the last $p$ components of $x$ match the first $p$ components of $v$. Then each such $(x, v)$ corresponds to a unique point of $\mathbb{C}^{r+s-p} = \mathbb{C}^\ell$ which, by the first two conditions, lies in $M(\mathcal{A})$. Under this identification, the projection $(x, v) \mapsto v$ coincides with $\pi$. This identifies $\pi|_{M(\mathcal{A})}$ with the pullback of $\pi_1|_{M(\mathcal{A}_1)}$, as claimed. □

**Corollary 4.3** *If $\mathcal{A}$ is a realization of the parallel connection of $\mathcal{A}_1$ and $\mathcal{A}_2$, then $M(\mathcal{A})$ is a trivial bundle over $M(\mathcal{A}_2)$ with fiber $M(d\mathcal{A}_1)$. In particular, $M(\mathcal{A}) \cong M(d\mathcal{A}_1) \times M(\mathcal{A}_2)$.*

*proof:* In case $X$ is a point, then $X$ is modular in $\mathcal{A}_1$, and the modular fibration $M(\mathcal{A}_1) \longrightarrow M(\mathcal{A}_X) = \mathbb{C}^*$ is a trivial bundle with fiber $d\mathcal{A}_1$, by [23, Proposition 5.1]. The pullback of a trivial bundle is trivial. □

This corollary clarifies the main construction of [7], which essentially established the diffeomorphism noted above. This argument shows in an alternate way that the diffeomorphisms among arrangements with different underlying matroids, constructed in [7], are all consequences of the triviality of the restriction of the Hopf bundle.

# 5 Examples

Corollary 3.3 of Section 3 can be used to identify $K(\pi, 1)$ arrangements of high rank, at least when the base arrangement $\mathcal{A}_X$ and (coned) fiber arrangement $c\mathcal{A}_{\overline{v}}$ are tractable. This will be the case, for instance, when $X$ is a modular coline, for then $c\mathcal{A}_{\overline{v}}$ will have rank three. In this section we present new families of examples of $K(\pi, 1)$ arrangements. Our results give some support for the conjecture [12], which was based primarily on rank-three phenomena, that factored arrangements of arbitrary rank are $K(\pi, 1)$. We also exhibit an interesting example with two different modular colines, allowing us to conclude the nontrivial result that one of the fiber arrangements is not $K(\pi, 1)$.

Let $\mathcal{B}_\ell$ denote the arrangement of reflecting hyperplanes in the Weyl group of type $B_\ell$. Thus $\mathcal{B}_\ell$ consists of the hyperplanes

$$H_{ij} = \{x \in \mathbb{C}^\ell \mid x_i = x_j\}, \text{ for } 1 \le i < j \le \ell,$$

$$\overline{H}_{ij} = \{x \in \mathbb{C}^\ell \mid x_i = -x_j\} \text{ for } 1 \le i < j \le \ell, \text{ and}$$

$$H_i = \{x \in \mathbb{C}^\ell \mid x_i = 0\}, \text{ for } 1 \le i \le \ell.$$

Let $\mathcal{A}_{\ell-1}$ denote the braid arrangement, consisting of the hyperplanes $H_{ij}$ above, for $1 \le i < j \le \ell$.

In [6], P. Edelman and V. Reiner used graphs to parametrize subarrangements of $\mathcal{B}_\ell$ containing $\mathcal{A}_{\ell-1}$, developing a calculus for combinatorial invariants of the arrangements in terms of the graphs. We find among these arrangements those which are not supersolvable, but have modular colines, for which the fiber arrangements are demonstrably $K(\pi,1)$. These examples coincide in large part with the arrangements between $\mathcal{A}_{\ell-1}$ and $\mathcal{B}_\ell$ which are factored, classified by D. Bailey in [1].

Let $\Gamma$ be a graph with vertex set $\{1,\ldots,\ell\}$, possibly with loops, but without multiple edges. Let edge($\Gamma$) and loop($\Gamma$) denote the sets of edges and loops of $\Gamma$, respectively. Let $\mathcal{A}_\Gamma$ be the arrangement defined by

$$\mathcal{A}_\Gamma = \mathcal{A}_{\ell-1} \ \cup\ \{\overline{H}_{ij} \mid ij \in \mathrm{edge}(\Gamma)\} \ \cup\ \{H_i \mid i \in \mathrm{loop}(\Gamma)\}.$$

The following results are proved in [6]. The notion of free arrangement plays little role in what follows; see [23] for a precise definition. A graph is *threshold* if it is built up by successively adjoining isolated and/or cone vertices, the latter being vertices which are adjacent to all preceding vertices.

**Theorem 5.1** *The arrangement $\mathcal{A}_\Gamma$ is free if and only if*

(i) $\Gamma$ *is a threshold graph, and*

(ii) $i \in \mathrm{loop}(\Gamma)$ *and* $\deg(j) > \deg(i)$ *implies* $j \in \mathrm{loop}(\Gamma)$. □

An edge $ij \in edge(\Gamma)$ is *loopless* if neither $i$ nor $j$ lies in loop($\Gamma$).

**Theorem 5.2** *Suppose $\mathcal{A}_\Gamma$ is free and $\Gamma$ has no loopless edges. Then $\mathcal{A}_\Gamma$ is supersolvable.* □

There are two families of exceptional graphs $\Gamma$ with loopless edges such that $\mathcal{A}_\Gamma$ is supersolvable; see [6]. Of course, any such arrangement $\mathcal{A}_\Gamma$ is $K(\pi,1)$.

Roughly speaking, an arrangement is *factored* [18, 11, 33, 23] if the cohomology of the complement is isomorphic as a $\mathbb{Z}$-module to the the tensor product of algebras with trivial multiplication generated by sets of hyperplanes of $\mathcal{A}$. For instance, Corollary 3.9 implies that supersolvable arrangements are factored. Such factorizations correspond to partitions of $\mathcal{A}$, properties of which are analyzed in [11, 33, 19]. D. Bailey [1] identified those arrangements $\mathcal{A}_\Gamma$ which are factored.

**Theorem 5.3** [1] *Suppose $\mathcal{A}_\Gamma$ is free and $\Gamma$ has at most one loopless edge. Then $\mathcal{A}_\Gamma$ is a factored arrangement.* □

Again there are two families of exceptional graphs with more than one loopless edge for which $\mathcal{A}_\Gamma$ is factored [1].

We establish criteria for $\mathcal{A}_\Gamma$ to have modular copoints or colines determined by coordinate subspaces $X$. The assertions below are easy to prove using Remark 2.5, by showing that the lattices of the coned fiber arrangements remain constant over $M(\mathcal{A}_X)$. (For an example, see the proof of Theorem 5.6.) Let $\Gamma'$ and $\Gamma''$ be the vertex-induced subgraphs of $\Gamma$ on vertices $\{2,\dots,\ell\}$ and $\{3,\dots,\ell\}$ respectively. Then $\mathcal{A}_{\Gamma'}$ and $\mathcal{A}_{\Gamma''}$ are flats of $G(\mathcal{A}_\Gamma)$ of corank one and two corresponding to the linear subspaces $x_2 = \cdots = x_\ell = 0$ and $x_3 = \cdots = x_\ell = 0$ respectively.

**Lemma 5.4** *The flat $\mathcal{A}_{\Gamma'}$ is a modular copoint of $G(\mathcal{A}_\Gamma)$ if and only if*

(i) $1 \in \mathrm{loop}(\Gamma)$ *implies* $\mathrm{loop}(\Gamma) = \{1,\dots,\ell\}$*, and*

(ii) $1j \in \mathrm{edge}(\Gamma)$ *implies* $j \in \mathrm{loop}(\Gamma)$ *and $j$ is adjacent to every vertex of* $\Gamma$*.* □

In particular, an isolated vertex of $\Gamma$ corresponds to a modular copoint.

**Lemma 5.5** *The flat $\mathcal{A}_{\Gamma''}$ is a modular coline of $G(\mathcal{A}_\Gamma)$ if and only if*

(i) $i \in \mathrm{loop}(\Gamma)$*, for $i = 1$ or $2$ implies* $\mathrm{loop}(\Gamma) \supseteq \{3,\dots,\ell\}$*,*

(ii) $ij \in \mathrm{edge}(\Gamma)$ *for $i = 1$ or $2$ implies* $j \in \mathrm{loop}(\Gamma)$ *and $j$ is adjacent to every vertex $k$ for $k \geq 3$, and*

(iii) $12 \in \mathrm{edge}(\Gamma)$ *implies* $\mathrm{loop}(\Gamma) \supseteq \{3,\dots,\ell\}$*.* □

The modular fibration corresponding to a modular copoint has as coned fiber arrangement $c\mathcal{A}_{\overline{v}}$ a central arrangement of rank two, which is $K(\pi,1)$, so $\mathcal{A}_\Gamma$ is $K(\pi,1)$ if and only if $\mathcal{A}_{\Gamma'}$ is $K(\pi,1)$. This is a "strictly linear fibration" as studied in [13, 32].

Factored arrangements of rank three were shown to be $K(\pi,1)$ in [27]. Supersolvable arrangements of arbitrary rank are factored, and are $K(\pi,1)$. In [12] we conjecture that factored arrangements of arbitrary rank are $K(\pi,1)$. The next result provides more support for this conjecture. The arrangements of this theorem are not supersolvable, by Theorem 5.2, but are factored, by Theorem 5.3. In fact, by an argument in [1], the examples described below are the only factored non-supersolvable arrangements $\mathcal{A}_\Gamma$ which have only two non-loop vertices.

**Theorem 5.6** *Suppose $\Gamma$ is the complete graph on $\ell$ vertices, and* $\mathrm{loop}(\Gamma) = \{3,\dots,\ell\}$*. Then $\mathcal{A}_\Gamma$ is a $K(\pi,1)$ arrangement.*

*proof:* Let $\mathcal{A} = \mathcal{A}_\Gamma$. Let $\Gamma''$ denote the vertex-induced subgraph of $\Gamma$ on vertices $\{3,\dots,\ell\}$. Then $\mathcal{A}_{\Gamma''}$ is a modular coline of $G(\mathcal{A})$ by Lemma 5.5, and $\mathcal{A}_{\Gamma''}$ is supersolvable, hence $K(\pi,1)$, by Theorem 5.2. Let $X = \bigcap \mathcal{A}_{\Gamma''}$, and let $\overline{v} = (v_3,\dots,v_\ell) \in M(\mathcal{A}_X)$. Then $v_i \neq \pm v_j$ for $3 \leq i < j \leq \ell$, and $v_j \neq 0$ for

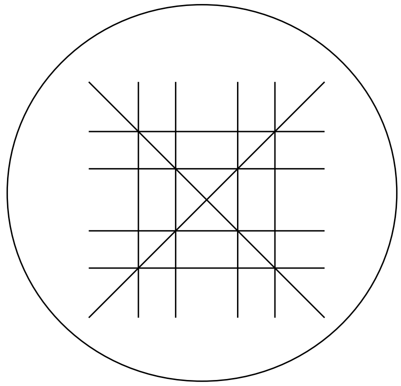

Figure 1: The coned fiber arrangement, for $\ell = 4$

$3 \leq j \leq \ell$. The fiber arrangement $\mathcal{A}_{\overline{v}}$ is the affine arrangement in $\mathbb{C}^2$ consisting of the lines

$$x_1 = \pm x_2, \;\; x_1 = \pm v_j, \;\text{ and }\; x_2 = \pm v_j$$

for $3 \leq j \leq \ell$.

The coned fiber arrangement $c\mathcal{A}_{\overline{v}}$ pictured in Figure 1 is a $K(\pi, 1)$ arrangement. Indeed $c\mathcal{A}_{\overline{v}}$ is precisely Example 3.13 of [10]. Alternatively, $c\mathcal{A}_{\overline{v}}$ is a factored arrangement of rank three, hence $K(\pi, 1)$ by [27]. We conclude by Corollary 3.3 that $\mathcal{A}$ is $K(\pi, 1)$. □

**Remark 5.7** The conclusion of the theorem also holds if $\text{loop}(\Gamma) = \emptyset$, for then $\mathcal{A}_\Gamma$ is a Coxeter arrangement of type $D_\ell$, which is simplicial. □

One can use Theorem 5.6 and Lemma 5.4 to build other examples of non-supersolvable (and non-simplicial) $K(\pi, 1)$ arrangements of high rank, by successively adding vertices satisfying 5.4 to the graphs of Theorem 5.6. See Figure 2 on the next page. The existence of loops in $\Gamma$ is essential: the same construction with the loopless complete graph (the $D_\ell$ arrangement) allows only the addition of isolated vertices.

There is one rank-three arrangement $\mathcal{A}_\Gamma$, a realization of the non-Fano matroid, which is not supersolvable, but is simplicial, hence $K(\pi, 1)$. This arrangement can also be used with Theorem 5.4 to construct non-supersolvable $K(\pi, 1)$ arrangements. This construction is illustrated in Figure 3 on the following page.

**Example 5.8** We exhibit in Figures 2 and 3 on the next page some other graphs $\Gamma$ for which $\mathcal{A}_\Gamma$ is $K(\pi, 1)$, using the constructions of the preceding paragraphs. These arrangements are factored, as is every arrangement arising from these constructions, by Theorem 5.3. □

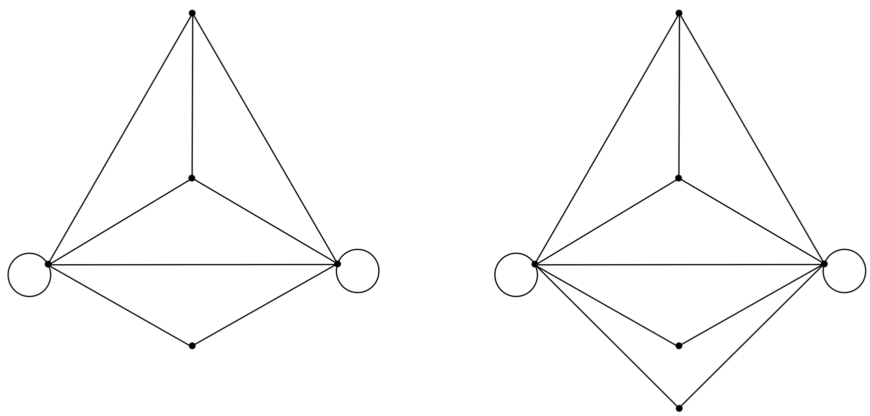

Figure 2: Extensions of Theorem 5.6

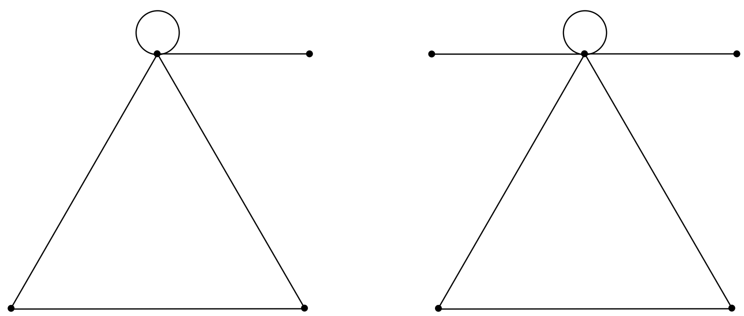

Figure 3: Graphs of $K(\pi,1)$ extensions of a non-Fano arrangement

We close with another interesting example from the class of "$A$-$B$ arrangements."

**Example 5.9** Let $\Gamma$ be the graph with vertex set $\{1,2,3,4\}$, edges 12, 13, and 24, and loops at vertices $1,2,$ and 3. Let $\Gamma_0''$ and $\Gamma_1''$ be the vertex-induced subgraphs of $\Gamma$ with vertex sets $\{1,2\}$ and $\{1,3\}$ respectively. Then both $\mathcal{A}_{\Gamma_0''}$ and $\mathcal{A}_{\Gamma_1''}$ are modular flats of $G(\mathcal{A}_\Gamma)$. The respective fiber arrangements are illustrated in Figure 4 on the following page.

The arrangement on the right, the (coned) fiber arrangement of $M(\mathcal{A}_\Gamma) \longrightarrow M(\mathcal{A}_{\Gamma_1''})$, is not $K(\pi,1)$ because it has a "simple triangle" [14, Corollary 3.3]. Now $\mathcal{A}_{\Gamma_1''}$ is a $K(\pi,1)$ arrangement, being a central arrangement of rank two. It follows that $M(\mathcal{A}_\Gamma)$ is not aspherical. But $\mathcal{A}_{\Gamma_0'}$ is also $K(\pi,1)$. We conclude that the (coned) fiber arrangement of $M(\mathcal{A}_\Gamma) \longrightarrow M(\mathcal{A}_{\Gamma_0'})$, shown on the left, cannot

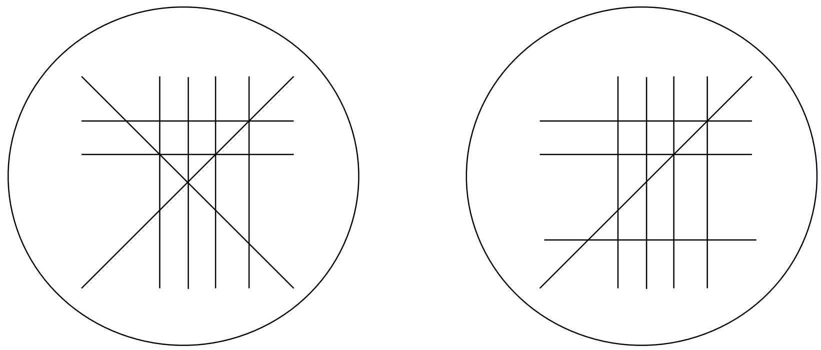

Figure 4: Two non-$K(\pi,1)$ arrangements

be $K(\pi,1)$. This is the only argument we know of to show this arrangement is not $K(\pi,1)$. □

The research presented here, and our general interest in $K(\pi,1)$ arrangements, was motivated in part by a suggestion of G. Ziegler of a straightforward construction of rank-three arrangements with the same underlying matroid but homotopy inequivalent complements. The argument avoids fundamental group computations, but relies on the existence of high-rank $K(\pi,1)$ arrangements with certain properties, whose existence has not yet been shown. Here is the precise problem, to which the methods of this paper may apply.

**Problem 5.10** (Ziegler) Find $K(\pi,1)$ arrangements whose matroids have the same flats of ranks one and two but have different characteristic polynomials.

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