The hypertoric intersection cohomology ring

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Abstract. We present a functorial computation of the equivariant intersection cohomology of a hypertoric variety, and endow it with a natural ring structure. When the hyperplane arrangement associated with the hypertoric variety is unimodular, we show that this ring structure is induced by a ring structure on the equivariant intersection cohomology sheaf in the equivariant derived category. The computation is given in terms of a localization functor which takes equivariant sheaves on a sufficiently nice stratified space to sheaves on a poset.

A hypertoric variety is a symplectic algebraic variety, equipped with a torus action, whose structure is determined by the geometry and combinatorics of a rational hyperplane arrangement in much the same way that a toric variety is determined by a rational convex polyhedron (or, more generally, a rational fan). Since hypertoric varieties were first introduced by Bielawski and Dancer [BD], many of their algebraic invariants have been computed in terms of the associated arrangements. In particular, combinatorial formulas have been given for the ordinary and equivariant cohomology rings of a smooth hypertoric variety [HS, Ko, HP], and for the intersection cohomology Betti numbers of a singular affine hypertoric variety [PW].

In this paper, we refine the results of [PW] to give a combinatorial computation of the equivariant intersection cohomology groups of a hypertoric variety $\mathfrak{M}_{\mathcal{H}}$ associated to an arbitrary rational hyperplane arrangement $\mathcal{H}$ (Theorem 2.7). We use this to prove a conjecture of [PW, 6.4], which states that the intersection cohomology of a hypertoric variety has a natural ring structure (Corollary 4.5). In the special case where $\mathcal{H}$ is central and unimodular (which is equivalent to saying that $\mathfrak{M}_{\mathcal{H}}$ is affine and has a hypertoric resolution of singularities), we show that this ring structure exists on the deepest possible level, namely on the equivariant IC sheaf in the equivariant derived category (Theorem 5.1). The unit element in this ring structure is given by the natural map from the constant equivariant sheaf, which implies that our ring structure "behaves like a cup product". In particular, for example, it implies that the restriction map from the equivariant intersection cohomology of $\mathfrak{M}_{\mathcal{H}}$ to the equivariant cohomology of the generic stratum will be a ring homomorphism.

We prove these results using a general notion, which we develop in Section 1, of localization from T-equivariant constructible sheaves on an equivariantly stratified T-space to sheaves on a poset whose elements index the strata, equipped with a linear structure that keeps track of the stabilizer on the associated stratum. In this framework, we identify the total equivariant intersection cohomology group $IH^{\bullet}_{T}(\mathfrak{M}_{\mathcal{H}})$ with the space of sections of a sheaf $\mathcal{L}$, called a minimal extension sheaf, on the poset $L_{\mathcal{H}}$ of flats of $\mathcal{H}$. The flats index the strata of a natural stratification of $\mathfrak{M}_{\mathcal{H}}$, and we show that the stalk of $\mathcal{L}$ at any flat is canonically isomorphic to the "local" equivariant intersection cohomology of $\mathfrak{M}_{\mathcal{H}}$ at any T-orbit in

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the corresponding stratum. Furthermore, the restriction maps in the sheaf coincide with the natural maps between local intersection cohomology groups of comparable strata (in fact, this last property is enough to characterize the isomorphism between the space of global sections of $\mathcal{L}$ and $IH^{\bullet}_{T}(\mathfrak{M})$ up to multiplication by a scalar). This formalism of localization functors provides a unified setting in which to understand our work along with a number of other theories involving localization of equivariant cohomology or equivariant sheaves, including [BBFK1, BBFK2, BraM, BreL, GKM].

The concept of a minimal extension sheaf was originally introduced for sheaves on fans in [BBFK1, BreL], where the sections of such sheaves give the equivariant intersection cohomology groups of toric varieties. We generalize this notion to arbitrary linear posets, focusing on the case of $L_{\mathcal{H}}$. This perspective turns out to be useful for understanding a number of important aspects of the topology of $\mathfrak{M}_{\mathcal{H}}$. In addition to our main theorem identifying $IH^{\bullet}_{T}(\mathfrak{M}_{\mathcal{H}})$ with the space of sections of $\mathcal{L}$, we show how to use $\mathcal{L}$ to compute the intersection cohomology Morse groups of $\mathfrak{M}_{\mathcal{H}}$ in the sense of stratified Morse theory for a generic projection of the moment map on our hypertoric variety. We also give a combinatorial version of the Beilinson-Bernstein-Deligne decomposition theorem for the canonical orbifold resolution of $\mathfrak{M}_{\mathcal{H}}$. Both of these computations are important for describing a duality relating perverse sheaves on hypertoric varieties defined by Gale dual arrangements, which will appear in the forthcoming paper [BLPW].

Part of our motivation for studying hypertoric varieties is that they share many geometric properties with other symplectic algebraic varieties that play prominent roles in representation theory and physics, such as quiver varieties, moduli spaces of Higgs bundles on a curve, and Hilbert schemes of points on symplectic surfaces. In light of Theorem 5.1, we are led to ask whether there is a broader class of symplectic varieties with natural ring structures on their IC sheaves. In [Pr, 3.4.4], the second author conjectures that many symplectic quotients of vector spaces, including all quiver varieties, admit such structures, and he uses the results of this paper to give an explicit conjectural presentation for the intersection cohomology rings that would arise in this way.

Finally, if $\mathcal{H}$ is an arrangement over a field other than $\mathbb{Q}$ and cannot be defined over $\mathbb{Q}$, then there is no associated hypertoric variety, but our theory of minimal extension sheaves still makes sense. For this reason the global sections of the minimal extension sheaf $\mathcal{L}$ can be called the intersection cohomology of the arrangement, regardless of whether there is a hypertoric interpretation. This is analogous to what happens for toric varieties, where Karu's theorem [Ka] implies that the intersection cohomology of a non-rational fan, defined by means of a minimal extension sheaf, satisfies all of the expected properties despite the lack of an associated toric variety.

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1 Linear posets and localization

In this first section, we construct a localization functor from the equivariant derived category of constructible sheaves on a sufficiently nice stratified space to the category of modules over the structure sheaf of a linear poset. We also give some basic definitions, results, and examples pertaining to minimal extension sheaves on a linear poset.

1.1 Sheaves on posets

Our results will be expressed in the language of sheaves on finite posets, which we now review. Suppose that $(P, \leq)$ is a finite poset. We put a topology on it by declaring $U \subset P$ to be open if for every $x \in U$ and $y \leq x$, we also have $y \in U$. Then for any $x \in P$, the set $U_x = \{y \in P \mid y \leq x\}$ is the smallest open set containing x. We can also think of P as a small category with the properties that each Hom set has at most one element, and no two distinct objects are isomorphic; here $x \leq y$ if and only if there exists a morphism $y \to x$.

Proposition 1.1 *Let $\mathcal{C}$ be an abelian category. The category of sheaves on P of objects in $\mathcal{C}$ is equivalent to the category of functors $P \to \mathcal{C}$. In other words, a sheaf $\mathcal{S}$ is given by an object $\mathcal{S}(x)$ for each $x \in P$, together with restriction maps $r_{xy} \colon \mathcal{S}(y) \to \mathcal{S}(x)$ for every $x, y \in X$ with $x \leq y$, satisfying $r_{xy} r_{yz} = r_{xz}$ whenever $x \leq y \leq z$.*

Proof: Given a sheaf $\mathcal{S}$, we obtain the data $\{\mathcal{S}(x), r_{xy}\}$ by setting $\mathcal{S}(x) = \mathcal{S}(U_x)$, the sections on the open set U_x, and letting r_{xy} be the restriction map. In the other direction, given the objects $\mathcal{S}(x)$ and maps r_{xy}, the sections $\mathcal{S}(U)$ of the sheaf $\mathcal{S}$ on an open set U is defined to be the projective limit $\varprojlim_{x \in U} \mathcal{S}(x)$. □

We will use the following shorthand: for $x \in P$, we let ∂x denote the "punctured neighborhood"

$$\partial x := U_x \smallsetminus \{x\} = \{y \in P \mid y < x\}.$$

If $\mathcal{S}$ is a sheaf on P, we let $\partial_x \colon \mathcal{S}(x) \to \mathcal{S}(\partial x)$ denote the restriction map from U_x to ∂x, and we put $\mathcal{S}(x, \partial x) = \ker \partial_x$.

Fix a field k (later we will take $k = \mathbb{R}$). We define a **linear poset** to be a pair (P, V) of a finite poset P together with a sheaf V of finite-dimensional k-vector spaces on it. In other words V is a functor $P \to \mathbf{Vect}_k$. In all of our examples the restriction maps r_{xy} will all be surjective. Given a linear poset (P, V), its **structure sheaf** $\mathcal{A} = \mathcal{A}_P$ is the sheaf of graded polynomial rings $\operatorname{Sym} V$, i.e. the sheaf whose stalk $\mathcal{A}(x)$ at $x \in P$ is equal to $\operatorname{Sym} V(x)$. We use the grading where elements of $V(x)$ have degree two. Let $\mathcal{A}{-}mod$ denote the category of finitely generated graded $\mathcal{A}$-modules; we will refer to them simply as $\mathcal{A}$-modules.

Our examples of linear posets will come from spaces with a torus action, in the following way. Let $\mathfrak{M}$ be a Hausdorff topological space on which a compact abelian Lie group T acts (in the examples we study T will be a torus, but these initial definitions make sense even when T is disconnected). We define a ***T*-decomposition** of $\mathfrak{M}$ indexed by a poset P to be a collection $\mathscr{S} = \{S_x\}_{x \in P}$ of locally closed T-invariant subspaces partitioning $\mathfrak{M}$ so that

- for every $x, y \in P$, $S_y \cap \overline{S_x} \neq \varnothing \iff S_y \subset \overline{S_x} \iff x \leq y$, and[3]
- for every $x \in P$, there is a Lie subgroup $T_x \subset T$ (possibly not connected) so that the stabilizer of any point of S_x is T_x.

Given a T-decomposition of $\mathfrak{M}$ indexed by P, we put a linear structure on P by letting $V(x) = (\mathfrak{t}_x)^*$, the dual of the Lie algebra of T_x. The definition of the poset structure ensures that $T_x \subset T_y$ if $x \leq y$, and we let the restriction $r_{xy} \colon V(y) \to V(x)$ be the map dual to the inclusion $\mathfrak{t}_x \subset \mathfrak{t}_y$.

The geometric meaning of the resulting structure sheaf $\mathcal{A} = \mathcal{A}_P$ comes from equivariant cohomology. For any $x \in P$ and any point $p \in S_x$, there is a canonical identification

$$H^\bullet_T(Tp) = H^\bullet_{T_x}(pt) = \operatorname{Sym} V(x) = \mathcal{A}(x)$$

of graded rings (all cohomology groups in this paper will be taken with coefficients in $\mathbb{R}$). If $x \leq y$, then the map $r_{xy} \colon \mathcal{A}(y) \to \mathcal{A}(x)$ is the pullback by any T-equivariant projection $Tp_x \to Tp_y$.

1.2 The localization functor

For a large class of T-spaces $\mathfrak{M}$ endowed with a T-decomposition $\mathscr{S}$, we can use modules over the structure sheaf $\mathcal{A}$ to study T-equivariant sheaves on $\mathfrak{M}$. Let $D^b_T(\mathfrak{M})$ denote the triangulated category of T-equivariant sheaves defined by Bernstein and Lunts [BerL], and let $D^b_{T,\mathscr{S}}(\mathfrak{M})$ be the full subcategory of "$\mathscr{S}$-constructible" objects: objects whose cohomology sheaves have finite-dimensional stalks, vanish outside of a finite range of degrees, and are locally constant on every S_x, $x \in P$. Let $\mu \colon \mathfrak{M} \to \mathfrak{M}/T$ denote the quotient map. We will construct, under suitable topological hypotheses on $\mathscr{S}$, a functor

$$\operatorname{Loc} \colon D^b_{T,\mathscr{S}}(\mathfrak{M}) \to \mathcal{A}{-}mod$$

so that

- for any $x \in P$ and any $p \in S_x$, there is a natural isomorphism

$$H^\bullet_T(Tp; B) \simeq (\operatorname{Loc} B)(x) \tag{1.1}$$

  of $\mathcal{A}(x) = H^\bullet_T(Tp)$-modules[4], and
- there is a natural homomorphism

$$\Gamma_B : H^\bullet_T(\mathfrak{M}; B) \to (\operatorname{Loc} B)(P) \tag{1.2}$$

  which is compatible under restriction with the isomorphism (1.1).

[3] Although the opposite convention on the partial order might seem more geometrically natural, this choice will agree with conventions regarding fans and hyperplane arrangements.

[4] Here and later in the paper we use the shorthand notation $H^\bullet_T(X; B)$ to denote the equivariant cohomology of the pullback j^*B, where $j \colon X \to \mathfrak{M}$ is the inclusion of a T-invariant subset X of $\mathfrak{M}$.

The first condition that we need the pair $(\mathfrak{M}, \mathscr{S})$ to satisfy is the following, which is an equivariant generalization of the notion of topological stratification as used by Goresky and MacPherson [GM, §1.1].

Definition 1.2 Let T be a compact Abelian Lie group, let $\mathfrak{M}$ be a T-space, and let $\mathscr{S} = \{S_x\}_{x\in P}$ be a T-decomposition of $\mathfrak{M}$ indexed by a poset P. Then we define the statement that $\mathscr{S}$ is a T**-stratification** inductively on $|P|$ to mean that for every $x \in P$, and every point $p \in S_x$, there exists

- an open neighborhood $U \subset \mathfrak{M}$ of the orbit $T \cdot p$,
- a T_x-space L, with a T_x-stratification $\{S^x_y \mid y < x\}$, and
- a T-equivariant homeomorphism

$$\phi\colon U \cong T \times_{T_x} \mathrm{cone}(L) \times D,$$

  where D is the unit disk in a Euclidean space, and $\mathrm{cone}(L)$ is the open topological cone $(L \times [0,1))/(L \times \{0\})$, with the induced T_x-action,

such that ϕ is compatible with the induced decompositions on both sides:

$$\phi(U \cap S_y) = T \times_{T_x} (S^x_y \times (0,1)) \times D$$

for all $y > x$, and $\phi(U \cap S_x) = T \times_{T_x} \{v\} \times D$, where $v \in L$ is the apex of the cone. When x is a minimal element of P, we take $L = \varnothing$, $\mathrm{cone}(L) = \{v\}$ (this takes care of the base case $|P| = 1$). The space $\mathrm{cone}(L)$ is called the **normal slice** to S_x at the point p, L itself is called the **link** of S_x, and U is called a **stratified tubular neighborhood** of S_x.

The following basic properties of this definition are easy to check.

Lemma 1.3 *If $\mathscr{S}$ is a T-stratification of $\mathfrak{M}$, then*

- *for any inclusion $j\colon \mathfrak{N} \to \mathfrak{M}$ of a locally closed union of some of the strata S_x, the functors j^*, j_*, $j^!$, $j_!$ preserve $\mathscr{S}$-constructibility,*
- *for every $x \in P$, S_x and S_x/T are both topological manifolds, and $S_x \to S_x/T$ is a principal T/T_x-fiber bundle, and*
- *for any Lie subgroup $T' \subset T$, the decomposition $\{S_x/T'\}$ defines a T/T'-stratification of $\mathfrak{M}/T'$.*

Given a T-space $\mathfrak{M}$ with a T-decomposition $\mathscr{S}$, in order to define our localization functor we will need to assume that

(A) $\mathscr{S}$ is a T-stratification,

(B) for each $x \in P$, the quotient S_x/T is simply connected, and

(C) for every $x, y \in P$, $y < x$, the space S^x_y given by Definition 1.2 is connected.

Condition (B) ensures that, for $B \in D^b_{T,\mathscr{S}}(\mathfrak{M})$, the spaces $H^\bullet_T(Tp;B)$ are all canonically isomorphic as the point p varies in S_x. Then, intuitively, we would like to define the map $(\operatorname{Loc} B)(y) \to (\operatorname{Loc} B)(x)$ for $x < y$ by letting a point $p \in S_x$ move to the boundary and degenerate to a point in $S_y \subset \overline{S_x}$. Condition (C) ensures that the resulting map does not depend on the path p takes.

More formally, we define the functor Loc by pushing forward sheaves from $\mathfrak{M}$ to P in two steps, first by the quotient map $\mu\colon \mathfrak{M} \to \mathfrak{M}/T$, and then by the natural map $\pi\colon \mathfrak{M}/T \to P$ which sends any point in S_x/T to x. The first step is a pushforward by the functor $Q_{\mu*}\colon D^b_T(\mathfrak{M}) \to D^+(\mathfrak{M}/T)$ which was defined by Bernstein and Lunts [BerL, 6.9]. We can describe this functor more concretely as follows. Let ET be an acyclic free T-space. The category $D^b_T(\mathfrak{M})$ is equivalent to the full subcategory of $D^b(\mathfrak{M} \times_T ET)$ consisting of objects whose pullback to $\mathfrak{M} \times ET$ is isomorphic to the pullback of an object of $D^b(\mathfrak{M})$ along the projection $\mathfrak{M} \times ET \to \mathfrak{M}$. Then $Q_{\mu*}$ is the (derived) pushforward $R\bar{\mu}_*$ along the quotient map $\bar{\mu}\colon \mathfrak{M} \times_T ET \to \mathfrak{M}/T$. Note the equivariant cohomology $H^\bullet_T(\mathfrak{M};B)$ of an object B of $D^b_T(\mathfrak{M})$ is just the ordinary hypercohomology of the corresponding object of $D^b(\mathfrak{M} \times_T ET)$.

We then define a graded sheaf B^μ to be the direct sum of the cohomology sheaves of $Q_{\mu*}B$. The following result shows that this sheaf has the stalks that we want.

Lemma 1.4 *For any point $p \in \mathfrak{M}$ and $B \in D^b_{T,\mathscr{S}}(\mathfrak{M})$, there is a natural isomorphism*[5]

$$H^\bullet_T(Tp;B) \cong (B^\mu)_{Tp}. \tag{1.3}$$

Proof: Base change gives a natural map $H^\bullet_T(Tp;B) \to (B^\mu)_{Tp}$. Suppose that $p \in S_x$. We have $(B^\mu)_{Tp} = \varinjlim H^\bullet(U \times_T ET; B)$, where the limit is over T-invariant open sets U containing Tp. If we let U be a stratified tubular neighborhood of S_x, as provided by Definition 1.2, then we get a saturated sequence $\{U_n\}$ of open neighborhoods of Tp by taking $\phi(U_n)$ to be the image of $T \times (L \times [0,1/n)) \times (1/n)D$ in $T \times_{T_x} \operatorname{cone}(L) \times D$. The group $H^\bullet(U_n \times_T ET; B)$ is independent of n, and so we see that

$$(B^\mu)_{Tp} = H^\bullet(U \times_T ET; B) = H^\bullet_T(U;B).$$

Note that $D^b_{T,\mathscr{S}}(\mathfrak{M})$ is generated as a triangulated category by the objects $j_{y!}\mathbb{R}_{S_y,T}$, $y \in P$, where $j_y\colon S_y \hookrightarrow \mathfrak{M}$ is the inclusion. Since both sides of Equation (1.3) are cohomological functors in B, it is enough to deal with the case $B = j_{y!}\mathbb{R}_{S_y,T}$. The map $U \times_T ET \to BT$ is a fiber bundle with fiber U; using this and the structure of the induced stratification on U it follows that when $x \neq y$, the derived pushforward of $B|_{U\times_T ET}$ to BT is zero, so $H^\bullet(U \times_T ET; B) = 0$. On the other hand, if $x = y$, then

$$H^\bullet_T(U;B) = H^\bullet_T(U \cap S_x) = H^\bullet_T(T \times_{T_x} \{v\} \times D) = H^\bullet_{T_x}(\{v\} \times D) = \mathcal{A}(x).$$

The result follows. □

[5] Note that on the left-hand side Tp is regarded as a subset of $\mathfrak{M}$, while on the right-hand side it is regarded as a point of the quotient $\mathfrak{M}/T$.

For the second step, we define the localization of B to be

$$\operatorname{Loc} B = \pi_* B^\mu.$$

Note that here π_* is the ordinary pushforward of sheaves, *not* the derived pushforward. Because of this, and because B^μ involves taking cohomology sheaves, this functor loses a lot of information. However, we shall see that it is well-behaved when applied to intersection cohomology sheaves, and it can be easily computed. Lemma 1.4 and the following lemma imply the existence of the isomorphism (1.1).

Lemma 1.5 *Let E be any sheaf on $\mathfrak{M}/T$ which is locally constant (hence constant) on S_x/T for every $x \in P$. Then the restriction*

$$(\pi_* E)(x) \to \Gamma(S_x/T, E)$$

is an isomorphism for all $x \in P$.

Proof: The stalk $(\pi_* E)(x)$ is the same as the space of sections $\Gamma(U_x; \pi_* E) = \Gamma(\pi^{-1}(U_x); E)$, where U_x is the smallest open set containing x. In other words, we need to show that

$$\Gamma(\pi^{-1}(U_x), E) \to \Gamma(S_x/T, E)$$

is an isomorphism. Our assumptions (B) and (C) on our stratification imply that this holds when $E = j_{y*}L$, where L is a local system on S_y/T, y is any element of P, and $j_y \colon S_y/T \to \mathfrak{M}/T$ is the inclusion. For general E, apply the exact sequence of sheaves

$$0 \to E \to \bigoplus_{y \in P} j_{y*} j_y^* E \to \bigoplus_{z \le w \in P} j_{w*} j_w^* j_{z*} j_z^* E.$$

We have shown that our map is an isomorphism for the second and third terms, so it is for E, as well. □

At this point $\operatorname{Loc} B$ is just a sheaf of graded vector spaces. To make it into an $\mathcal{A}$-module, first note that there is an identification of the ring $A = H_T^\bullet(pt)$ with the graded endomorphisms of the constant equivariant sheaf $\mathbb{R}_{pt,T}$ in $D_T^b(pt)$. Pulling back $\mathbb{R}_{pt,T}$ to $\mathfrak{M}$ and tensoring gives an action of A on any object $B \in D_{T,\mathscr{S}}^b(\mathfrak{M})$, and hence on the localized sheaf $\operatorname{Loc} B$. The resulting action of of A on the stalk of $(\operatorname{Loc} B)(x)$ agrees with the action on the equivariant hypercohomology $H_T^\bullet(Tp)$, $p \in S_x$, under the identification $(\operatorname{Loc} B)(x) \simeq H_T^\bullet(Tp; B)$. In particular, it factors through the quotient $\mathcal{A}(x)$, making $\operatorname{Loc} B$ into an $\mathcal{A}$-module, as required.

Finally, to define the map $\Gamma_B : H_T^\bullet(\mathfrak{M}; B) \to (\operatorname{Loc} B)(P)$, note that the hypercohomology of $Q_{\mu*}B$ is just the global equivariant cohomology $H_T^\bullet(\mathfrak{M}; B)$. This then has a natural graded homomorphism to the global sections of the sheaf B^μ, which is isomorphic to the global sections of $\operatorname{Loc} B$. It is easy to check that this is a map of A-modules.

Remark 1.6 In certain exceptional cases Γ_B will be an isomorphism; one of our main results is that this happens when $\mathfrak{M}$ is a hypertoric variety and B is the equivariant intersection cohomology sheaf on $\mathfrak{M}$. More examples in which Γ_B is an isomorphism are given in the following section.

1.3 Relations with other theories

The localization functor we have just defined generalizes constructions that have been used to study equivariant cohomology and equivariant sheaves in a number of different settings. In order to put our results in context, we point out some of these connections in this section.

One case that has been extensively studied is when $\mathfrak{M}$ is a toric variety defined by a rational fan Σ, $\mathscr{S}$ is the stratification by orbits of the complex torus $T_{\mathbb{C}}$, and T is the maximal compact subgroup of $T_{\mathbb{C}}$. The poset P is the fan Σ itself, ordered by inclusion of cones. The linear structure is given by $V(\sigma) = \mathrm{span}(\sigma)^*$ for any cone $\sigma \in \Sigma$. The resulting structure sheaf $\mathcal{A}$ is also known as the sheaf of "conewise polynomial functions" on Σ: its sections on an open set (subfan) $\Sigma' \subset \Sigma$ is the ring of real-valued functions on the support of Σ' which restrict to polynomial functions on each cone.

For any space satisfying the assumptions of §1.2, the structure sheaf $\mathcal{A}$ will be isomorphic to the localization of the equivariant constant sheaf $\mathbb{R}_T$ on $\mathfrak{M}$. When $\mathfrak{M}$ is a toric variety, the ring $\mathcal{A}(\Sigma)$ of global sections is naturally isomorphic to the equivariant Chow cohomology ring of $\mathfrak{M}$ [Pa, Thm. 1]. If the fan Σ is simplicial, then $\mathfrak{M}$ is rationally smooth (that is, it has at worst orbifold singularities), and the map $\Gamma_{\mathbb{R}_T}$ of (1.2) will be an isomorphism. For arbitrary rational fans, if we take B to be the equivariant intersection cohomology sheaf $\mathbf{IC}_T(\mathfrak{M})$, the map $\Gamma_B : IH^\bullet_T(\mathfrak{M}) \to \mathrm{Loc}(B)(\Sigma)$ is an isomorphism [BBFK1, Theorem 2.2].

Remark 1.7 Because toric varieties have such simple geometry, the authors of [BBFK1] were able to use a simpler (but equivalent) construction in place of our localization functor, defining a presheaf on a fan by taking equivariant intersection cohomology on open unions of $T_{\mathbb{C}}$-orbits. The statement that Γ_B is an isomorphism is equivalent to saying that this presheaf is a sheaf. These papers also gave an algorithm for computing the sheaf $\mathrm{Loc}(\mathbf{IC}_T(\mathfrak{M}))$ by showing that it is a minimal extension sheaf, a concept which we discuss in the next section.

Our localization functor can also be used to express aspects of the theory of moment graphs. Suppose X is a proper normal algebraic variety X over $\mathbb{C}$ endowed with an algebraic action of a torus $T_{\mathbb{C}}$, and let $\mathfrak{M} \subset X$ be the subvariety which is the union of all the zero and one-dimensional orbits of X. If $\dim_{\mathbb{C}} \mathfrak{M} = 1$, it will consist of a collection of projective lines joined at $T_{\mathbb{C}}$-fixed points. The resulting linear poset can be viewed as a graph with directions assigned to an edge, which has sometimes been called the **moment graph** of X. Goresky, Kottwitz and MacPherson showed that if X is **equivariantly formal** [GKM, §1.2] (this will hold for instance if X is smooth and projective, or if it admits a paving by affines), then a theorem of Chang and Skjelbred [CS, Lemma 2.3] implies that $H^\bullet_T(X)$ is isomorphic as a ring to $\mathcal{A}(P)$, via the composition of the restriction $H^\bullet_T(X) \to H^\bullet_T(\mathfrak{M})$ with $\Gamma_{\mathbb{R}_{\mathfrak{M},T}} : H^\bullet_T(\mathfrak{M}) \to \mathcal{A}(P)$. Guillemin and Zara [GZ1, GZ2, GZ3] have studied many aspects of the geometry of moment graphs for smooth varieties X (using a stronger definition reflecting more of the geometry of X) and their cohomology.

If X is singular, then it may fail to be equivariantly formal, but at least when it is projective X will be equivariantly formal for intersection cohomology, which implies that $IH^\bullet_T(X)$

is isomorphic as an $H_T(pt)$-module to the global sections of the sheaf $\mathrm{Loc}(\mathbf{IC}_{X,T}|_{\mathfrak{M}})$. When X is a Schubert variety in a flag variety, the first author and MacPherson [BraM, Theorem 1.5] showed that this sheaf can be computed using a universal property similar to the one satisfied by minimal extension sheaves.

1.4 Pure $\mathcal{A}$-modules and minimal extension sheaves

The concept of pure sheaves and minimal extension sheaves will be central throughout this paper. Such objects have previously been defined and studied only in the special case of a fan, described in Section 1.3. Nonetheless, the definitions and the basic results concerning them generalize without difficulty, and we refer the reader to [BBFK2, BreL] for proofs of Proposition 1.10 and Lemma 1.16.

Definition 1.8 For an arbitrary linear poset (P, V), we call an $\mathcal{A}$-module $\mathcal{M}$ **pure** if

- it is pointwise free: for every $x \in P$, $\mathcal{M}(x)$ is a free $\mathcal{A}(x)$-module, and
- it is flabby: for any $U \subset P$ open, the restriction $\mathcal{M}(P) \to \mathcal{M}(U)$ is surjective.

Definition 1.9 If M is a graded module over a polynomial ring $\mathrm{Sym}\, V$, we define

$$\overline{M} = M/VM,$$

and for a homomorphism $\phi\colon M_1 \to M_2$ of graded modules, we let $\overline{\phi}\colon \overline{M_1} \to \overline{M_2}$ denote the induced homomorphism of graded vector spaces.

The following proposition gives a classification of pure sheaves on a linear poset. See [BBFK2, Proposition 1.3 and Theorem 2.3], [BreL, Theorem 5.3] for proofs.

Proposition 1.10 *For each $x \in P$, there is an indecomposable pure $\mathcal{A}_P$-module $\mathcal{L}_x$, unique up to isomorphism, with the property that $\mathcal{L}_x(y) = 0$ if $x \not\leq y$ and $\mathcal{L}_x(x) \cong \mathcal{A}(x)$. For any pure sheaf $\mathcal{M}$, there is an isomorphism*

$$\mathcal{M} \cong \bigoplus_{x \in P} \ker \overline{\partial_x} \otimes_k \mathcal{L}_x. \tag{1.4}$$

The sheaf $\mathcal{L}_x$ is known as a **minimal extension sheaf**.

Remark 1.11 The isomorphisms in Proposition 1.10 are not in general canonical.

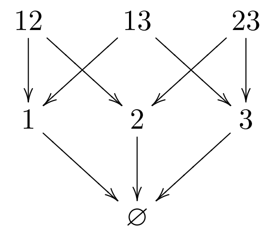
Remark 1.12 Note that a sheaf $\mathcal{M}$ is flabby if and only if the restriction $\partial_y\colon \mathcal{M}(y) \to \mathcal{M}(\partial y)$ is surjective for all $y \in P$. The fact that $\mathcal{L}_x$ is both flabby and indecomposable means that

$$\overline{\partial_y}\colon \overline{\mathcal{L}_x(y)} \to \overline{\mathcal{L}_x(\partial y)}$$

is surjective when $y = x$ and an isomorphism for all other $y \in P$.

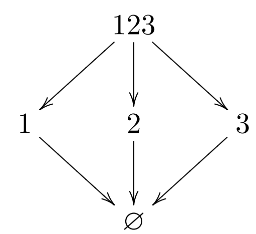
We next give a pair of examples that illustrate the phenomena that we will encounter in Section 3. Let V be a two-dimensional real vector space, and let ℓ_1, ℓ_2, and ℓ_3 be three distinct lines in V.

Example 1.13 Consider the poset



with linear structure given by $V(\varnothing) = 0$, $V(i) = V/\ell_i$, and $V(ij) = V$, with the obvious restriction maps. It is easy to check that the structure sheaf $\mathcal{A}$ is flabby, and is therefore a minimal extension sheaf $\mathcal{L}_\varnothing$ for the minimal element $\varnothing$. See Lemma 3.2 for a generalization of this example.

Example 1.14 On the other hand, consider the poset



obtained by collapsing the three maximal elements of the poset in Example 1.13 into a single element 123, and putting $V(123) = V$. Now the structure sheaf $\mathcal{A}$ is no longer flabby: the restriction map

$$\mathcal{A}(123) \to \mathcal{A}(\partial 123)$$

is not surjective. Thus $\mathcal{L}_\varnothing(123)$ must have some "extra stuff" to correct this problem. Indeed, we have

$$\mathcal{L}_\varnothing(\varnothing) = \mathcal{A}(\varnothing) = \operatorname{Sym} V(\varnothing) = \mathbb{R},$$

$$\mathcal{L}_\varnothing(i) = \mathcal{A}(i) = \operatorname{Sym} V(i) \quad \text{for } i = 1, 2, 3,$$

$$\text{and} \quad \mathcal{L}_\varnothing(123) = \operatorname{Sym} V \oplus \operatorname{Sym} V[-2].$$

To define the restriction maps, we must specify the image of the generator of $\operatorname{Sym} V[-2]$ in $\mathcal{L}_\varnothing(\partial 123)_2 \cong V/\ell_1 \oplus V/\ell_2 \oplus V/\ell_3$. The only requirement is that it should not come from a single element of V. In fact, any two choices satisfying this condition will define the same sheaf up to a unique isomorphism; the fact that there is no natural choice reflects the fact that there is no natural basis of the free $\operatorname{Sym} V$-module $\mathcal{L}_\varnothing(123)$.

Remark 1.15 When the linear poset comes from a rational fan as described in §1.3, Proposition 1.10 is a combinatorial version of the decomposition theorem of Beilinson, Bernstein, and Deligne [BBD] for toric resolutions of singularities (or more precisely its equivariant version proved by Bernstein and Lunts [BerL, §5.3]). A resolution $\varpi\colon \widetilde{X} \to X$ of toric varieties arises from a subdivision of a rational fan Σ into a smooth fan $\widetilde{\Sigma}$. The decomposition theorem says that the pushforward $\varpi_* \mathbb{R}_{\widetilde{X},T}$ of the constant equivariant sheaf splits as a direct sum of shifted intersection cohomology sheaves of subvarieties of X. Because ϖ is $T_{\mathbb{C}}$-equivariant, these subvarieties must be closures of $T_{\mathbb{C}}$-orbits.

If O_σ is the orbit corresponding to $\sigma \in \Sigma$, then the localization of $\mathbf{IC}_T(\overline{O_\sigma})$ is the minimal extension sheaf $\mathcal{L}_\sigma$. On the other hand, pushing forward the structure sheaf $\mathcal{A}_{\widetilde{\Sigma}} = \operatorname{Loc} \mathbb{R}_{\widetilde{X},T}$ to Σ gives an $\mathcal{A}_\Sigma$-module $\mathcal{E}$, which is easily seen to be pure. It is not hard to show that localization commutes with pushforwards, so $\mathcal{E} = \operatorname{Loc} \varpi_* \mathbb{R}_{\widetilde{X},T}$. The decomposition theorem thus implies that $\mathcal{E}$ splits into a direct sum of shifts of the minimal extension sheaves $\mathcal{L}_\sigma$, $\sigma \in \Sigma$, as guaranteed by Proposition 1.10.

We say that an $\mathcal{A}_P$-module $\mathcal{E}$ is **rigid** if the group of (grading-preserving) automorphisms of $\mathcal{E}$ consists only of multiplications by nonzero scalars. This notion is important because it tells us that if $\mathcal{E}$ is rigid and another $\mathcal{A}_P$-module $\mathcal{F}$ is isomorphic to $\mathcal{E}$, then that isomorphism is (nearly) canonical. In our applications the groups $\mathcal{E}(P)$ and $\mathcal{F}(P)$ will have degree zero parts that are naturally identified with the base field k, which will allow us to make our isomorphisms completely canonical. We will use the following criterion to establish rigidity in the proof of our first main result, Theorem 2.7. See [BBFK2, Remark 1.8] for a proof.

Lemma 1.16 *A minimal extension sheaf $\mathcal{L}_x$ is rigid if and only if for each $y \in P$ there exists a number d so that $\mathcal{L}_x(y)$ is generated in degrees $\leq d$ and $\mathcal{L}_x(y, \partial y)$ is generated in degrees $> d$.*

2 Hyperplane arrangements and hypertoric varieties

In this section we review a number of constructions related to hyperplane arrangements. We explain how to associate to an arrangement a linear poset (the lattice of flats) and an algebraic variety (the hypertoric variety). Our main purpose is to state Theorem 2.7, which relates the equivariant intersection cohomology sheaf on a hypertoric variety to a sheaf on the lattice of flats via the localization functor of Section 1.

2.1 Hyperplane arrangements

We briefly describe the notation and main constructions for hyperplane arrangements that we will use. Let I be a finite indexing set, and let V be an affine linear subspace of $\mathbb{R}^I$ which is not contained in any translate of a coordinate subspace $\mathbb{R}^J$, $J \subsetneq I$. Then we consider the collection $\mathcal{H}$ of affine hyperplanes in V formed by intersecting V with the coordinate hyperplanes of $\mathbb{R}^I$:

$$H_i := V \cap \{x \in \mathbb{R}^I \mid x_i = 0\}.$$

Note that this is technically a multi-set; there is no reason why the H_i must be distinct, and in fact the phenomenon of repeated hyperplanes will be forced upon us by one of the constructions that we will define presently. The hyperplanes are also cooriented, meaning that they come equipped with normal vectors, namely the restrictions of the coordinate linear forms on $\mathbb{R}^I$. When we refer to an **arrangement** in V, we will always mean a multi-set of cooriented affine hyperplanes in V whose normal vectors span V^*.

A **flat** of $\mathcal{H}$ is a subset of I of the form $\{i \in I \mid x \in H_i\}$ for some $x \in V$. Given a flat $F \subset I$, we set

$$H_F := \bigcap_{i \in F} H_i.$$

This gives a bijection between the set of flats and the set of all possible nonempty intersections of the hyperplanes H_i. Let $L_{\mathcal{H}}$ denote the poset of all flats of $\mathcal{H}$ ordered by inclusion: $E \leq F$ whenever $E \subset F$, or equivalently $H_E \supset H_F$. It is a ranked poset; the rank $\operatorname{rk} F$ of a flat F is the codimension of H_F in V, and the rank of $\mathcal{H}$ is $\dim V$, since a maximal flat F has $H_F = 0$.

If V is a vector subspace of $\mathbb{R}^I$ (that is if it contains the origin), the arrangement $\mathcal{H}$ will be called **central**. Note that a central arrangement has a unique maximal flat, namely I itself, and $H_I = \{0\}$. At the other end of the spectrum, if $V \subset \mathbb{R}^I$ is generic with respect to translation, then $|F| = \operatorname{codim} H_F$ for every flat F. Such an arrangement will be called **simple**.

For any flat F of $\mathcal{H}$, we may define two auxiliary arrangements as follows. The **restriction** of $\mathcal{H}$ at F, denoted $\mathcal{H}^F$, is the arrangement defined by the inclusion of H_F into

$$\mathbb{R}^{I \smallsetminus F} := \{x \in \mathbb{R}^I \mid x_i = 0 \text{ for all } i \in F\}.$$

It is the arrangement in H_F with hyperplanes $H_i \cap H_F$ for all $i \notin F$. Its lattice of flats is isomorphic to the ideal $\{E \in L_{\mathcal{H}} \mid E \geq F\}$.

The **localization** of $\mathcal{H}$ at F, denoted $\mathcal{H}_F$, is the arrangement given by the inclusion of $\pi_F(V)$ into

$$\mathbb{R}^F := \mathbb{R}^I / \mathbb{R}^{I \smallsetminus F},$$

where $\pi_F \colon \mathbb{R}^I \to \mathbb{R}^F$ is the coordinate projection. Its hyperplanes are $\pi_F(H_i)$ for all $i \in F$, i.e. the images of the hyperplanes of $\mathcal{H}$ which contain H_F. The lattice of flats $L_{\mathcal{H}_F}$ is isomorphic to the interval $[\varnothing, F]$ of $L_{\mathcal{H}}$. The localization $\mathcal{H}_F$ is always central, with F as its unique maximal flat; this is because $\pi_F(H_F) = \{0\}$, so the origin is contained in $\pi_F(V)$.

For any flat F, let $\langle F \rangle$ denote the **linearization** of H_F, by which we mean the linear subspace of $\mathbb{R}^I$ obtained by translating H_F back to the origin. Let $V_0 := \langle \varnothing \rangle$ be the linearization of V, so that $\langle F \rangle = V_0 \cap \mathbb{R}^{I \smallsetminus F}$ for any flat F. Let

$$V(F) := \pi_F(V) = \pi_F(V_0) = V_0 / \langle F \rangle;$$

it is the normal space of the inclusion of H_F into V.

The poset $L_{\mathcal{H}}$ naturally becomes a linear poset if we associate to each flat F the vector space $V(F)$, and to each pair $E \leq F$ the natural quotient map $V(F) \to V(E)$. We will denote the structure sheaf of this linear poset by $\mathcal{A}$ when the arrangement is clear from the context and $\mathcal{A}_{\mathcal{H}}$ when it isn't. Note that every stalk $\mathcal{A}(F)$ of this sheaf is a quotient of a single polynomial ring $A = \operatorname{Sym} V_0$.

Remark 2.1 The arrangement $\mathcal{H}$ is simple if and only if every localization $\mathcal{H}_F$ is a normal crossings arrangement, defined by the inclusion of $\mathbb{R}^F$ into itself. In other words, simplicity is equivalent to the property that the inclusion $V(F) \hookrightarrow \mathbb{R}^F$ is an isomorphism for each flat

F.

For any arrangement $\mathcal{H}$ defined by $V \subset \mathbb{R}^I$, we can find a translation $\widetilde{V}$ of $V \subset \mathbb{R}^I$ with the property that the associated arrangement $\widetilde{\mathcal{H}}$ is simple. If we identify $\widetilde{V}$ with V by an affine transformation of $\mathbb{R}^I$, the hyperplane $\widetilde{H}_i$ of $\widetilde{\mathcal{H}}$ is a translation of the hyperplane H_i of $\mathcal{H}$, and simplicity of $\widetilde{H}$ means that these translations are maximally generic with respect to intersections. We then have a canonical surjection

$$\pi : L_{\widetilde{\mathcal{H}}} \to L_{\mathcal{H}},$$

taking each flat of $\widetilde{\mathcal{H}}$ to the minimal flat of $\mathcal{H}$ that contains it, with the property that

$$\pi^* \mathcal{A}_{L_{\mathcal{H}}} = \mathcal{A}_{L_{\widetilde{\mathcal{H}}}}.$$

We will refer to the arrangement $\widetilde{\mathcal{H}}$ as a **simplification** of $\mathcal{H}$. Observe that a simplification $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ induces a simplification $\widetilde{\mathcal{H}_F}$ of the localization $\mathcal{H}_F$ at any flat F.

Finally, suppose that $V \subset \mathbb{R}^I$ is defined over the rational numbers. We say that the resulting arrangement $\mathcal{H}$ is **unimodular** if for each subset $S \subset I$, the projection of the lattice $V \cap \mathbb{Z}^I$ to $\mathbb{Z}^S$ has no cotorsion. Unimodularity depends only on V_0 (rather than its translation V), and it is preserved by restriction and localization.

Remark 2.2 Seymour [Se] gives what amounts to a classification of unimodular arrangements up to repetition and translation of hyperplanes. More precisely, he shows that every central unimodular arrangement can be built out of graphic arrangements (subarrangements of the braid arrangement), their Gale duals, and one exceptional example, using three elementary gluing constructions. See [W, 1.2.5, 3.1.1] for a detailed statement of this result.

Remark 2.3 The linear poset structure on $L_{\mathcal{H}}$ and the one on the set of cones in a fan have a common generalization. Let W be a finite dimensional vector space, and take a collection $\{u_i \mid i \in S\}$ of nonzero vectors in W indexed by a finite set S. Then for any subset $E \subset S$, we put $V(E) := \operatorname{span}\{u_i \mid i \in E\}^*$. If $P \subset 2^S$ is a collection of subsets of S, ordered by inclusion, this defines a linear poset structure on P whose restriction maps are the natural quotient morphisms.

If Σ is a fan in W, the associated linear poset is given by taking S to be the collection of 1-cones, u_ρ any nonzero vector in ρ, and P the collection of all sets of the form $\{\rho \in S \mid \rho \subset \sigma\}$ where $\sigma \in \Sigma$. If $\mathcal{H}$ is an arrangement in V with indexing set I, we get our linear poset structure on $L_{\mathcal{H}}$ by taking $W = V_0^*$, $S = I$, $P = L_{\mathcal{H}}$, and letting u_i be the normal vector to the ith hyperplane (or in other words the restriction of the coordinate function $x_i \colon \mathbb{R}^I \to \mathbb{R}$ to V_0) for any $i \in I$.

It is then easy to see that Example 1.14 is the linear poset associated to a central arrangement of three lines in the plane whose normal vectors sum to zero, while Example 1.13 comes from any simplification of this arrangement.

2.2 Matroid and broken circuit complexes

Here we collect a few definitions and well-known results from algebraic combinatorics that we will need in Section 3. A **simplicial complex** Δ on the ground set I is a nonempty collection of subsets of I, called **faces**, that is closed under inclusion: $S' \subset S \in \Delta \Rightarrow S' \in \Delta$. Given such a Δ, its **face ring** $\mathbb{R}[\Delta]$ is defined to be the quotient of the polynomial ring $\operatorname{Sym}\mathbb{R}^I = \mathbb{R}[e_i]_{i\in I}$ by the ideal generated by square-free monomials of the form $e_S := \prod_{i\in S} e_i$ for $S \subset I$ *not* a face. To fit with our interpretation of this ring as equivariant cohomology, we place the generators e_i in degree two.

At several points we will make use of the fact that an inclusion $\Delta' \subset \Delta$ of simplicial complexes induces a canonical homomorphism $\mathbb{R}[\Delta] \to \mathbb{R}[\Delta']$ of face rings by sending the generator e_i of $\mathbb{R}[\Delta]$ to the corresponding generator e'_i of $\mathbb{R}[\Delta']$ if $\{i\} \in \Delta'$, and to 0 otherwise. We will refer to this as the restriction homomorphism dual to the inclusion.

The simplicial complex Δ is called **Cohen-Macaulay** if there exists a vector subspace $W \subset \mathbb{R}^I$ such that $\mathbb{R}[\Delta]$ is a finitely generated free module over the polynomial ring $\operatorname{Sym} W$. In particular this implies that all of the maximal faces of Δ must have cardinality $d = \dim W$. If Δ is Cohen-Macaulay, its **h-polynomial** $h_\Delta(q)$ is defined to the Hilbert series of

$$\mathbb{R}[\Delta] \otimes_{\operatorname{Sym} W} \mathbb{R}$$

with degrees reduced by half. In other words, the coefficient of q^k is the number of generators of $\mathbb{R}[\Delta]$ over $\operatorname{Sym} W$ in degree $2k$. It can be computed by the formula

$$h_\Delta(q) = \sum_{k\geq 0} f_k(\Delta)\, q^k\, (1-q)^{d-k},$$

where $f_k(\Delta)$ is the number of faces of Δ of cardinality k (i.e. simplices of dimension $k-1$) [St, §II.2].

There are two classes of simplicial complex that will interest us. The first is the **matroid complex** associated to an affine linear subspace $V \subset \mathbb{R}^I$, where a set $S \subset I$ is a face if and only if the composition $V \hookrightarrow \mathbb{R}^I \twoheadrightarrow \mathbb{R}^S$ is surjective. We denote this complex $\Delta_{\mathcal{H}}$. Note that with our conventions, a central arrangement and its simplification have the same associated matroid complex: a subset $S \subset I$ is a face if and only if the normal vectors to the corresponding hyperplanes form an independent set. If the arrangement is simple, then faces of $\Delta_{\mathcal{H}}$ may equivalently be characterized as sets of hyperplanes with nonempty intersection. In other words, the posets $\Delta_{\mathcal{H}}$ and $L_{\mathcal{H}}$ are the same for a simple arrangement $\mathcal{H}$.

The second type of simplicial complex that we need is the **broken circuit complex** of a matroid complex. Given a matroid complex Δ, a **circuit** is a minimal subset of I that is not a face of Δ. Given an ordering σ of I, a **broken circuit** is a set obtained by removing the σ-minimal element of a circuit, and the broken circuit complex $\Delta^{bc}_{\mathcal{H}}$ is defined to be subcomplex of Δ consisting of those faces that do not contain any broken circuit.

Both matroid complexes and their broken circuit complexes are Cohen-Macaulay, with the subspace $V_0 \subset \mathbb{R}^I$ serving as an appropriate W (see, for example, [HS, §4] and [PS, Prop 1]). While the broken circuit complex depends on the ordering σ, its h-polynomial does not

[B, §7.4]. We will denote the h-polynomials of $\Delta_{\mathcal{H}}$ and any of its broken circuit complexes by $h_{\mathcal{H}}(q)$ and $h^{bc}_{\mathcal{H}}(q)$, respectively. The degree of $h_{\mathcal{H}}(q)$ is less than or equal to the rank of $\mathcal{H}$, while the degree of $h^{bc}_{\mathcal{H}}(q)$ is strictly less than the rank of $\mathcal{H}$ [B, §7.4].

2.3 Hypertoric varieties

In this section we assume that the affine linear subspace $V \subset \mathbb{R}^I$ is spanned by its rational points, and we explain how to use it to construct a complex algebraic variety called a **hypertoric variety**, originally introduced by Bielawski and Dancer [BD]. We will mainly be interested in how hypertoric varieties look topologically, as stratified spaces, using the stratification introduced in [PW, §2].

Consider the coordinate torus $T^I = U(1)^I$. We will think of T^I as the torus whose Lie algebra is *dual* to $\mathbb{R}^I$; in other words, we will identify T^I with the quotient of $(\mathbb{R}^I)^*$ by the standard lattice $(\mathbb{Z}^I)^*$. Define an action of T^I on the complex symplectic vector space $\mathbb{H}^I = T^*\mathbb{C}^I = \mathbb{C}^I \times (\mathbb{C}^I)^*$ by letting T^I act by coordinate multiplication on $\mathbb{C}^I$, and by the contragredient action on $(\mathbb{C}^I)^*$. Then $\mathbb{H}^I$ carries a natural hyperkähler structure so that this action is hyperhamiltonian with moment map

$$\Psi = (\Psi_{\mathbb{R}}, \Psi_{\mathbb{C}}) : \mathbb{H}^I \to \mathbb{R}^I \times \mathbb{C}^I \;\cong\; \mathbb{R}^I \otimes_{\mathbb{R}} \operatorname{Im}\mathbb{H}.$$

The components $\Psi_{\mathbb{R}}$ and $\Psi_{\mathbb{C}}$ of Ψ are given by the formulas

$$\Psi_{\mathbb{R}}(z,w) = \sum_{i\in I}(|z_i|^2 - |w_i|^2)\delta_i \quad\text{and}\quad \Psi_{\mathbb{C}}(z,w) = \sum_{i\in I} 2 z_i w_i \delta_i,$$

where δ_i is the i^{th} standard basis vector in $\mathbb{R}^I$ or $\mathbb{C}^I$.

The map Ψ is T^I-invariant; in fact, the fibers are single T^I-orbits, so Ψ identifies the quotient $\mathbb{H}^I/T^I$ with the target $\mathbb{R}^I \otimes_{\mathbb{R}} \operatorname{Im}\mathbb{H}$. The stabilizer of a point $(z,w) \in \mathbb{H}^I$ is the coordinate subtorus

$$T_{(z,w)} := \{t \in T^I \mid t_i \neq 1 \implies z_i = w_i = 0 \text{ for all } i \in I\}.$$

The hypertoric variety $\mathfrak{M}_{\mathcal{H}}$ is a hyperkähler quotient of $\mathbb{H}^I$ by a subtorus of T^I. More precisely, let

$$T_{V_0} = {V_0}^{\perp}/(V_0^{\perp} \cap (\mathbb{Z}^I)^*) \subset (\mathbb{R}^I)^*/(\mathbb{Z}^I)^*$$

be the subtorus of T^I whose Lie algebra is $V_0^{\perp}$, and let

$$V^{\text{hk}} := V \times V_0^{\mathbb{C}} \subset \mathbb{R}^I \times \mathbb{C}^I.$$

Then

$$\mathfrak{M}_{\mathcal{H}} := \Psi^{-1}(V^{\text{hk}})/T_{V_0}$$

is defined to be the quotient of the preimage of V^{hk} by the action of T_{V_0}. The hypertoric variety $\mathfrak{M}_{\mathcal{H}}$ is rationally smooth if and only if the arrangement $\mathcal{H}$ is simple, and it is smooth if and only if $\mathcal{H}$ is simple and unimodular [BD, 3.2 & 3.3].

Remark 2.4 We have just defined hypertoric varieties in the manner most convenient for our purposes here, but there are also more algebraic ways to think about these spaces. First, the hyperkähler quotient may be thought of as a holomorphic symplectic quotient, in which the torus T_{V_0} is replaced by its complexification and the real moment map equation is replaced by a stability condition. Alternatively, a smooth hypertoric variety may be defined as a union of the cotangent bundles of the toric varieties that appear in Remark 2.10. See [Pr, §1.1 & Rmk. 2.1.6] for more details.

Consider the affine subspace arrangement of $V^{\rm hk}$ consisting of the subspaces

$$H_i^{\rm hk} := H_i \times (H_i)_0^{\mathbb{C}}$$

for all $i \in I$, where $(H_i)_0^{\mathbb{C}} \subset V_0^{\mathbb{C}}$ is the complexification of the linearization of H_i. For all flats F of $\mathcal{H}$, let

$$H_F^{\rm hk} = \bigcap_{i\in F} H_i^{\rm hk} \quad \text{and} \quad \mathring{H}_F^{\rm hk} := H_F^{\rm hk} \smallsetminus \bigcup_{E<F} H_E^{\rm hk}.$$

Let $T = T^I/T_{V_0}$ be the torus with Lie algebra V_0^*. The action of T^I on $\mathbb{H}^I$ induces an action of T on $\mathfrak{M}_{\mathcal{H}}$ with hyperkähler moment map $\mu\colon \mathfrak{M}_{\mathcal{H}} \to V^{\rm hk}$ induced by Ψ. Each fiber of μ is a single T-orbit, and the Lie algebra stabilizer of a point of $\mu^{-1}(p)$ for $p \in \mathring{H}_F^{\rm hk}$ is $V(F)^* = H_F^{\perp} \subset V^*$.

The space $\mathfrak{M}_{\mathcal{H}}$ has a decomposition $\mathscr{S}$ into the pieces

$$S_F := \mu^{-1}(\mathring{H}_F^{\rm hk}),\ F \in L_{\mathcal{H}}.$$

Equivalently, S_F is the image of the points $(z, w) \in \Psi^{-1}(V^{\rm hk})$ for which $z_i = w_i = 0$ if and only if $i \in F$. It is easy to see that $S_E \subset \overline{S_F}$ if and only if $F \subset E$ for any flats E, F. Every point in S_F has the same stabilizer T_F, namely the subtorus of T with Lie algebra $V(F)^*$. Thus if we induce a linear poset structure on the lattice of flats $L_{\mathcal{H}}$ by applying the construction of Section 1.1 to the pair $(\mathfrak{M}, \mathscr{S})$, we get exactly the one given in Section 2.1.

The following result, which is similar to Lemmas 2.4 and 2.5 of [PW], describes the local structure of this decomposition. Note that for any flat F the torus T_F is naturally isomorphic to the torus which acts on $\mathfrak{M}_{\mathcal{H}_F}$, and T/T_F is naturally isomorphic to the torus which acts on $\mathfrak{M}_{\mathcal{H}^F}$.

Proposition 2.5 *$\mathscr{S}$ is a T-stratification of $\mathfrak{M}$; the normal slice to a point in the stratum S_F can be taken to be isomorphic as a stratified T_F-space to $\mathfrak{M}_{\mathcal{H}_F}$. Furthermore, the closure $\overline{S_F}$ of a stratum is isomorphic as a stratified T/T_F-space to $\mathfrak{M}_{\mathcal{H}^F}$.*

Proof: By Lemma 1.3, to show that $\mathscr{S}$ is a T-stratification it is enough to work "upstairs" and show that the decomposition of $\Psi^{-1}(V^{\rm hk})$ into the sets $\Psi^{-1}(\mathring{H}_F^{\rm hk})$, $F \in L_{\mathcal{H}}$ is a T^I-stratification. To see this, fix F and take a point $p \in \Psi^{-1}(\mathring{H}_F^{\rm hk})$. Choose an $\mathbb{R}$-vector subspace $N \subset V_0^{\rm hk} = V_0 \times V_0^{\mathbb{C}}$ which is complementary to $H_F^{\rm hk}$. Then we can find an open disk $D \subset H_F^{\rm hk}$ centered at p and (shrinking D if necessary) an open disk $B \subset N$ centered at 0 so that $B + D$ meets $H_E^{\rm hk}$ if and only if $E \leq F$.

Let $U = \Psi^{-1}(B + D)$. If $\Psi_F\colon \mathbb{H}^F \to (\operatorname{Im}\mathbb{H})^F$ is the hyperkähler moment map for $\mathbb{H}^F$, then we have a map $U \to \Psi_F^{-1}(V(F)^{\mathrm{hk}})$ given by restricting the projection $\mathbb{H}^I \to \mathbb{H}^F$. Its image is $\Psi_F^{-1}(B_F)$, where $B_F \cong B$ is the image of B under the projection $\pi_F \otimes 1_{\mathrm{Im}\,\mathbb{H}}$. This will be the normal slice to the stratum $\Psi^{-1}(\mathring{H}_F^{\mathrm{hk}})$.

Define a map $\tau \colon U \to T^{I \setminus F}$ as follows. Suppose that the ith coordinate of p is $z_i + w_i\,\mathbf{j}$, where $z_i, w_i \in \mathbb{C}$. Then if a point $q \in U$ has coordinates $z_i' + w_i'\,\mathbf{j}$, we let the ith coordinate of $\tau(q)$ be $z_i'/|z_i'|$ if $z_i \neq 0$ and $w_i'/|w_i'|$ otherwise (we can shrink the disks B and D if necessary to ensure that $z_i' \neq 0$ for all points in U in the first case, and $w_i' \neq 0$ in the second case).

Consider the map $U \to \Psi_F^{-1}(B_F) \times D$ which sends q to $(\psi(q), d)$, where $\Psi(q) = b + d$, $b \in B$, $d \in D$. Its restriction to $\tau^{-1}(1)$ is a continuous, proper bijection, and so it is a homeomorphism. If σ denotes its inverse, then $(t, x, y) \mapsto t \cdot \sigma(x, y)$ defines the required homeomorphism

$$T^I \times_{T^F} \Psi_F^{-1}(B_F) \times D \to U.$$

Since $V(F)$ is a linear subspace of $\mathbb{R}^F$, the stratifications of $V(F)^{\mathrm{hk}}$ and $\Psi_F^{-1}(V(F)^{\mathrm{hk}})$ are invariant under the the multiplication action of $\mathbb{R}^+$. Thus we have a stratum-preserving homeomorphism $\Psi_F^{-1}(B_F) \cong \Psi_F^{-1}(V(F)^{\mathrm{hk}})$, and the normal slice is topologically a cone, as required. In addition, this implies that the normal slice to S_F in $\mathfrak{M}$ is isomorphic as a T_F-stratified space to $\Psi_F^{-1}(V(F)^{\mathrm{hk}})/T_{V(F)_0} = \mathfrak{M}_{\mathcal{H}_F}$.

Finally, the identification of $\overline{S_F}$ with $\mathfrak{M}_{\mathcal{H}^F}$ follows easily by restricting to $\mathbb{H}^{I \setminus F}$. □

Corollary 2.6 *The stratification $\mathscr{S}$ of $\mathfrak{M}_{\mathcal{H}}$ satisfies the conditions used in Section 1.2 to define the localization functor.*

Proof: We have already shown that it is a topological T-stratification. The condition (B) follows from the fact that for every flat F, the space

$$S_F/T \cong \mathring{H}_F^{\mathrm{hk}}$$

is a complement of a collection of codimension three subspaces of V^{hk}, and is therefore simply connected. Condition (C) follows from Proposition 2.5 and the fact that strata of hypertoric varieties are connected. □

2.4 Intersection cohomology of hypertoric varieties

We can now state our first main theorem, which says that the intersection cohomology groups of the hypertoric variety $\mathfrak{M}_{\mathcal{H}}$ may be computed in terms of sheaves on $L_{\mathcal{H}}$. Let $\mathcal{L} = \mathcal{L}_{\varnothing}$ be the minimal extension sheaf on $L_{\mathcal{H}}$ with maximal support, and let $\mathbf{IC}_T(\mathfrak{M}_{\mathcal{H}})$ be the equivariant intersection cohomology sheaf of $\mathfrak{M}_{\mathcal{H}}$.

Theorem 2.7 *The sheaf $\mathcal{L}$ is rigid. There is an isomorphism*

$$\mathrm{Loc}\,\mathbf{IC}_T(\mathfrak{M}_{\mathcal{H}}) \cong \mathcal{L}$$

of graded $\mathcal{A}_{\mathcal{H}}$-modules, and the map

$$\Gamma : IH_T^{\bullet}(\mathfrak{M}_{\mathcal{H}}) \to \mathcal{L}(L_{\mathcal{H}})$$

of Equation (1.2) *is an isomorphism of graded* $\operatorname{Sym} V_0$*-modules.*

Remark 2.8 Since $\mathcal{L}$ is rigid, the isomorphism of sheaves in Theorem 2.7 is necessarily unique up to scalar multiplication. It can be made completely unique using the fact that the two graded vector spaces of global sections are each canonically isomorphic to $\mathbb{R}$ in degree zero.

We conclude the section with a proposition that will be essential to the proof of Theorem 2.7. It is proved in [PW, §2], using an alternative, algebro-geometric construction of hypertoric varieties.

Proposition 2.9 *Let $\mathcal{H}$ be an arrangement, and $\widetilde{\mathcal{H}}$ a simplification of $\mathcal{H}$. There is a natural semismall projective map $\varpi : \mathfrak{M}_{\widetilde{\mathcal{H}}} \to \mathfrak{M}_{\mathcal{H}}$. This map restricts to an equivariant fiber bundle over each stratum of $\mathfrak{M}_{\mathcal{H}}$, and the fiber over the stratum S_F is T_F-equivariantly homotopy equivalent to $\mathfrak{M}_{\widetilde{\mathcal{H}_F}}$.*

Remark 2.10 In particular, if $\mathcal{H}$ is central, then the fiber of $\mathfrak{M}_{\widetilde{\mathcal{H}}} \to \mathfrak{M}_{\mathcal{H}}$ over the point stratum of $\mathfrak{M}_{\mathcal{H}}$ is equivariantly homotopy equivalent to $\mathfrak{M}_{\widetilde{\mathcal{H}}}$ itself. This subspace is called the **core** of $\mathfrak{M}_{\widetilde{\mathcal{H}}}$, and is equivariantly homeomorphic to a union of toric varieties, one for each bounded chamber of $\widetilde{\mathcal{H}}$, glued together along toric subvarieties [BD, 6.5]. If $\mathfrak{M}_{\widetilde{\mathcal{H}}}$ is smooth, i.e. $\mathcal{H}$ is unimodular, it is the union of open subsets which are isomorphic to the cotangent bundles to these toric varieties.

Remark 2.11 Alternatively, it is possible to understand any hypertoric variety, smooth or not, in terms of a single toric variety inside of which it sits. From the definition of $\mathfrak{M}_{\mathcal{H}}$ in Section 2.3, one can see that $\mathfrak{M}_{\mathcal{H}}$ is a complete intersection inside of the toric variety

$$\Psi^{-1}(V \times \mathbb{C}^I)/T_{V_0} = \Psi_{\mathbb{R}}^{-1}(V)/T_{V_0}.$$

This toric variety is known as the **Lawrence toric variety** associated to $\mathcal{H}$. When $\mathfrak{M}_{\mathcal{H}}$ is smooth or an orbifold, Hausel and Sturmfels [HS, §6] prove that the embedding of $\mathfrak{M}_{\mathcal{H}}$ into the Lawrence toric variety induces an isomorphism on cohomology, and use that fact to compute the cohomology ring of $\mathfrak{M}_{\mathcal{H}}$. This method fails, however, when dealing with the more refined invariant of intersection cohomology of a singular variety.

3 Minimal extension sheaves on the lattice of flats

Our first step toward the proof of Theorem 2.7 is an algebraic study of the minimal extension sheaves on the linear poset $L_{\mathcal{H}}$. The results of this section do not require the existence of a hypertoric variety, so we do not need to assume that the arrangement is rational. In fact, the results are true as stated for arrangements over an arbitrary field of characteristic zero[6], but we will continue to work with real arrangements so as to keep our notation from Section 2 in place.

To simplify matters, we observe that for any flat F of $\mathcal{H}$, the closure $\overline{\{F\}}$ of $\{F\}$ in our topology on $L_{\mathcal{H}}$ is isomorphic to the lattice of flats of the restricted arrangement $\mathcal{H}^F$, and

[6]The characteristic zero hypothesis is needed for [PS, Prop. 7], which we use to prove Proposition 3.11.

the restriction of $\mathcal{A}_{\mathcal{H}}$ to $\overline{\{F\}}$ is $\mathcal{A}_{\mathcal{H}^F} \otimes_{\mathrm{Sym}\langle F\rangle} \mathrm{Sym}\, V_0$. It follows that the minimal extension sheaf $\mathcal{L}_F$ with support $\overline{\{F\}}$ is obtained by extension of scalars from the minimal extension sheaf on $\mathcal{H}^F$ whose support is all of $L_{\mathcal{H}^F}$. Thus we can concentrate on describing $\mathcal{L}_F$ in the case of the flat $F = \varnothing$ whose closure is $L_{\mathcal{H}}$.

3.1 Simple arrangements

Let $\widetilde{\mathcal{H}}$ be a simplification of $\mathcal{H}$ in the sense of Section 2.1; in other words, the hyperplanes of $\widetilde{\mathcal{H}}$ are generic translates of the hyperplanes of $\mathcal{H}$. Minimal extension sheaves on $L_{\widetilde{\mathcal{H}}}$ are easy to understand, and this will help us to understand those on $L_{\mathcal{H}}$.

There is a natural way besides the one that we have already described in §2.1 to make the lattice $L_{\mathcal{H}}$ into a linear poset: associate to the flat F the vector space $\mathbb{R}^F$, with restrictions given by the obvious quotient maps. Let $\mathcal{A}'_{\mathcal{H}}$ be the resulting sheaf of rings on $L_{\mathcal{H}}$.

Lemma 3.1 *There is a natural homomorphism $\mathcal{A}_{\mathcal{H}} \to \mathcal{A}'_{\mathcal{H}}$ of sheaves of rings, which is an isomorphism if and only if $\mathcal{H}$ is simple. In particular, we have an isomorphism $\mathcal{A}_{\widetilde{\mathcal{H}}} \cong \mathcal{A}'_{\widetilde{\mathcal{H}}}$.*

Proof: The natural inclusions $V(F) \to \mathbb{R}^F$ described in §2.1 are compatible with the maps $V(E) \to V(F)$ and $\mathbb{R}^E \to \mathbb{R}^F$ for $E \leq F$, so they induce a homomorphism $\mathcal{A}_{\mathcal{H}} \to \mathcal{A}'_{\mathcal{H}}$. It will be an isomorphism if and only if each map $V(F) \to \mathbb{R}^F$ is an isomorphism, which is equivalent to the simplicity of $\mathcal{H}$, by Remark 2.1. □

The following lemma generalizes Example 1.13.

Lemma 3.2 *The minimal extension sheaf $\mathcal{L}_\varnothing$ corresponding to the maximal flat $\varnothing$ of $L_{\widetilde{\mathcal{H}}}$ is isomorphic to the structure sheaf $\mathcal{A}_{\widetilde{\mathcal{H}}}$.*

Proof: By Definition 1.8 and Proposition 1.10, we need only show that $\mathcal{A}_{\widetilde{\mathcal{H}}} \cong \mathcal{A}'_{\widetilde{\mathcal{H}}}$ is flabby. But for any $F \in L_{\widetilde{\mathcal{H}}}$ the map $\mathcal{A}'_{\widetilde{\mathcal{H}}}(F) \to \mathcal{A}'_{\widetilde{\mathcal{H}}}(\partial F)$ is surjective, with kernel $(\prod_{i\in F} e_i)\mathcal{A}'_{\widetilde{\mathcal{H}}}(F)$. □

We can also give a nice description of the space of global sections of this sheaf, which, by Lemma 3.2, is naturally a ring.

Lemma 3.3 *The ring of global sections $\mathcal{A}(L_{\widetilde{\mathcal{H}}})$ is canonically isomorphic to the face ring $\mathbb{R}[\Delta_{\widetilde{\mathcal{H}}}]$.*

Proof: Because $\widetilde{\mathcal{H}}$ is simple, we have $\mathcal{A}_{\widetilde{\mathcal{H}}} \cong \mathcal{A}'_{\widetilde{\mathcal{H}}}$ and $L_{\widetilde{\mathcal{H}}} = \Delta_{\widetilde{\mathcal{H}}}$. It is an easy exercise to check that the maps $\mathbb{R}[\Delta_{\widetilde{\mathcal{H}}}] \to \mathcal{A}'_{\widetilde{\mathcal{H}}}(F) = \mathrm{Sym}\,\mathbb{R}^F$ which send e_i to the ith standard basis vector if $i \in F$, and to zero if $i \notin F$, induce an isomorphism $\mathbb{R}[\Delta_{\widetilde{\mathcal{H}}}] \cong \mathcal{A}'_{\widetilde{\mathcal{H}}}(L_{\widetilde{\mathcal{H}}})$ of graded rings. □

3.2 The pushforward of the structure sheaf

Recall from Section 2.1 that we have a natural map of posets $\pi\colon L_{\widetilde{\mathcal{H}}} \to L_{\mathcal{H}}$ with $\mathcal{A}_{\widetilde{\mathcal{H}}} \cong \pi^*\mathcal{A}_{\mathcal{H}}$. It follows that the pushforward π_* takes $\mathcal{A}_{\widetilde{\mathcal{H}}}$-modules to $\mathcal{A}_{\mathcal{H}}$-modules, and we may therefore define the $\mathcal{A}_{\mathcal{H}}$-module

$$\mathcal{E} := \pi_*\mathcal{A}_{\widetilde{\mathcal{H}}}.$$

Proposition 3.4 *$\mathcal{E}$ is a pure $\mathcal{A}_{\mathcal{H}}$-module.*

Proof: The pushforward of a flabby sheaf is always flabby, so we need only show $\mathcal{E}$ is pointwise free. Let $F \in L_{\mathcal{H}}$ be any flat, and consider the stalk $\mathcal{E}(F) = \mathcal{E}(U_F) = \mathcal{A}_{\widetilde{\mathcal{H}}}(\pi^{-1}(U_F))$ of $\mathcal{E}$ at F. The linear poset $\pi^{-1}(U_F)$ is isomorphic to the linear poset of the simplification $\widetilde{\mathcal{H}_F}$ of $\mathcal{H}_F$ induced by the simplification $\widetilde{\mathcal{H}}$ of $\mathcal{H}$. Thus, by Lemma 3.3, we have $\mathcal{E}(U_F) \cong \mathbb{R}[\Delta_{\widetilde{\mathcal{H}_F}}]$, which is a free $\mathcal{A}_{\mathcal{H}}(F)$-module by the discussion of matroid complexes in Section 2.2. □

Lemma 3.5 *For any flat F of $\mathcal{H}$, we have the following three facts.*

(a) $\mathcal{E}(F, \partial F)$ is a free $\mathcal{A}_{\mathcal{H}}(F)$-module generated in degree $2 \operatorname{rk} F$, with a natural basis in one-to-one correspondence with $\pi^{-1}(F)$.

(b) The map $\overline{\mathcal{E}(F, \partial F)} \to \overline{\mathcal{E}(F)}$ is surjective in degree $2 \operatorname{rk} F$ (the top nonzero degree).

(c) There is an isomorphism $\mathcal{E} \cong \bigoplus_{F \in L_{\mathcal{H}}} \overline{\mathcal{E}(F)}_{2 \operatorname{rk} F} \otimes_k \mathcal{L}_F$, where we consider $\overline{\mathcal{E}(F)}_{2 \operatorname{rk} F}$ as a graded vector space concentrated in degree $2 \operatorname{rk} F$.

Proof: Let $\Delta = \Delta_{\widetilde{\mathcal{H}_F}}$ and let $\Delta^\circ = \Delta_{<\operatorname{rk} F}$ be the subcomplex of Δ with all top-dimensional simplices removed. We have already seen in the proof of Proposition 3.4 that $\mathcal{E}(F) = \mathbb{R}[\Delta]$. Since the map π preserves rank, we have $\pi^{-1}(\partial F) = \{E \in \pi^{-1}(U_F) \mid \operatorname{rk} E < \operatorname{rk} F\}$. Thus we have $\mathcal{E}(\partial F) = \mathbb{R}[\Delta^\circ]$, and the map $\mathcal{E}(F) \to \mathcal{E}(\partial F)$ is given by the natural restriction of face rings. It follows that $\mathcal{E}(F, \partial F)$ has a basis consisting of elements $e_E = \prod_{i \in E} e_i$ for $E \in \Delta \setminus \Delta^\circ = \pi^{-1}(F)$. This proves Statement (a).

Next, take a shelling order $E_1, \ldots, E_r$ of the top-dimensional simplices of Δ. Then $\mathcal{E}(F) = \mathbb{R}[\Delta]$ has an $\mathcal{A}_{\mathcal{H}}(F)$-module basis $e_{S_1}, \ldots, e_{S_r}$, where

$$S_k := \{i \in E_k \mid E_k \setminus \{i\} \subset E_j \text{ for some } j < k\}.$$

The degree $2 \operatorname{rk} F$ elements of this basis are a subset of our basis for $\mathcal{E}(F, \partial F)$, which implies Statement (b).

Last, reduce the short exact sequence

$$0 \to \mathcal{E}(F, \partial F) \to \mathcal{E}(F) \to \mathcal{E}(\partial F) \to 0$$

to obtain a right exact sequence

$$\overline{\mathcal{E}(F, \partial F)} \to \overline{\mathcal{E}(F)} \overset{\overline{\partial_F}}{\to} \overline{\mathcal{E}(\partial F)} \to 0.$$

Statements (a) and (b) imply that $\ker \overline{\partial_F}$ is isomorphic to $\overline{\mathcal{E}(F)}_{2 \operatorname{rk} F}$. This fact, along with Propositions 1.10 and 3.4, gives us Statement (c). □

Remark 3.6 If the arrangement $\mathcal{H}$ is rational, this decomposition can be deduced from the geometric decomposition theorem of [BBD, 6.2.5] along with the fact that the map $\varpi \colon \mathfrak{M}_{\widetilde{\mathcal{H}}} \to \mathfrak{M}_{\mathcal{H}}$ is semismall, as in the toric case (see Remark 1.15). As we will see in Proposition 4.2 below, $\mathcal{E}$ is the localization of the pushforward $\varpi_* \mathbb{R}_{\mathfrak{M}_{\widetilde{\mathcal{H}}}, T}$. The decomposition theorem says that this splits into a direct sum of intersection cohomology sheaves, whose localizations are the various minimal extension sheaves $\mathcal{L}_F$. The fact that each sheaf $\mathcal{L}_F$ can only appear in the decomposition in a single degree $2 \operatorname{rk} F$ follows from the semismallness of ϖ.

Semi-smallness also implies that the decomposition of $\varpi_* \mathbb{R}_{\mathfrak{M}_{\widetilde{\mathcal{H}}},T}$ into IC sheaves is unique. In contrast, the isomorphism of Lemma 3.5(c) is not canonical, since it involves choosing representatives for $\overline{\mathcal{E}(F)}$ in $\mathcal{E}(F)$. The summand $\mathcal{L}_\varnothing$, however, *does* sit inside $\mathcal{E}$ canonically.

Corollary 3.7 *The minimal extension sheaf $\mathcal{L}_\varnothing$ is rigid.*

Proof: Lemma 3.5 tells us that for any flat $F \neq \varnothing$, the costalk $\mathcal{E}(F, \partial F)$ is generated in degree $2 \operatorname{rk} F$, thus so is its summand $\mathcal{L}_\varnothing(F, \partial F)$. The corollary will follow from Lemma 1.16 if we can show that $\mathcal{L}_\varnothing(F)$ is generated in degrees strictly less than $2 \operatorname{rk} F$.

This can be deduced from Corollary 3.10 below, but we can also give a simple direct proof. We observed in Section 2.2 that the face ring $\mathbb{R}[\Delta_{\widetilde{\mathcal{H}_F}}]$ is generated as an $\mathcal{A}(F)$-module in degrees less than or equal to $2 \operatorname{rk} \mathcal{H}_F = 2 \operatorname{rk} F$. Since $\mathcal{L}_\varnothing(F)$ is a summand of $\mathcal{E}(F) \cong \mathbb{R}[\Delta_{\widetilde{\mathcal{H}_F}}]$, it is also generated in degrees at most $2 \operatorname{rk} F$. In fact, we can do one better: Lemma 3.5(c) tells us that

$$\mathcal{E} \cong \mathcal{L}_\varnothing \oplus \overline{\mathcal{E}(F)}_{2 \operatorname{rk} F} \otimes_k \mathcal{L}_F \oplus \text{other terms}.$$

Passing to the stalk at F, reducing, and taking the part of degree $2 \operatorname{rk} F$, we obtain the equation

$$\overline{\mathcal{E}(F)}_{2 \operatorname{rk} F} \cong \overline{\mathcal{L}_\varnothing(F)}_{2 \operatorname{rk} F} \oplus \overline{\mathcal{E}(F)}_{2 \operatorname{rk} F} \otimes_k \overline{\mathcal{L}_F(F)}_0 \oplus \text{other terms}.$$

Since $\overline{\mathcal{L}_F(F)}_0 \cong \mathbb{R}$, we must have $\overline{\mathcal{L}_\varnothing(F)}_{2 \operatorname{rk} F} = 0$, proving our statement. □

3.3 The sheaves $\mathcal{R}^{bc}$ and $\mathcal{R}$

In this section we construct two explicit models of the minimal extension sheaf $\mathcal{L}$ on $L_\mathcal{H}$. As an application we compute the Betti numbers of the stalk of $\mathcal{L}$ at every flat, a necessary step toward the proof of Theorem 2.7.

Fix an ordering of the indexing set I, and define a sheaf of rings $\mathcal{R}^{bc}$ on $L_\mathcal{H}$ by letting

$$\mathcal{R}^{bc}(F) := \mathbb{R}[\Delta^{bc}_{\mathcal{H}_F}]$$

for any flat $F \in L_\mathcal{H}$, and letting the map $\mathcal{R}^{bc}(F) \to \mathcal{R}^{bc}(E)$ be the restriction induced by the inclusion $\Delta^{bc}_{\mathcal{H}_E} \subset \Delta^{bc}_{\mathcal{H}_F}$ for $E \leq F$. (Note that in fact $\Delta^{bc}_{\mathcal{H}_E}$ consists of all simplices of $\Delta^{bc}_{\mathcal{H}_F}$ which are contained in E.) There is a unique sheaf with these properties by Proposition 1.1.

This sheaf $\mathcal{R}^{bc}$ is a quotient (as a sheaf of rings) of the sheaf $\mathcal{A}'$ defined in the proof of Lemma 3.3, since $\mathbb{R}[\Delta^{bc}_{\mathcal{H}_F}]$ is a quotient of the polynomial ring $\operatorname{Sym} \mathbb{R}^F$. The map $\mathcal{A} \to \mathcal{A}'$ makes $\mathcal{R}^{bc}$ into an algebra over $\mathcal{A}$, and in particular into an $\mathcal{A}$-module.

Lemma 3.8 *For any open set U, $\mathcal{R}^{bc}(U)$ is isomorphic to the face ring of the complex $\bigcup_{F \in U} \Delta^{bc}_{\mathcal{H}_F}$.*

Proof: Recall from Proposition 1.1 that

$$\mathcal{R}^{bc}(U) = \varprojlim_{F \in U} \mathcal{R}^{bc}(F).$$

It is clear that the face ring of $\bigcup_{F\in U}\Delta^{bc}_{\mathcal{H}_F}$ maps surjectively to each $\mathcal{R}^{bc}(F)$, and therefore to the inverse limit. To see that this map is an isomorphism, note that $\mathcal{R}^{bc}(F)$ has a vector space basis consisting of monomials in the e_i whose support S is contained in F and contains no broken circuit. We call such a set S *allowable* for F. Suppose a monomial has support S which is allowable for two flats F_1 and F_2. Then for any section of $\mathcal{R}^{bc}$ on a set U containing both flats the coefficients of this monomial on F_1 and F_2 must be the same, since S is allowable for $F_1 \cap F_2 \in U$. □

Proposition 3.9 *The sheaf $\mathcal{R}^{bc}$ is a minimal extension sheaf.*

Proof: The fact that $\mathcal{R}^{bc}$ is pointwise free follows from the fact that broken circuit complexes are Cohen-Macaulay, and flabbiness is a consequence of Lemma 3.8. Thus we only need to show that $\mathcal{R}^{bc}$ is indecomposable as an $\mathcal{A}$-module. By Proposition 1.10, it is enough to show that for any $F \neq \varnothing$, the map $\overline{\mathcal{R}^{bc}(F,\partial F)} \to \overline{\mathcal{R}^{bc}(F)}$ is zero. The target $\overline{\mathcal{R}^{bc}(F)}$ is zero in degrees $\geq 2\,\mathrm{rk}\,F$, since $\deg h^{bc}_{\mathcal{H}_F}(q) < \mathrm{rk}\,\mathcal{H}_F = \mathrm{rk}\,F$. On the other hand, $\overline{\mathcal{R}^{bc}(F,\partial F)}$ is concentrated in degree $2\,\mathrm{rk}\,F$, since $\bigcup_{E<F}\Delta^{bc}_{\mathcal{H}_E}$ is the $(\mathrm{rk}\,F-2)$-skeleton of $\Delta^{bc}_{\mathcal{H}_F}$. Thus the map vanishes for degree reasons. □

Corollary 3.10 *For any arrangement $\mathcal{H}$ and any flat $F \in L_{\mathcal{H}}$, the Hilbert series of the reduction of the stalk of $\mathcal{L}_\varnothing$ at F coincides with the h-polynomial of the broken circuit complex of the localized arrangement $\mathcal{H}_F$ with degrees doubled:*

$$\mathrm{Hilb}(\overline{\mathcal{L}_\varnothing(F)}, t) = h^{bc}_{\mathcal{H}_F}(t^2).$$

We have now constructed an explicit minimal extension sheaf $\mathcal{R}^{bc}$ on $\mathcal{L}$. There is, however, something unsatisfying about this construction. Our definition of the sheaf $\mathcal{R}^{bc}$ depends on the notion of a broken circuit, which in turn depends on an ordering of the indexing set I. We would rather have a construction that does not depend on such a choice.

For $\mathcal{H}$ a central arrangement, David Speyer and the second author defined a ring $R(\mathcal{H})$ that doesn't depend on any choices, and admits a flat degeneration to the face ring $\mathbb{R}[\Delta^{bc}_{\mathcal{H}}]$ for any choice of ordering of I [PS, Thm. 4]. This ring is defined to be the subring of rational functions on V generated by x_i^{-1}, $i \in I$, where x_i is the restriction of the i^{th} coordinate function to $V \subset \mathbb{R}^I$. More explicitly, if we put $e_i = x_i^{-1}$, the ring has a presentation of the form

$$R(\mathcal{H}) = \mathbb{R}[e_i \mid i \in I] \Big/ \left\langle \sum_{i\in I_0} a_i e_{I_0 \smallsetminus \{i\}} \;\middle|\; I_0 \subset I \text{ and } \sum_{i \in I_0} a_i x_i|_V = 0 \right\rangle.$$

(Note that this ideal is generated by those terms for which I_0 is a circuit.) We now define a sheaf of $\mathcal{A}$-algebras $\mathcal{R}$ on $L_{\mathcal{H}}$ by putting

$$\mathcal{R}(F) := R(\mathcal{H}_F)$$

for all $F \in L_{\mathcal{H}}$, with restriction maps $\mathcal{R}(F) \to \mathcal{R}(G)$ given by setting the variables $\{e_i \mid i \in F \smallsetminus G\}$ to zero. The $\mathcal{A}(F)$-module structure on $\mathcal{R}(F)$ is given by the inclusion $V(F) \hookrightarrow \mathbb{R}^F$

defined in Section 2.1. Note that $\mathcal{H}_F$ is always central, so the fact that the ring $R(\mathcal{H})$ is only defined for central arrangements does not pose any problems.

Proposition 3.11 *The sheaf $\mathcal{R}$ is a minimal extension sheaf.*

Proof: We need to show that the stalks of $\mathcal{R}$ are free modules over the stalks of $\mathcal{A}$, that $\mathcal{R}$ is flabby, and that it is indecomposable as an $\mathcal{A}$-module. The first statement is proved in [PS, Prop. 7]. To prove the second and third statements, we will show that $\mathcal{R}$ is isomorphic to $\mathcal{R}^{bc}$ as a sheaf of graded vector spaces. Since we have already shown in Proposition 3.9 that $\mathcal{R}^{bc}$ is flabby and that it is indecomposable *for degree reasons*, this will be sufficient.

By [PS, Thm. 4] the ring $\mathcal{R}(F)$ has an $\mathbb{R}$-basis given by all monomials in the variables e_i whose support is contained in F and contains no broken circuit. We have already observed that these monomials form a basis for the ring $\mathcal{R}^{bc}(F)$, hence they may be used to define a vector space isomorphism $\psi_F\colon \mathcal{R}^{bc}(F) \to \mathcal{R}(F)$ for each F. These isomorphisms are compatible with the restriction maps, and therefore define an isomorphism of sheaves of graded vector spaces. □

Remark 3.12 Propositions 3.9 and 3.11 say that the sheaves $\mathcal{R}^{bc}$ and $\mathcal{R}$ are both minimal extension sheaves on $L_{\mathcal{H}}$, and are therefore canonically isomorphic as sheaves of $\mathcal{A}$-modules by Corollary 3.7. They are *not*, however, isomorphic as sheaves of $\mathcal{A}$-algebras. Propositions 3.9 and 3.11, when coupled with Theorem 2.7, give two different ways to put a ring structure on the equivariant intersection cohomology group $IH^\bullet_T(\mathfrak{M}_{\mathcal{H}})$. For now, we will say that only the ring structure coming from the sheaf $\mathcal{R}$ is canonical, because this did not involve making any unnatural choices. Later we will give this assertion more precise mathematical meaning by showing that, if $\mathcal{H}$ is unimodular and central, the group equivariant IC sheaf of $\mathfrak{M}_{\mathcal{H}}$ admits a unique ring structure in the equivariant derived category, and the induced ring structure on $IH^\bullet_T(\mathfrak{M}_{\mathcal{H}})$ is the one coming from $\mathcal{R}$.

Example 3.13 We illustrate the sheaves $\mathcal{R}$ and $\mathcal{R}^{bc}$ using a central arrangement of three lines in the plane. More precisely, let $V = V_0 \subset \mathbb{R}^3$ be the vector subspace spanned by $v_1 = (1,0,1)$ and $v_2 = (0,1,1)$. The resulting linear poset is the one described in Example 1.14, with $V(1) = V/\mathbb{R}\cdot v_2$, $V(2) = V/\mathbb{R}\cdot v_1$, and $V(3) = V/\mathbb{R}\cdot(v_1 - v_2)$. For all flats F except the maximal flat $\{1,2,3\}$, we have

$$\mathcal{R}(F) = \mathcal{R}^{bc}(F) \cong \mathcal{A}(F) = \operatorname{Sym} V(F),$$

since $\mathcal{H}_F$ is simple for these flats.

At the flat $F = \{1,2,3\}$, however, we see a difference between $\mathcal{R}$ and $\mathcal{R}^{bc}$. We have $\mathcal{R}(F) = \mathbb{R}[e_1,e_2,e_3]/\langle e_2e_3 + e_1e_3 - e_1e_2\rangle$, while $\mathcal{R}^{bc} = \mathbb{R}[e_1,e_2,e_3]/\langle e_2e_3\rangle$; these are clearly not isomorphic as rings. The sheaves $\mathcal{R}$ and $\mathcal{R}^{bc}$ *are* canonically isomorphic as modules over $\mathcal{A}$, since they are both minimal extension sheaves. This gives an isomorphism $\mathcal{R}(F) \simeq \mathcal{R}^{bc}(F)$ of $\mathcal{A}(F) = \operatorname{Sym}(V)$-modules; it is the unique isomorphism which induces the identity on the spaces of linear forms.

4 The hypertoric IC sheaf

We now prove Theorem 2.7 and explore some of its consequences.

4.1 Proof of Theorem 2.7

Let $\mathcal{H}$ be an arbitrary rational arrangement, and $\widetilde{\mathcal{H}}$ a simplification of $\mathcal{H}$. Let

$$\mathfrak{M} := \mathfrak{M}_{\mathcal{H}} \quad \text{and} \quad \widetilde{\mathfrak{M}} := \mathfrak{M}_{\widetilde{\mathcal{H}}}.$$

Since $\widetilde{\mathcal{H}}$ is simple, $\widetilde{\mathfrak{M}}$ is rationally smooth, and $\mathbf{IC}_{\widetilde{\mathfrak{M}},T} = \mathbb{R}_{\widetilde{\mathfrak{M}},T}$. Our strategy will be to first prove the theorem for $\widetilde{\mathcal{H}}$, and then tackle the general case by exploiting the relationship between the maps

$$\pi : L_{\widetilde{\mathcal{H}}} \to L_{\mathcal{H}}$$

of Section 2.1 and

$$\varpi : \mathfrak{M}_{\widetilde{\mathcal{H}}} \to \mathfrak{M}_{\mathcal{H}}$$

of Proposition 2.9.

Proposition 4.1 *There is an isomorphism* $\operatorname{Loc}\mathbb{R}_{\widetilde{\mathfrak{M}},T} \cong \mathcal{A}_{\widetilde{\mathcal{H}}}$ *of graded* $\mathcal{A}_{\widetilde{\mathcal{H}}}$*-modules, and the map*

$$\Gamma : H^\bullet_T(\widetilde{\mathfrak{M}}) \to \mathcal{A}_{\widetilde{\mathcal{H}}}(L_{\widetilde{\mathcal{H}}})$$

of Equation (1.2) *is an isomorphism of graded* $\operatorname{Sym} V_0$*-modules.*

Proof: The stalk of $\operatorname{Loc}\mathbb{R}_{\widetilde{\mathfrak{M}},T}$ at a point $x \in L_{\widetilde{\mathcal{H}}}$ is the T-equivariant cohomology of any fiber of $S_x \to S_x/T$, which is isomorphic to $\mathcal{A}(x)$. The restriction maps in $\operatorname{Loc}\mathbb{R}_{\widetilde{\mathfrak{M}},T}$ are module maps over $\operatorname{Sym} V_0$, thus to see that they are the natural quotient maps it is enough to know that they are isomorphisms in degree zero. This follows from the existence of the map Γ.

It remains only to show that Γ is an isomorphism. Lemma 3.3 tells us that $\mathcal{A}(L_{\widetilde{\mathcal{H}}})$ is isomorphic to the face ring $\mathbb{R}[\Delta_{\widetilde{\mathcal{H}}}]$. The fact that $H^\bullet_T(\widetilde{\mathfrak{M}})$ is also isomorphic to $\mathbb{R}[\Delta_{\widetilde{\mathcal{H}}}]$ can be inferred from [Ko] in the case where $\mathcal{H}$ is unimodular, and from [HS, 1.1] in the general case. For an explicit statement and proof of this result, see [Pr, 3.2.2]. The values of Γ on the generators of $H^\bullet_T(\widetilde{\mathfrak{M}})$ can be computed via the stalk maps, from which we can conclude that Γ is an isomorphism. □

The following proposition says that localization commutes with pushing forward.

Proposition 4.2 *There is a natural isomorphism of* $\mathcal{A}_{\mathcal{H}}$*-modules*

$$\pi_*\mathcal{A}_{\widetilde{\mathcal{H}}} \cong \operatorname{Loc}\varpi_*\mathbb{R}_{\widetilde{\mathfrak{M}},T}.$$

Proof: Let $\mathcal{E} = \pi_*\mathcal{A}_{\widetilde{\mathcal{H}}}$ and $\mathcal{F} = \operatorname{Loc}\varpi_*\mathbb{R}_{\widetilde{\mathfrak{M}},T}$. As was explained in the proof of Proposition 3.4, the stalk of $\mathcal{E}$ at a flat F is isomorphic to the face ring $\mathbb{R}[\Delta_{\widetilde{\mathcal{H}_F}}]$. The restriction map

$\mathcal{E}(E) \to \mathcal{E}(F)$ for $E \leq F$ sends a generator e_i, $i \in E$ to the corresponding generator of $\mathcal{E}(F)$ if $i \in F$, and to 0 otherwise.

On the topological side, the stalk of $\mathcal{F}$ at a flat $F \in L_{\mathcal{H}}$ is the equivariant cohomology

$$H^\bullet_T(\varpi^{-1}(Tp)) \cong H^\bullet_{T_F}(\varpi^{-1}(p)),$$

where p is any point in the stratum $S_F \subset \mathfrak{M}$ and $T_F \subset T$ is the stabilizer of p. Proposition 2.9 tells us that $\varpi^{-1}(p)$ is T_F-equivariantly homotopy equivalent to $\mathfrak{M}_{\widetilde{\mathcal{H}_F}}$, thus the stalk $\mathcal{F}_F$ is isomorphic to $H^\bullet_{T_F}(\mathfrak{M}_{\widetilde{\mathcal{H}_F}})$, which in turn is isomorphic to $\mathbb{R}[\Delta_{\widetilde{\mathcal{H}_F}}]$ by [Pr, 3.2.2]. What remains is to show that the restriction maps and the $\mathcal{A}_{\mathcal{H}}$-module structure are the same as those of $\mathcal{E}$.

For any $i \in F$, let S_i be the stratum of $\mathfrak{M}_{\widetilde{\mathcal{H}_F}}$ corresponding to the singleton flat $\{i\}$. The isomorphism of [Pr, 3.2.2] identifies the generator $e_i \in \mathbb{R}[\Delta_{\widetilde{\mathcal{H}_F}}]$ with a class in $H^\bullet_{T_F}(\mathfrak{M}_{\widetilde{\mathcal{H}_F}})$ which restricts to a nonzero class on any T_F-orbit in $S_{\{i\}}$, but restricts to zero on any T_F-orbit in $S_{\{j\}}$ for $j \neq i$. It follows that, for any $E \leq F \in L_{\mathcal{H}}$, our restriction maps

$$\mathbb{R}[\Delta_{\widetilde{\mathcal{H}_F}}] \cong \mathcal{E}(F) \to \mathcal{E}(E) \cong \mathbb{R}[\Delta_{\widetilde{\mathcal{H}_E}}]$$

are given by setting e_i to zero for all $i \in F \smallsetminus E$, which agree with the maps

$$\mathbb{R}[\Delta_{\widetilde{\mathcal{H}_F}}] \cong \mathcal{F}(F) \to \mathcal{F}(E) \cong \mathbb{R}[\Delta_{\widetilde{\mathcal{H}_E}}]$$

given in the proof of Proposition 3.4. The agreement of the $\mathcal{A}_{\mathcal{H}}$-module structures may be verified in a similar manner. □

The final ingredient to the proof of Theorem 2.7 is the following result of [PW, 4.3].

Theorem 4.3 *If $\mathcal{H}$ is a central arrangement defined over the rational numbers, then the intersection cohomology Poincaré polynomial of $\mathfrak{M}_{\mathcal{H}}$ coincides with the h-polynomial of the broken circuit complex with degrees doubled:*

$$\mathrm{Hilb}(IH^\bullet(\mathfrak{M}_{\mathcal{H}}), t) = h^{bc}_{\mathcal{H}}(t^2).$$

Corollary 4.4 *Let $\mathcal{H}$ be any arrangement defined over the rational numbers, and let F be a flat of $\mathcal{H}$. For any point x_F on the stratum S_F, the Hilbert series of the cohomology of the stalk of the* $\mathbf{IC}$ *sheaf at x_F coincides with the h-polynomial of the broken circuit complex of the localized arrangement $\mathcal{H}_F$ with degrees doubled:*

$$\mathrm{Hilb}(H^\bullet_{x_F}(\mathbf{IC}_{\mathfrak{M}_{\mathcal{H}}}), t) = h^{bc}_{\mathcal{H}_F}(t^2).$$

Proof: This follows immediately from Proposition 2.5 and Theorem 4.3, the latter applied to the central arrangement $\mathcal{H}_F$. □

The decomposition theorem for perverse sheaves of [BBD] tells us that $\mathrm{Loc}\,\mathbf{IC}_T(\mathfrak{M})$ is a summand of $\mathrm{Loc}\,\varpi_*\mathbb{R}_{\widetilde{\mathfrak{M}},T}$, and therefore of $\mathcal{E}$ by Proposition 4.2. Using the computations of the stalk Betti numbers of the minimal extension sheaf $\mathcal{L}$ (Corollary 3.10) and of the intersection cohomology sheaf (Corollary 4.4), we see that this summand must be $\mathcal{L}$, in its unique embedding into $\mathcal{E}$ described in Lemma 3.5(c).

To see that the map

$$\Gamma : IH^\bullet_T(\mathfrak{M}) \to \mathcal{L}(L_\mathcal{H})$$

is an isomorphism, consider the class of objects $B \in D^b_{T,\mathscr{S}}(\mathfrak{M})$ for which

$$\Gamma_B : H^\bullet_T(\mathfrak{M}; B) \to (\operatorname{Loc} B)(L_\mathcal{H})$$

is an isomorphism. Lemma 3.3 and Proposition 4.2 combine to show that $\varpi_*\mathbb{R}_{\widetilde{\mathfrak{M}},T}$ belongs to this class. Since this class of sheaves is clearly closed under taking summands, $\mathbf{IC}_T(\mathfrak{M})$ also belongs to it. This completes the proof of Theorem 2.7.

As a consequence of Theorem 2.7 and Proposition 3.11 we obtain the following corollary, which was conjectured in [PW, 6.4].

Corollary 4.5 *There is a natural isomorphism of graded vector spaces $IH^\bullet_T(\mathfrak{M}) \cong R(\mathcal{H})$.*

The second half of that conjecture, which states that the ensuing ring structure on $IH^\bullet_T(\mathfrak{M})$ may be interpreted as an intersection pairing, is the subject of Section 5.

4.2 The Morse stalk

Fix a rational arrangement $\mathcal{H}$, and let $L = L_\mathcal{H}$ and $\mathfrak{M} = \mathfrak{M}_\mathcal{H}$. Although Theorem 2.7 gives a topological interpretation for the global sections of a minimal extension sheaf on L, it is more difficult to understand the sections on other open sets. In this section we study the space of sections over a particular open set, and discover its topological meaning. The results of this section are not needed in the rest of the paper; we include them because they will be of fundamental importance in the forthcoming paper [BLPW].

A naive guess would be that the sections on an open set $U \subset L_\mathcal{H}$ give the intersection cohomology of $\bigcup_{F\in U} S_F$, but this is *not* correct. Too much information has been lost by the localization functor, since by (1.1) it essentially treats each stratum as if it were a single T-orbit. For instance, $\mathcal{L}(\{\varnothing\}) = \mathbb{R}$, which is not isomorphic to

$$H^\bullet_T(S_\varnothing) \cong H^\bullet(S_\varnothing/T) \cong H^\bullet\left(V^{\mathrm{hk}} \smallsetminus \bigcup_{i\in I} H_i^{\mathrm{hk}}\right).$$

Later, in Section 5, we will introduce a more refined localization functor which uses the cohomology along entire strata and which can capture the intersection cohomology of any open union of strata.

There is one other case in which we can understand the space of sections $\mathcal{L}(U)$, namely when $U = L_{>0}$ is the set of flats of corank > 0, corresponding to strata of positive dimension. We will interpret the spaces $\mathcal{L}(L_{>0})$ and $\mathcal{L}(L, L_{>0})$ in terms of the intersection cohomology of certain subspaces of $\mathfrak{M}$, as follows.

We begin by recording some more properties of hypertoric varieties. The algebro-geometric construction to which we alluded in Section 2.3 allows us to extend the action of T on $\mathfrak{M}$ to an algebraic action of the complexified torus $T_{\mathbb{C}}$. This action is hamiltonian with respect to a complex symplectic form on $\mathfrak{M}$, with moment map

$$\mu_{\mathbb{C}} : \mathfrak{M} \to \mathfrak{t}_{\mathbb{C}}^* \cong V_0^{\mathbb{C}}$$

obtained by composing the hyperkähler moment map $\mu : \mathfrak{M} \to V^{\mathrm{hk}}$ with the projection $V^{\mathrm{hk}} = V \times V_0^{\mathbb{C}} \to V_0^{\mathbb{C}}$. For any polyhedron $\Delta \subset V_0$, let X_Δ be the associated toric variety along with its natural $T_{\mathbb{C}}$-action. The following proposition first appeared in [HP, §2].

Proposition 4.6 *The subvariety $\mu_{\mathbb{C}}^{-1}(0)$ of $\mathfrak{M}$ is $T_{\mathbb{C}}$-equivariantly isomorphic to a union of toric varieties X_Δ indexed by the chambers of $\mathcal{H}$, where X_Δ is glued to $X_{\Delta'}$ along the toric subvariety $X_{\Delta\cap\Delta'}$.*

Remark 4.7 The subvariety $\mu_{\mathbb{C}}^{-1}(0)$ is called the **extended core** of $\mathfrak{M}$. Sitting inside the extended core is the ordinary core, which has already been mentioned (when $\mathcal{H}$ is simple) in Remark 2.10. The ordinary core may be defined as the union of all compact toric subvarieties of the extended core. If at least one chamber Δ is bounded, then the core may be described as the union of those X_Δ for which Δ is bounded. But this is not always the case. For example if $\mathcal{H}$ is central, then every chamber is unbounded, and the core consists of a single point.

Fix a cocharacter $\rho\colon U(1) \to T$ for which $\mathfrak{M}^{\rho(U(1))} = \mathfrak{M}^T$, and let $\rho_{\mathbb{C}}\colon \mathbb{C}^* \to T_{\mathbb{C}}$ denote its complexification. Define two subspaces $\mathfrak{M}^{\pm}$ of $\mathfrak{M}$ by

$$\mathfrak{M}^{\pm} := \left\{ x \in \mathfrak{M} \;\middle|\; \lim_{t\to 0} \rho_{\mathbb{C}}(t)^{\pm 1} x \text{ exists in } \mathfrak{M} \right\}.$$

Both of these subspaces are unions of extended core components of $\mathfrak{M}$. Specifically, we can view the derivative $d\rho$ of ρ as an element of $\operatorname{Lie} T = V^*$, and $\mathfrak{M}^+$ (respectively $\mathfrak{M}^-$) is the union of those X_Δ for which $d\rho$ has a minimum (respectively a maximum) on Δ. The intersection $\mathfrak{M}^0 = \mathfrak{M}^+ \cap \mathfrak{M}^-$ is the ordinary core of $\mathfrak{M}$.

The space $\mathfrak{M} \setminus \mathfrak{M}^+$ inherits a stratification by restricting the strata from $\mathfrak{M}$. All the strata intersect non-trivially with $\mathfrak{M} \setminus \mathfrak{M}^+$ except for the zero-dimensional strata, so the poset of strata is $L_{>0}$. We would like to consider the localization functor associated with this stratified space, but to do so we must first check that conditions (A) through (C) of Section 1.2 are satisfied. Conditions (A) and (C) are local, so they are preserved when we pass to the open subset $\mathfrak{M} \setminus \mathfrak{M}^+$ of $\mathfrak{M}$. For condition (B), suppose that $\dim S_F \neq 0$. Then the quotient $(S_F \setminus (S_F \cap \mathfrak{M}^+))/T$ may be identified (via the hyperkähler moment map) with a subset of H_F^{hk}. The part of H_F^{hk} that we do *not* get includes the subspaces $H_i^{\mathrm{hk}} \cap H_F^{\mathrm{hk}}$, $i \notin F$, which have real codimension three, along with a finite number of polyhedral subspaces of real codimension $2 \operatorname{rk} \mathcal{H}^F$. If $\operatorname{rk} \mathcal{H}^F > 1$, then we are done. If $\operatorname{rk} \mathcal{H}^F = 1$, we get the complement of a finite number of line segments and a ray joined end-to-end in $\mathbb{R}^3$, so again the space is simply connected.

Thus we have a localization functor $\operatorname{Loc}\colon D^b_{T,\mathscr{S}}(\mathfrak{M} \setminus \mathfrak{M}^+) \to (\mathcal{A}|_{L_{>0}})\text{--}mod$. It is easy to see that restricting from $\mathfrak{M}$ to $\mathfrak{M} \setminus \mathfrak{M}^+$ and then localizing is the same as localizing and then restricting to $L_{>0}$. It follows that for any $B \in D^b_{T,\mathscr{S}}(\mathfrak{M})$, if we put $\mathcal{B} = \operatorname{Loc} B$, then we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} & & H_T^\bullet(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+; B) & \longrightarrow & H_T^\bullet(\mathfrak{M}; B) & \longrightarrow & H_T^\bullet(\mathfrak{M} \setminus \mathfrak{M}^+; B) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B}(L, L_{>0}) & \longrightarrow & \mathcal{B}(L) & \longrightarrow & \mathcal{B}(L_{>0}) \end{array} \tag{4.1}$$

where the upper row comes from the long exact sequence of the pair $(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+)$. Note that $H_T^\bullet(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+; B) = H_T^\bullet(\mathfrak{M}^+; j^! B)$, where $j\colon \mathfrak{M}^+ \hookrightarrow \mathfrak{M}$ is the inclusion.

Definition 4.8 We say that B satisfies ρ-localization if $H_T^\bullet(\mathfrak{M}; B) \to H_T^\bullet(\mathfrak{M} \setminus \mathfrak{M}^+; B)$ is surjective and the vertical maps in (4.1) are isomorphisms.

Theorem 4.9 *The intersection cohomology sheaf* $\mathbf{IC}_T(\mathfrak{M})$ *satisfies* ρ*-localization.*

Proof: This property is preserved under taking direct summands, so it is enough to show that $\varpi_* \mathbb{R}_{\widetilde{\mathfrak{M}},T}$ satisfies ρ-localization. Since ϖ is proper and we have the identities

$$\widetilde{\mathfrak{M}}^+ = \varpi^{-1}(\mathfrak{M}^+) \text{ and } \pi^{-1}(L_{>0}) = \widetilde{L}_{>0},$$

this is equivalent to showing that the sheaf $\mathbb{R}_{\widetilde{\mathfrak{M}},T}$ on $\widetilde{\mathfrak{M}}$ satisfies ρ-localization. Thus we have reduced the proof to the case where $\mathcal{H}$ is simple, so $\mathfrak{M}$ is rationally smooth and $\mathbf{IC}_T(\mathfrak{M})$ is the equivariant constant sheaf.

In this case, (4.1) becomes the following diagram.

$$\begin{array}{ccccccccc} & & H_T^\bullet(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+) & \longrightarrow & H_T^\bullet(\mathfrak{M}) & \longrightarrow & H_T^\bullet(\mathfrak{M} \setminus \mathfrak{M}^+) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}(L, L_{>0}) & \longrightarrow & \mathcal{A}(L) & \longrightarrow & \mathcal{A}(L_{>0}) & \longrightarrow & 0 \end{array}$$

The bottom row is right exact because $\mathcal{A}$ is flabby. By Theorem 2.7 the middle vertical map is an isomorphism. It is enough to show that the left vertical map is an isomorphism, since this will imply the vanishing of the connecting homomorphisms for the pair $(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+)$, and the five-lemma will take care of the rest.

Let $\mathfrak{M}^T = \{x_1, \ldots, x_r\}$, and let F_i denote the flat of $\mathcal{H}$ for which $S_{F_i} = \{x_i\}$. Let

$$C_i := \{x \mid \lim_{t \to 0} \rho(t) x = x_i\},$$

so that $\mathfrak{M}^+ = \bigcup_{i=1}^r C_i$. Each C_i is an $2d$-dimensional rational homology cell (a quotient of $\mathbb{C}^d$ by a finite group). Choose an ordering of the x_i so that $\mathfrak{M}_i = \mathfrak{M} \setminus \bigcup_{j>i} C_j$ is open for all i; thus $\mathfrak{M}_0 = \mathfrak{M} \setminus \mathfrak{M}^+$ and $\mathfrak{M}_r = \mathfrak{M}$. By excision, $H_T^\bullet(\mathfrak{M}_i, \mathfrak{M}_{i-1})$ is isomorphic to $H_T^\bullet(D_i, D_i \setminus C_i)$, where D_i is a quotient of $\mathbb{C}^{2d}$ by a finite group and the inclusion $C_i \hookrightarrow D_i$ comes from a coordinate inclusion $\mathbb{C}^d \hookrightarrow \mathbb{C}^{2d}$. Thus it is a free $H_T^\bullet(pt)$-module of rank 1, generated in degree $2d$.

It follows that $H_T^\bullet(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+)$ is a free $H_T^\bullet(pt)$-module of rank r generated in degree $2d$, and that the map to $H_T^\bullet(\mathfrak{M})$ is an inclusion. This implies that the left vertical map is injective, and since $\mathcal{A}(L, L_{>0})$ is also free of rank r and generated in degree $2d$, this proves the theorem. □

Remark 4.10 When $\mathcal{H}$ is a central arrangement, so the hypertoric variety $\mathfrak{M}_{\mathcal{H}}$ has a unique T-fixed point p, the functor $B \mapsto H^\bullet_T(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+; B)$ is the cohomology of an equivariant version of the hyperbolic localization functor explored in [Br]. It can also be viewed as the Morse group of B for the stratified Morse function obtained by composing $d\rho \in V^*$ with the real moment map $\mu_{\mathbb{R}} \colon \mathfrak{M}_{\mathcal{H}} \to V$.

Essentially the same functor was used by Mirkovic and Vilonen in [MV] to give the weight space in their construction of the geometric Satake correspondence identifying IC sheaves of Schubert varieties in the loop Grassmannian of a reductive group G with representations of the Langlands dual group $G^\vee$. They were able to prove the main properties of this functor without using the decomposition theorem; since the singularities of hypertoric varieties have many similarities to singularities of loop Grassmannian Schubert varieties, we hope that similar ideas may make it possible to prove Theorem 2.7 and Theorem 4.9 without using the decomposition theorem.

5 Lifting the ring structure to the derived category

Let $\mathcal{H}$ be a unimodular central arrangement. In [PW, 6.4], it was conjectured that there is a natural isomorphism of $\operatorname{Sym} V_0$-modules between $IH^\bullet_T(\mathfrak{M})$ and $R(\mathcal{H})$, and that this isomorphism may be interpreted as an intersection pairing. In Corollary 4.5 we have already produced such a natural isomorphism, but it is not clear what our isomorphism has to do with intersection theory. This is partly explained by the following theorem, which is the main result of the remainder of the paper.

Let $\mathfrak{M} = \mathfrak{M}_{\mathcal{H}}$ be a hypertoric variety defined by a *unimodular* central arrangement $\mathcal{H}$. Let $\mathbf{IC} = \mathbf{IC}_{\mathfrak{M},T}$ be the equivariant intersection cohomology sheaf, and let $u : \mathbb{R}_{\mathfrak{M},T} \to \mathbf{IC}$ be the natural map.

Theorem 5.1 *The object* $\mathbf{IC}$ *can be made into a commutative ring object in* $D^b_T(\mathfrak{M})$ *with unit* u*. More precisely, there is a commutative and associative morphism*

$$m\colon \mathbf{IC} \otimes \mathbf{IC} \to \mathbf{IC}$$

such that the natural map

$$\mathbf{IC} \;\cong\; \mathbb{R}_{\mathfrak{M},T} \otimes \mathbf{IC} \xrightarrow{u\otimes id} \mathbf{IC} \otimes \mathbf{IC} \xrightarrow{m} \mathbf{IC}$$

is the identity. This ring structure is unique.

Applying the localization functor Loc *to this ring structure gives the ring structure on the minimal extension sheaf* $\mathcal{L}$ *coming from Theorem 2.7 and Proposition 3.11. In particular, the ring structure induced by* m *on* $IH^\bullet_T(\mathfrak{M})$ *is that of Corollary 4.5.*

Remark 5.2 A ring structure on an object in the derived category induces a ring structure on $H^\bullet_T(Y; \mathbf{IC}|_Y)$ for any T-invariant subspace $Y \subset M$, and it does so in a functorial way. In particular, the restriction map

$$R(\mathcal{H}) \cong IH^\bullet_T(\mathfrak{M}) \to H^\bullet_T(S_\varnothing) \tag{5.1}$$

is a ring homomorphism, where the target has the usual ring structure. Thus Theorem 5.1 provides another, deeper sense in which our ring structure can be called natural, at least for unimodular arrangements. In contrast, the analogous natural map from $R^{bc}(\mathcal{H})$ to $H^\bullet_T(S_\varnothing)$ is *not* a ring homomorphism.

Remark 5.3 The fact that we need to assume $\mathcal{H}$ is unimodular is somewhat puzzling, and deserves some explanation. The intersection cohomology groups we are calculating are taken with rational coefficients, and their dimensions are independent of any lattice structure. In fact, the results of Section 3 hold over an arbitrary field of characteristic zero, even if there is no hypertoric variety in the picture. This is similar to the situation for toric varieties, where a theorem of Karu [Ka] (see also [BreL2]) can be used to show that statements about intersection cohomology or even more general equivariant perverse sheaves on toric varieties can be proved for fans without any rationality hypothesis (see [BraL, §5 and §6] for example).

However, something unexpected happens for hypertoric varieties. The action of T on the open stratum $S_\varnothing$ is quasi-free, and the quotient $S_\varnothing/T$ is homeomorphic to the complement in V^{hk} of the codimension 3 subspaces H_i^{hk} (see the notation in §2.3). The cohomology of this space is given by [dLS, 5.6] as a quotient of a polynomial ring where among the relations are $\sum_{i\in C} \mathrm{sgn}(a_i) e_{C\setminus\{i\}}$ where $\sum_{i\in C} a_i x_i|_V = 0$ is the relation for a circuit C. For general $\mathcal{H}$ this differs from the relation $\sum a_i e_{C\setminus\{i\}}$ which holds in the ring $R(\mathcal{H})$, but it is the same if $\mathcal{H}$ is unimodular. This means that (5.1) cannot be a ring homomorphism (with the ring structure we have described on $IH^\bullet_T(\mathfrak{M})$) if $\mathcal{H}$ is not unimodular, and so Theorem 5.1 cannot hold. Note that for some non-unimodular arrangements there may still exist a ring structure on $\mathbf{IC}$, but the resulting ring structure on $IH^\bullet_T(\mathfrak{M})$ will not agree with that of Corollary 4.5.

We do not have a good explanation for this situation, but since non-unimodularity of $\mathcal{H}$ is equivalent to the orbifold resolution $\mathfrak{M}_{\widetilde{\mathcal{H}}}$ having singular points, we speculate that there may be corrections coming from orbifold cohomology which would make a statement like Theorem 5.1 possible for general rational arrangements.

Our strategy for proving Theorem 5.1 is to compute the multiplication map m in terms of its localization in a combinatorial category of sheaves, and then lift this combinatorial multiplication to a derived category morphism. We already have a ring structure on $\mathrm{Loc}\,\mathbf{IC}_{\mathfrak{M},T}$, but the functor Loc is not fully faithful, so we cannot lift this ring structure to the derived category. The problem can be seen, for instance, by noting that the decomposition of $\mathcal{E} = \mathrm{Loc}\,\varpi_* \mathbb{R}_{\mathfrak{M}_{\widetilde{\mathcal{H}},T}}$ is not canonical, whereas the decomposition of $\varpi_* \mathbb{R}_{\mathfrak{M}_{\widetilde{\mathcal{H}},T}}$ into intersection cohomology sheaves is canonical, since $\varpi\colon \mathfrak{M}_{\widetilde{\mathcal{H}}} \to \mathfrak{M}_{\mathcal{H}}$ is semismall.

To solve this problem we construct a richer localization $\widehat{\mathrm{Loc}}$ which takes cohomology of a derived category object along entire strata rather than single T-orbits. We describe this functor in Section 5.2 and show that it completely captures homomorphisms between objects satisfying a parity vanishing condition. In Section 5.3 we describe $\widehat{\mathrm{Loc}}(\mathbf{IC})$, and show that under suitable hypotheses it is a "generalized minimal extension sheaf" or GMES, and that a ring structure on a GMES induces a ring structure on **IC**. All of this is done for a general stratified T-space satisfying certain parity vanishing conditions; we hope that there will be other interesting spaces satisfying our hypotheses. In Sections 5.4 and 5.5 we work out the specific case of hypertoric varieties and construct the required ring structure, thus proving Theorem 5.1.

5.1 Stratum-by-stratum Homs in derived categories

Computing Hom-spaces between objects in the derived category (or equivariant derived category) is often difficult, in part because the derived category is not a stack: in general morphisms cannot be built from local data. In some special cases, however, morphisms can be localized. This happens for instance in the subcategory of perverse sheaves. We will focus on another such case, in which the objects satisfy strong parity vanishing conditions along strata. We show that homomorphisms between such objects can be described by their restriction to strata.

We start with a connected T-space $\mathfrak{M}$, endowed with a finite T-decomposition $\mathscr{S}$. For any locally closed union $\mathfrak{N}$ of strata, let $j_{\mathfrak{N}}\colon \mathfrak{N} \to \mathfrak{M}$ denote the inclusion. We make the following assumptions on $\mathscr{S}$. The first two are exactly the same as the first two assumptions from Section 1.2.

(A) $\mathscr{S}$ is a T-stratification in the sense of Definition 1.2.

(B) For all $S \in \mathscr{S}$, the quotient S/T is simply connected (so T-equivariant local systems on S are trivial).

(C′) For any $R, S \in \mathscr{S}$ with $R \subset \overline{S}$, we have $H^k_T(R; j_R^* j_{S*}\mathbb{R}_{S,T}) = 0$ for k odd.

In particular, when $R = S$, (C′) says that the odd equivariant cohomology groups of S itself vanish.

Remark 5.4 The cohomology groups in (C′) can be described more geometrically as follows. Suppose that there exists a T-invariant open neighborhood $U = U_R$ of R and a T-equivariant deformation retraction $U \times [0,1] \to U$ onto R so that for all points p and $t \in [0,1)$, the image of (p,t) is contained in the same stratum as p (the assumption that $\mathscr{S}$ is a T-stratification implies that such a neighborhood exists locally near every orbit in R). Then there is an isomorphism

$$H^\bullet_T(R; j_R^* j_{S*}\mathbb{R}_{S,T}) \cong H^\bullet_T(U \cap S).$$

We want to consider objects in the equivariant derived category of $\mathfrak{M}$ whose cohomology along each stratum vanishes in odd degrees. First consider the case of a single stratum S. Let $D^{ev}_T(S)$ be the full subcategory of $D^b_T(S)$ consisting of objects whose cohomology sheaves are locally constant (hence constant by (B)) and vanish in odd degrees.

Lemma 5.5 *The following statements hold for any object B of $D_T^{ev}(S)$.*

(a) The equivariant cohomology $H_T^\bullet(S;B)$ is a free $H_T^\bullet(S)$-module which vanishes in odd degrees. For any $x \in S$, restricting cohomology induces isomorphisms

$$H_T^\bullet(S;B) \otimes_{H_T^\bullet(S)} H_T^\bullet(Tx) \cong H_T^\bullet(Tx;B), \text{ and}$$

$$H_T^\bullet(S;B) \otimes_{H_T^\bullet(S)} \mathbb{R} \cong H^\bullet(\{x\};B).$$

(b) For any stratum $R \in \mathscr{S}$, the natural map

$$H_T^\bullet(S;B) \otimes_{H_T^\bullet(S)} H_T^\bullet(R; j_R^* j_{S*} \mathbb{R}_{S,T}) \to H_T^\bullet(R; j_R^* j_{S*} B)$$

is an isomorphism (in particular the right hand side vanishes in odd degrees).

(c) For any $C \in D_T^b(S)$, the natural maps

$$\mathrm{Hom}^\bullet_{D_T^b(S)}(B,C) \to \mathrm{Hom}_{H_T^\bullet(S)}(H_T^\bullet(S;B), H_T^\bullet(S;C))$$

and

$$H_T^\bullet(S; B \otimes C) \to H_T^\bullet(S;B) \otimes_{H_T^\bullet(S)} H_T^\bullet(S;C)$$

are isomorphisms.

Proof: First we prove (a). Let $[2l, 2m]$ be the largest interval on which $H^\bullet(B)$ is supported; we proceed by induction on the difference $m-l$. If $m-l=0$, then B is a T-equivariant local system placed in degree $2m$. Our condition (B) says it must be a constant local system, in which (a) is obvious. Otherwise, consider the distinguished triangle $\tau_{<2m}B \to B \to \tau_{\geq 2m}B$. The statement holds for $\tau_{<2m}B = \tau_{<2m-1}B$ and $\tau_{\geq 2m}B = H^{2m}(B)[-2m]$ by the inductive hypothesis. Taking equivariant cohomology of this triangle gives a long exact sequence; since the extreme terms are free modules vanishing in odd degrees, the middle is as well.

The other statements are now proved by a similar induction, using the five-lemma and the freeness of $H_T^\bullet(S;B)$. □

When the space $\mathfrak{M}$ has multiple strata, we consider sheaves which satisfy parity vanishing along each stratum.

Definition 5.6 Let $B \in D_T^b(\mathfrak{M})$. We say that B has "$*$-parity vanishing" (respectively "!-parity vanishing") if $j_S^* B$ (resp. $j_S^! B$) is in $D_T^{ev}(S)$ for all $S \in \mathscr{S}$. Let $D_T^*(\mathfrak{M})$ (respectively $D_T^!(\mathfrak{M})$) denote the full subcategory of $D_T^b(\mathfrak{M})$ consisting of such objects.

Note that $*$-parity vanishing is equivalent to $\mathscr{S}$-constructibility plus the vanishing of the ordinary cohomology sheaves in odd degrees. The constant equivariant sheaf $\mathbb{R}_{\mathfrak{M},T}$ has $*$-parity vanishing, but does not have !-parity vanishing in general unless $\mathfrak{M}$ is smooth. If $\mathfrak{N}$ is a locally closed union of strata of $\mathfrak{M}$ and $j\colon \mathfrak{N} \to \mathfrak{M}$ is the inclusion, then j^* and $j_!$ preserve $*$-parity vanishing, while $j^!$ and j_* preserve !-parity vanishing.

Lemma 5.7 *If $B \in D^b_T(\mathfrak{M})$ is $\mathscr{S}$-constructible and satisfies !-parity vanishing, then for any stratum S the equivariant cohomology $H^\bullet_T(S; j_S^* B)$ vanishes in odd degrees.*

Proof: Choose an ordering $S_1, \ldots, S_r$ of the strata so that $U_k = \bigcup_{i=1}^k S_i$ is open in $\mathfrak{M}$ for all $k = 1, \ldots, r$, and let $B_k = f_{k*} f_k^* B$, where $f_k \colon U_k \to \mathfrak{M}$ is the inclusion. In particular $B_0 = 0$ and $B_r = B$. Since $j^!_{S_k} B \in D^{ev}_T(S_k)$, Lemma 5.5(b) implies that $H^\bullet_T(S; j_S^* j_{S_k *} j^!_{S_k} B)$ vanishes in odd degrees. The lemma follows by induction on k using the distinguished triangles $j_{S_k *} j^!_{S_k} B \to B_k \to B_{k-1}$. □

For any T-space X and objects $B, C \in D^b_T(X)$, let

$$\mathrm{Hom}^k_X(B, C) = \mathrm{Hom}_{D^b_T(X)}(B, C[k])$$

and

$$\mathrm{Hom}^\bullet_X(B, C) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k_X(B, C).$$

We want to describe the space of homomorphisms $B \to C$ by looking at their restrictions $j_S^* B \to j_S^* C$ to each stratum S. These restrictions for different strata are constrained by a compatibility condition which can be described formally using the adjunction

$$\mathrm{Hom}^\bullet_S(j_S^* B, j_S^* C) = \mathrm{Hom}^\bullet_{\mathfrak{M}}(B, j_{S*} j_S^* C).$$

To simplify notation, define $\Phi_S = j_{S*} j_S^*$. Then the restriction of a morphism to S can be rephrased as the map

$$\mathrm{Hom}^\bullet_{\mathfrak{M}}(B, C) \to \mathrm{Hom}^\bullet_{\mathfrak{M}}(B, \Phi_S C) \tag{5.2}$$

obtained by composing with the adjunction map $C \to \Phi_S C$. The compatibility condition arises because the functoriality of Φ_S gives rise to a commutative square

$$\begin{array}{ccc} \mathrm{Hom}^\bullet_{\mathfrak{M}}(B, C) & \longrightarrow & \mathrm{Hom}^\bullet_{\mathfrak{M}}(B, \Phi_S C) \\ \downarrow & & \downarrow \\ \mathrm{Hom}^\bullet_{\mathfrak{M}}(B, \Phi_R C) & \longrightarrow & \mathrm{Hom}^\bullet_{\mathfrak{M}}(B, \Phi_R \Phi_S C) \end{array} \tag{5.3}$$

for any pair of strata R, S with $R \subset \overline{S}$.

Theorem 5.8 *Consider a pair of objects $B \in D^*_T(\mathfrak{M})$ and $C \in D^!_T(\mathfrak{M})$. The group $\mathrm{Hom}^k_{\mathfrak{M}}(B, C)$ vanishes when k is odd, and the localization* (5.2) *identifies $\mathrm{Hom}^\bullet_{\mathfrak{M}}(B, C)$ with the set of tuples*

$$(f_S) \in \bigoplus_{S \in \mathscr{S}} \mathrm{Hom}^\bullet_{\mathfrak{M}}(B, \Phi_S C)$$

such that for any strata R and S with $R \subset \overline{S}$, f_R and f_S map under (5.3) *to the same element of $\mathrm{Hom}^\bullet_{\mathfrak{M}}(B, \Phi_R \Phi_S C)$.*

Remark 5.9 We can express the second statement of the theorem in another way that will be useful. Fix B, and consider the commutative square (5.3) as a functor of C. Negating one of the arrows in (5.3) gives a functor

$$\mathrm{Hom}^\bullet_{\mathfrak{M}}(B,-) \to \bigoplus_S \mathrm{Hom}^\bullet_{\mathfrak{M}}(B,\Phi_S(-)) \to \bigoplus_{\substack{R\subset \overline{S}\\ R\neq S}} \mathrm{Hom}^\bullet_{\mathfrak{M}}(B,\Phi_R\Phi_S(-)) \tag{5.4}$$

from $D^b_T(\mathfrak{M})$ to the category of three-term complexes of graded vector spaces. Theorem 5.8 asserts that the evaluation of this functor on C is left-exact.

Proof: We use induction on the number of strata in $\mathscr{S}$. If $|\mathscr{S}| = 1$, then the parity vanishing of $\mathrm{Hom}^k_{\mathfrak{M}}(B,C)$ follows from Lemma 5.5(c), while the second statement is trivial.

Now suppose $|\mathscr{S}| > 1$, and assume the theorem is true for all spaces with fewer strata. let S_0 be a closed stratum of $\mathscr{S}$, and let $U = \mathfrak{M} \setminus S_0$. Applying (5.4) to the distinguished triangle $j_{S_0*}j^!_{S_0}C \to C \to j_{U*}j^*_UC$ gives a long exact sequence of chain complexes. In fact, it breaks into short exact sequences of chain complexes. For the term $\mathrm{Hom}^\bullet_{\mathfrak{M}}(B,-)$, this follows from the induction hypothesis, since by adjunction $\mathrm{Hom}^\bullet_{\mathfrak{M}}(B, j_{S_0*}j^!_{S_0}C)$ and $\mathrm{Hom}^\bullet_{\mathfrak{M}}(B, j_{U*}j^*_UC)$ can be expressed as homomorphisms on S_0 and U between objects which satisfy the hypotheses of the theorem, and so they vanish in odd degrees. For the second term, we have $\mathrm{Hom}^\bullet_{\mathfrak{M}}(B,\Phi_SC) = \mathrm{Hom}^\bullet_S(j^*_SB, j^*_SC)$, which vanishes in odd degrees whenever $C \in D^!_T(\mathfrak{M})$, by Lemma 5.5 and Lemma 5.7. For the third term, just note that $\Phi_R\Phi_Sj_{S_0*}j^!_{S_0}C = 0$ whenever $R \neq S$.

The theorem now follows from the snake lemma if we can show that the chain complexes (5.4) coming from $j_{S_0*}j^!_{S_0}C$ and $j_{U*}j^*_UC$ are left exact. In both cases this follows from applying the inductive hypothesis, again using adjunction to express the first term in (5.4) as homomorphisms on S_0 and U. Note that the last entry of the chain complex for $j_{U*}j^*_UC$ will contain extra terms of the form $\mathrm{Hom}^\bullet_{\mathfrak{M}}(B,\Phi_{S_0}\Phi_S(C))$ which do not appear in the complex for j^*_UC, but they do not affect exactness on the left. □

5.2 Localization

We now use Theorem 5.8 to describe some cohomology and homomorphism groups of $D^b_T(\mathfrak{M})$ in terms of modules over a sheaf of rings on a finite poset $\widehat{\mathscr{S}}$. Elements of this poset are of two types: (1) strata $S \in \mathscr{S}$, and (2) pairs (R,S) of strata with $R \subset \overline{S}$, $R \neq S$. The partial order $\preceq$ on this set is given by letting $(R,S) \preceq R$ and $(R,S) \preceq S$ for all (R,S) of type (2); two different elements which are both of type (1) or of type (2) are incomparable. Note that the closure relations among the strata do not appear in this partial order. We consider sheaves on this poset as described in Section 1.1. We will abuse notation slightly and denote the stalk of a sheaf $\mathcal{S}$ on $\widehat{\mathscr{S}}$ at (R,S) by $\mathcal{S}(R,S)$ rather than $\mathcal{S}((R,S))$. We will not need the earlier notation $\mathcal{S}(U,V) = \ker(\mathcal{S}(U) \to \mathcal{S}(V))$, so this should not cause confusion.

Define a localization functor $\widehat{\mathrm{Loc}}$ from $D^b_T(\mathfrak{M})$ to sheaves of $H^\bullet_T(pt)$-modules on $\widehat{\mathscr{S}}$ by defining its stalks to be

$$(\widehat{\mathrm{Loc}}\, B)(S) := H^\bullet_T(S;B) = H^\bullet_T(\mathfrak{M};\Phi_S B)$$

and
$$(\widehat{\operatorname{Loc}}\, B)(R,S) := H_T^\bullet(\mathfrak{M}; \Phi_R\Phi_S B) = H_T^\bullet(R; j_R^* j_{S*} j_S^* B)$$
and letting the maps between the stalks be induced by the natural maps $\Phi_R B \to \Phi_R\Phi_S B$ and $\Phi_S B \to \Phi_R\Phi_S B$. Note that if there exists a T-invariant tubular neighborhood of R as in Remark 5.4, then $(\widehat{\operatorname{Loc}}\, B)(R,S)$ can be described more geometrically as $H_T^\bullet(U_R \cap S; B)$.

Let $\underline{A}$ denote the constant sheaf on $\widehat{\mathscr{S}}$ with stalk the polynomial ring $A = H_T^\bullet(pt)$.

Lemma 5.10 *For any $B, C \in D_T^b(\mathfrak{M})$, there is a natural morphism*
$$\phi_{B,C}\colon\ \widehat{\operatorname{Loc}}\, B \otimes_{\underline{A}} \widehat{\operatorname{Loc}}\, C \to \widehat{\operatorname{Loc}}(B \otimes C). \tag{5.5}$$
If D is another object in $D_T^b(\mathfrak{M})$, then the two maps
$$\phi_{B\otimes C,D} \circ (\phi_{B,C} \otimes \mathrm{id}_D), \phi_{B,C\otimes D} \circ (\mathrm{id}_B \otimes \phi_{C,D})\colon\ \widehat{\operatorname{Loc}}\, B \otimes_{\underline{A}} \widehat{\operatorname{Loc}}\, C \otimes_{\underline{A}} \widehat{\operatorname{Loc}}\, D \to \widehat{\operatorname{Loc}}(B \otimes C \otimes D)$$
are equal. If $B = \mathbb{R}_{\mathfrak{M},T}$, the resulting maps
$$H_T^\bullet(S) \otimes_{H_T^\bullet(pt)} H_T^\bullet(S; C) \to H_T^\bullet(S; C), \quad S \in \mathscr{S}$$
give the usual action of cohomology.

Proof: A more precise statement of the associativity constraint on ϕ should include the natural isomorphisms between the different ways of associating the triple tensor products. This, together with the existence of a morphism $\underline{A} \to \widehat{\operatorname{Loc}}\, \mathbb{R}_{\mathfrak{M},T}$ compatible with the isomorphisms $\underline{A} \otimes_{\underline{A}} \mathcal{B} \cong \mathcal{B} \cong \mathcal{B} \otimes_{\underline{A}} \underline{A}$ and $\mathbb{R}_{\mathfrak{M},T} \otimes B \cong B \cong B \otimes \mathbb{R}_{\mathfrak{M},T}$, is the definition of the statement that $\widehat{\operatorname{Loc}}$ is (or forms part of) a modular functor from $D_T^b(\mathfrak{M})$ to $\underline{A}$-modules. The functors j_{S*}, j_S^*, and $H_T^\bullet$ are all modular functors, which implies that $\widehat{\operatorname{Loc}}$ is also. □

In particular this means that the sheaf $\widehat{\mathcal{A}} = \widehat{\operatorname{Loc}}\, \mathbb{R}_{\mathfrak{M},T}$ is a sheaf of graded rings on $\widehat{\mathscr{S}}$ with multiplication given by the cup product, and for any $B \in D_T^b(\mathfrak{M})$ the sheaf $\widehat{\operatorname{Loc}}\, B$ is naturally a (left) graded $\widehat{\mathcal{A}}$-module. Our assumptions on parity vanishing imply that $\widehat{\mathcal{A}}$ is commutative and the left and right module structures on $\widehat{\operatorname{Loc}}\, B$ coincide.

Applying the associativity constraint on ϕ to
$$B \otimes \mathbb{R}_{\mathfrak{M},T} \otimes C \cong B \otimes C$$
(note that the two ways of constructing this isomorphism are equal), we conclude that $\phi_{B,C}$ descends to a natural map
$$\widehat{\operatorname{Loc}}\, B \otimes_{\widehat{\mathcal{A}}} \widehat{\operatorname{Loc}}\, C \to \widehat{\operatorname{Loc}}(B \otimes C). \tag{5.6}$$
(Again the more precise statement is that $\widehat{\operatorname{Loc}}$ is a modular functor from $D_T^b(\mathfrak{M})$ to $\widehat{\mathcal{A}}-mod$.) Lemma 5.5 implies that this map is an isomorphism if both B and C are in $D_T^*(\mathfrak{M})$: the second part of statement (c) implies that it is an isomorphism at points of $\mathscr{S}$, and (b) implies that it is an isomorphism at points $(R,S) \in \widehat{\mathscr{S}} \smallsetminus \mathscr{S}$. Theorem 5.8 can now be reformulated

as follows.

Theorem 5.11 *If $B \in D^*_T(\mathfrak{M})$, $C \in D^!_T(\mathfrak{M})$, then the natural map*

$$\mathrm{Hom}^\bullet_{\mathfrak{M}}(B, C) \to \mathrm{Hom}^\bullet_{\widehat{\mathcal{A}}-mod}(\widehat{\mathrm{Loc}}\, B, \widehat{\mathrm{Loc}}\, C)$$

is an isomorphism. In particular, taking $B = \mathbb{R}_{\mathfrak{M},T}$, the global sections of $\widehat{\mathrm{Loc}}\, C$ are canonically isomorphic to $H^\bullet_T(\mathfrak{M}; C)$.

Proof: Let $\mathcal{B} = \widehat{\mathrm{Loc}}\, B$ and $\mathcal{C} = \widehat{\mathrm{Loc}}\, C$. A map $\mathcal{B} \to \mathcal{C}$ consists of maps $\mathcal{B}(S) \to \mathcal{C}(S)$ and $\mathcal{B}(R, S) \to \mathcal{C}(R, S)$ over all S and (R, S) in $\widehat{\mathscr{S}}$, compatible with the restriction maps in the sheaves $\mathcal{B}$ and $\mathcal{C}$. Because $B \in D^*_T(\mathfrak{M})$, Lemma 5.5(b) implies that the map $\mathcal{B}(S) \otimes_{\widehat{\mathcal{A}}(S)} \widehat{\mathcal{A}}(R, S) \to \mathcal{B}(R, S)$ is an isomorphism. Thus the maps $\mathcal{B}(R, S) \to \mathcal{C}(R, S)$ are determined by the maps $\mathcal{B}(S) \to \mathcal{C}(S)$ for $S \in \mathscr{S}$. These maps will determine a map of sheaves if and only if for each $(R, S) \in \widehat{\mathscr{S}}$, the composition

$$\mathcal{B}(R) \to \mathcal{C}(R) \to \mathcal{C}(R, S)$$

is equal to

$$\mathcal{B}(R) \to \mathcal{B}(R, S) \xleftarrow{\sim} \mathcal{B}(S) \otimes_{\widehat{\mathcal{A}}(S)} \widehat{\mathcal{A}}(R, S) \to \mathcal{C}(S) \otimes_{\widehat{\mathcal{A}}(S)} \widehat{\mathcal{A}}(R, S) \to \mathcal{C}(R, S).$$

But by Lemma 5.5(c) we have isomorphisms

$$\mathrm{Hom}^\bullet_{\widehat{\mathcal{A}}(S)}(\mathcal{B}(S), \mathcal{C}(S)) \cong \mathrm{Hom}^\bullet_S(B, \Phi_S(C))$$

and

$$\mathrm{Hom}^\bullet_{\widehat{\mathcal{A}}(R)}(\mathcal{B}(R), \mathcal{C}(R, S)) \cong \mathrm{Hom}^\bullet_R(B, \Phi_R \Phi_S(C)),$$

and with these identifications Theorem 5.8 shows that the space of homomorphisms $\mathcal{B} \to \mathcal{C}$ is isomorphic to $\mathrm{Hom}^\bullet_{\mathfrak{M}}(B, C)$. □

5.3 Generalized minimal extension sheaves

We want to use our functor $\widehat{\mathrm{Loc}}$ to describe intersection cohomology. We need to assume that $\mathbf{IC}_{\mathfrak{M},T} \in D^*_T(\mathfrak{M})$, or equivalently that the cohomology sheaves of $\mathbf{IC}_{\mathfrak{M}}$ vanish in odd degrees. Since $\mathbf{IC}_{\mathfrak{M},T}$ is Verdier self-dual, this also implies that $\mathbf{IC}_{\mathfrak{M},T} \in D^!_T(\mathfrak{M})$. This condition holds for many interesting singular spaces, including toric varieties, Schubert varieties, and hypertoric varieties; some general theorems which imply it can be found in [BJ]. (Note, however, that many varieties covered by these results will *not* have a stratification satisfying our condition (C′).)

We now define a class of $\widehat{\mathcal{A}}$-modules called *generalized minimal extension sheaves* (GMES) in analogy with the minimal extension sheaves of [BBFK2, BreL] described in Section 1.4. The definition is also very similar to the indecomposable pure sheaves on moment graphs which were defined in [BraM] to compute torus-equivariant intersection cohomology of Schubert varieties.

For any stratum $S \in \mathscr{S}$, let $\partial S = \{R \in \mathscr{S} \mid S \subset \overline{R}, R \neq S\}$.

Definition 5.12 We say an $\widehat{\mathcal{A}}$-module $\widehat{\mathcal{L}}$ is a GMES if the following conditions hold:

1. $\widehat{\mathcal{L}}(S_0) \cong \widehat{\mathcal{A}}(S_0)$ for the open stratum S_0.
2. For each $S \in \mathscr{S}$, $\widehat{\mathcal{L}}(S)$ is a free $\widehat{\mathcal{A}}(S)$-module.
3. For each $(R, S) \in \widehat{\mathscr{S}}$, we have
$$\widehat{\mathcal{L}}(R, S) \cong \widehat{\mathcal{L}}(S) \otimes_{\widehat{\mathcal{A}}(S)} \widehat{\mathcal{A}}(R, S),$$
and the map $\widehat{\mathcal{L}}(S) \to \widehat{\mathcal{L}}(R, S)$ is the natural one coming from the tensor product.
4. For each $S \in \mathscr{S}$ the restriction maps from $\widehat{\mathcal{L}}(S)$ and $\widehat{\mathcal{L}}(p^{-1}(\partial S))$ to $\bigoplus_{R \in \partial S} \widehat{\mathcal{L}}(S, R)$ have the same image.
5. $\widehat{\mathcal{L}}$ is minimal with respect to conditions 1-4.

Remark 5.13 Condition 4 is equivalent to saying that $\widehat{\mathcal{L}}$ is: (1) generated by global sections, i.e. the restriction $\widehat{\mathcal{L}}(\widehat{\mathscr{S}}) \to \widehat{\mathcal{L}}(S)$ is surjective for every $S \in \mathscr{S}$, and (2) flabby for the coarser topology on $\widehat{\mathscr{S}}$ whose open sets are $p^{-1}(U)$ for $U \subset \mathscr{S}$ open in the order topology, where the map $p \colon \widehat{\mathscr{S}} \to \mathscr{S}$ is defined by $p(S) = S$ and $p((R, S)) = S$.

Proposition 5.14 *Any two generalized minimal extension sheaves are isomorphic.*

The proof of this result is essentially the same as for ordinary minimal extension sheaves; see [BBFK2, Theorem 2.3] and [BreL, Theorem 5.3].

Theorem 5.15 *Suppose that* $\mathbf{IC}_{\mathfrak{M},T} \in D^*_T(\mathfrak{M})$ *and that*

$$\textit{for every } (R, S) \in \widehat{\mathscr{S}}, \textit{ the map } \widehat{\mathcal{A}}(S) \to \widehat{\mathcal{A}}(R, S) \textit{ is surjective.} \quad (*)$$

Then $\widehat{\mathcal{L}} := \widehat{\mathrm{Loc}}\, \mathbf{IC}_{\mathfrak{M},T}$ *is a generalized minimal extension sheaf. Furthermore it is rigid, meaning that it has only scalar automorphisms as a graded* $\widehat{\mathcal{A}}$*-module.*

Proof: Put $\mathbf{IC} = \mathbf{IC}_{\mathfrak{M},T}$. Condition 1 of Definition 5.12 is immediate, since $\mathbf{IC}|_{S_0} = \mathbb{R}_{S_0,T}$. Conditions 2 and 3 follow from Lemma 5.5. To prove conditions 4 and 5, take any $S \in \mathscr{S}$, and let $U = \bigcup_{R \in \partial S} R$. We need to show that $\widehat{\mathcal{L}}(S) = H^\bullet_T(S; j_S^* \mathbf{IC})$ is the minimal free $\widehat{\mathcal{A}}(S)$-module which surjects onto the image of

$$\widehat{\mathcal{L}}(p^{-1}(\partial S)) \longrightarrow \bigoplus_{R \in \partial S} \widehat{\mathcal{L}}(S, R). \quad (5.7)$$

The long exact sequence

$$\cdots \to H^\bullet_T(S; j_S^! \mathbf{IC}) \to H^\bullet_T(S; j_S^* \mathbf{IC}) \to H^\bullet_T(S; j_S^* j_{U*} j_U^* \mathbf{IC}) \to \dots$$

breaks into short exact sequences, since the first and third terms vanish in odd degrees by Lemmas 5.5 and 5.7. Combining Lemma 5.5(a) with the degree restrictions on stalks and costalks of $\mathbf{IC}$, we see that $H^\bullet_T(S; j_S^* \mathbf{IC})$ is generated in degrees less than the complex codimension of S, while $H^\bullet_T(S; j_S^! \mathbf{IC})$ is generated in degrees greater than that. This implies that $\widehat{\mathcal{L}}(S)$ is the smallest free $\widehat{\mathcal{A}}_S$-module which surjects onto $H^\bullet_T(S; j_S^* j_{U*} j_U^* \mathbf{IC})$.

Since $\widehat{\mathcal{L}}(p^{-1}(\partial S)) \cong IH^\bullet_T(U)$ by Theorem 5.11, the map (5.7) can be factored as

$$IH^\bullet_T(U) \to H^\bullet_T(S; j_S^* j_{U*} j_U^* \mathbf{IC}) \to \bigoplus_{R \in \partial S} H^\bullet_T(S; j_S^* \Phi_R \mathbf{IC}) = \bigoplus_{R \in \partial S} \widehat{\mathcal{L}}(S, R),$$

so our result will follow if we can show that the first map is surjective and the second map is injective. The surjectivity follows from the more general fact that $H^\bullet_T(U; B) \to H^\bullet_T(S; j_S^* B)$ is surjective for any $B \in D^!_T(U)$, which can be proved by an induction similar to the proof of Lemma 5.7; the fact that the statement holds for $j_{R*} j_R^! B$ if $R \in \partial S$ follows from the property (*) and Lemma 5.5(b). Injectivity of the second map also follows from a more general statement, that $H^\bullet_T(S; j_S^* j_{U*} j_U^* B) \to \bigoplus_{R \in \partial S} H^\bullet(\mathfrak{M}; \Phi_S \Phi_R B)$ is injective for any object $B \in D^!_T(\mathfrak{M})$. This again can be proved along the lines of Lemma 5.7.

The fact that the sheaf $\widehat{\mathcal{L}}$ has only scalar automorphisms follows by induction on the number of strata, using the degree constraints on the generators of $\widehat{\mathcal{L}}_S$ and $H^\bullet_T(S; j_S^! \mathbf{IC})$. □

Remark 5.16 It also follows from our proof that the equivariant intersection cohomology $IH^\bullet_{T,S}(\mathfrak{M}) = H^\bullet_T(S; j_S^! \mathbf{IC})$ with supports along a stratum S is isomorphic to the kernel of the map $\widehat{\mathcal{L}}(S) \to \bigoplus_{\partial S} \widehat{\mathcal{L}}(S, R)$.

Remark 5.17 One example which satisfies all our hypotheses is when $\mathfrak{M} = X_\Sigma$ is the toric variety defined by a fan Σ, and T is the maximal compact subgroup of the natural complex torus $T_\mathbb{C}$. In that case, we recover the theory of minimal extension sheaves defined in [BBFK2, BreL]. The strata S_σ are just the $T_\mathbb{C}$-orbits indexed by cones $\sigma \in \Sigma$. For any pair of strata S_σ, S_τ with $S_\tau \subset S_\sigma$, there is an equivariant deformation retraction of S_σ to its intersection with a tubular neighborhood of S_τ, which implies that the sheaf $\widehat{\mathcal{A}}$ is the pullback of the structure sheaf $\mathcal{A}$ of conewise polynomial functions on Σ via the continuous map $p\colon \widehat{\Sigma} \to \Sigma$, $\sigma \mapsto \sigma$, $(\tau, \sigma) \mapsto \sigma$, and the generalized minimal extension sheaf $\widehat{\mathcal{L}}$ is the pullback of the toric minimal extension sheaf $\mathcal{L}$.

Remark 5.18 If the property (*) does not hold, it is still possible to calculate the sheaf $\widehat{\mathrm{Loc}}\, \mathbf{IC}$, but definition 5.12 must be modified. The problem is that although $\widehat{\mathcal{L}}(S)$ surjects onto $H^\bullet_T(S; j_S^* j_{U*} j_U^* \mathbf{IC})$, this module may not be the same as the image of the map $IH^\bullet_T(U) \to \bigoplus_{R>S} \widehat{\mathcal{L}}_{(S,R)}$. Instead, it must be computed as a submodule of $\bigoplus_{R>S} \widehat{\mathcal{L}}_{(S,R)}$ cut out by compatibility relations coming from maps to $H^\bullet_T(S; \Phi_S \Phi_R \Phi_{R'} \mathbf{IC})$. For an example of a space where (*) fails, consider the nilpotent cone in $\mathfrak{sl}_3(\mathbb{C})$ with the stratification by adjoint orbits, and the action of the maximal torus T in $SL_3(\mathbb{C})$.

We now use our results to study maps $\mathbf{IC} \otimes \mathbf{IC} \to \mathbf{IC}$. Suppose that such a map makes $\mathbf{IC}$ into a commutative ring object, as in Theorem 5.1. Then Lemma 5.10 implies that $\widehat{\mathcal{L}} = \widehat{\mathrm{Loc}}\, \mathbf{IC}$ is a sheaf of commutative $\widehat{\mathcal{A}}$-algebras. With our hypotheses, we will show the converse.

Theorem 5.19 *Suppose that our stratified T-space $(\mathfrak{M}, \mathscr{S})$ satisfies properties (A), (B) and (C′) of Section 5.1 and property (*) of Theorem 5.15, and that $\mathbf{IC} \in D^*_T(\mathfrak{M})$. Then any commutative $\widehat{\mathcal{A}}$-algebra structure on $\widehat{\mathcal{L}} = \widehat{\mathrm{Loc}}\,\mathbf{IC}$ lifts uniquely to a commutative ring structure on $\mathbf{IC}$.*

Proof: Since $\mathbf{IC}$ is self-dual, it also lies in $D^!_T(\mathfrak{M})$. Since the tensor product commutes with taking stalks, we have $\mathbf{IC} \otimes \mathbf{IC} \in D^*_T(\mathfrak{M})$. Applying (5.6) gives a natural isomorphism $\widehat{\mathcal{L}} \otimes_{\widehat{\mathcal{A}}} \widehat{\mathcal{L}} \cong \widehat{\mathrm{Loc}}(\mathbf{IC} \otimes \mathbf{IC})$. Thus Theorem 5.11 gives an isomorphism

$$\mathrm{Hom}^\bullet_{D^b_T(\mathfrak{M})}(\mathbf{IC} \otimes \mathbf{IC}, \mathbf{IC}) \to \mathrm{Hom}^\bullet_{\widehat{\mathcal{A}}-mod}(\widehat{\mathcal{L}} \otimes_{\widehat{\mathcal{A}}} \widehat{\mathcal{L}}, \widehat{\mathcal{L}}).$$

Let $m\colon\ \mathbf{IC} \otimes \mathbf{IC} \to \mathbf{IC}$ be the morphism corresponding to the multiplication map of $\widehat{\mathcal{L}}$. Associativity of this product is expressed as the equality of two maps $\mathbf{IC} \otimes \mathbf{IC} \otimes \mathbf{IC} \to \mathbf{IC}$, so another application of Theorem 5.11 allows us to deduce associativity for m from associativity of the multiplication on $\widehat{\mathcal{L}}$. The compatibility of the unit map $\mathbb{R}_{\mathfrak{M},T} \to \mathbf{IC}$ with m follows in a similar way, since its localization gives the natural map $\widehat{\mathcal{A}} \to \widehat{\mathcal{L}}$, and the isomorphism $\mathbb{R}_{\mathfrak{M},T} \otimes \mathbf{IC} \to \mathbf{IC}$ localizes to give the natural isomorphism $\widehat{\mathcal{A}} \otimes_{\widehat{\mathcal{A}}} \widehat{\mathcal{L}} \to \widehat{\mathcal{L}}$ (see Lemma 5.10). □

Remark 5.20 Note that because $\widehat{\mathcal{L}}$ is isomorphic to any generalized minimal extension sheaf by a unique isomorphism, to construct a commutative ring structure on $\mathbf{IC}$ it is enough to produce any sheaf of commutative $\widehat{\mathcal{A}}$-algebras which is a GMES.

5.4 The structure sheaf for hypertoric varieties

Let $\mathcal{H}$ be a central, unimodular arrangement in the vector space V, with hyperplanes indexed by a finite set I, and consider the hypertoric variety $\mathfrak{M}_\mathcal{H}$ along with the stratification described in Section 2.3, in which the strata S_F are indexed by flats $F \in L = L_\mathcal{H}$. To simplify notation, from now on we will identify the stratum with the flat that names it in our notation. Thus our extended poset $\widehat{L}$ will consist of single flats and pairs (E, F) of flats with $E < F$, rather than strata and pairs of strata. In this section we give an explicit description of the structure sheaf $\widehat{\mathcal{A}}$ of $\widehat{L}$ and use it to conclude that $\mathfrak{M}_\mathcal{H}$ satisfies all of the hypotheses of Theorem 5.19, thus reducing Theorem 5.1 to a statement about the minimal extension sheaf $\widehat{\mathcal{L}}$ on $\widehat{L}$.

We begin by fixing some notation. For any subset $I' \subset I$, let $\mathbb{R}[I']$ denote the polynomial ring with generators e_i for $i \in I'$, and let $Q_{I'}$ be the ideal in this ring generated by e_i^2, $i \in I'$. Let $J_\mathcal{H} \subset \mathbb{R}[I]$ be the ideal

$$\left\langle \sum_{i \in I_0} a_i e_{I_0 \smallsetminus \{i\}} \;\middle|\; I_0 \subset I \text{ and } \sum_{i \in I_0} a_i x_i|_V = 0 \right\rangle.$$

Thus the ring $R(\mathcal{H})$ introduced in Section 3.3 is equal to $\mathbb{R}[I]/J_\mathcal{H}$. For each flat F, we get ideals $J_{\mathcal{H}^F}$ and $J_{\mathcal{H}_F}$ in the rings $\mathbb{R}[I \setminus F]$ and $\mathbb{R}[F]$, respectively. We will abuse notation and use the same symbols $Q_{I'}$, $J_{\mathcal{H}^F}$, $J_{\mathcal{H}_F}$ to refer to the ideals that they generate in $\mathbb{R}[I]$, using the obvious inclusions $\mathbb{R}[I'] \subset \mathbb{R}[I]$ for any $I' \subset I$.

Lemma 5.21 *For any flat $f \in L_{\mathcal{H}}$, we have the following equality of ideals in $\mathbb{R}[I]$:*

$$J_{\mathcal{H}} + J_{\mathcal{H}^F} + Q_{I \smallsetminus F} = J_{\mathcal{H}_F} + J_{\mathcal{H}^F} + Q_{I \smallsetminus F}.$$

Proof: One inclusion is clear, since the generators of $J_{\mathcal{H}_F}$ form a subset of the generators of $J_{\mathcal{H}}$. For the other direction, consider a generator $f = \sum_{i \in I_0} a_i e_{I_0 \smallsetminus \{i\}}$ of $J_{\mathcal{H}}$ with $a_i \neq 0$ for all $i \in I_0$. Partition I_0 into $I_1 = I_0 \cap F$ and $I_2 = I_0 \setminus I_1$. If $I_2 = \varnothing$ then $f \in J_{\mathcal{H}_F}$. Otherwise, let

$$g = \sum_{i \in I_1} a_i e_{I_1 \smallsetminus \{i\}} \text{ and } h = \sum_{j \in I_2} a_j e_{I_2 \smallsetminus \{j\}},$$

so that

$$f = e_{I_2} g + e_{I_1} h.$$

Then $h \in J_{\mathcal{H}^F}$, and for any $j \in I_2$, we have

$$a_j e_{I_2} = e_j h \;\; - \sum_{k \in I_2 \smallsetminus \{j\}} a_k e_j e_{I_2 \smallsetminus \{k\}} \;\in\; J_{\mathcal{H}^F} + Q_{I \smallsetminus F}.$$

Dividing by a_j, we conclude that $e_{I_2} \in J_{\mathcal{H}^F} + Q_{I \smallsetminus F}$, and therefore $f \in J_{\mathcal{H}^F} + Q_{I \smallsetminus F}$. This completes the proof. □

Recall that for any flat $F \in L_{\mathcal{H}}$, the stabilizer of any point of S_F is the subtorus $T_F \subset T$ with Lie algebra $\langle F \rangle^{\perp} \subset V^*$. Then T/T_F is the natural torus that acts on the closure $\overline{S_F} \cong \mathfrak{M}_{\mathcal{H}^F}$. For this reason, we may also think of T/T_F as a quotient of the coordinate torus $T^{I \smallsetminus F}$. Let

$$A^F = \operatorname{Sym}\langle F \rangle = H^{\bullet}_{T/T_F}(pt).$$

Then we have natural inclusions

$$A^F \subset A = \operatorname{Sym} V \text{ and } A^F \subset \mathbb{R}[I \setminus F].$$

Let $\widehat{\mathcal{A}}$ be the structure sheaf on $\widehat{L}$ induced by the hypertoric variety $\mathfrak{M}_{\mathcal{H}}$, as described in Section 5.2. The following proposition gives an explicit description of this sheaf. Let

$$S(\mathcal{H}) = \mathbb{R}[I]/(J_{\mathcal{H}} + Q_I).$$

Proposition 5.22 *There are canonical isomorphisms*

$$\widehat{\mathcal{A}}(F) \cong A \otimes_{A^F} S(\mathcal{H}^F)$$

and

$$\widehat{\mathcal{A}}(E, F) \cong A \otimes_{A^F} S(\mathcal{H}^F) \otimes_{\mathbb{R}[I \smallsetminus E]} S(\mathcal{H}^E) = A \otimes_{A^F} \mathbb{R}[I \setminus F]/(J_{\mathcal{H}^F} + J_{\mathcal{H}^E} + Q_{I \smallsetminus F}),$$

and the restriction maps $\widehat{\mathcal{A}}(F) \to \widehat{\mathcal{A}}(E, F)$ and $\widehat{\mathcal{A}}(E) \to \widehat{\mathcal{A}}(E, F)$ are the obvious ones.

Proof: First, note that tensoring with A over A^F is just the change of coefficients that gives

T-equivariant cohomology from $T(F)$-equivariant cohomology, so the general case follows from the case where $F = \varnothing$ is the minimal flat.

We obtain the identification of $\widehat{\mathcal{A}}(\varnothing) = H^\bullet_T(S_\varnothing)$ with $S(\mathcal{H})$ by noting that T acts freely on $S_\varnothing$, so there is an isomorphism of rings $H^\bullet_T(S_\varnothing) \cong H^\bullet(S_\varnothing/T)$. The moment map μ identifies $S_\varnothing/T$ with the complement

$$M(\mathcal{H}) := V^{\mathrm{hk}} \setminus \bigcup_{i \in I} H_i^{\mathrm{hk}}$$

of the tripled arrangement. By [dLS, 5.6] this ring is naturally isomorphic to the quotient $\mathbb{R}[I]/(J_{\mathcal{H}} + Q_I)$.[7] Note that the formula in [dLS] has $\sum_{i \in I_0} \operatorname{sgn}(a_i) e_{I_0 \smallsetminus \{i\}}$ in place of the generator $\sum_{i \in I_0} a_i e_{I_0 \smallsetminus \{i\}}$ of $J_{\mathcal{H}}$, but since we assume $\mathcal{H}$ is unimodular, these are the same.

In order to pin down the restriction maps in the structure sheaf we need to describe this isomorphism more precisely. In [dLS] the generator e_i corresponding to the i^{th} hyperplane is the pullback of a generating class in the degree two cohomology of $M(\mathcal{H}_{\{i\}}) \cong \mathbb{R}^3 \setminus \{0\}$ via the natural quotient map. This class can be given another way, using the construction of $\mathfrak{M}_{\mathcal{H}} = \Psi^{-1}(V^{\mathrm{hk}})/T_V$ as a hyperkähler quotient of $\mathbb{H}^I$. The open stratum $S_\varnothing$ is a quotient of $\Psi^{-1}(V^{\mathrm{hk}}) \cap (\mathbb{H}^\times)^I$ by the action of the subtorus $T_V \subset T^I$, which acts freely. Thus we have a ring isomorphism $\widehat{\mathcal{A}}(\varnothing) \cong H^\bullet_{T^I}(\Psi^{-1}(V^{\mathrm{hk}}) \cap (\mathbb{H}^\times)^I)$, giving rise to a natural homomorphism $\mathbb{R}[I] = H^\bullet_{T^I}(pt) \to \widehat{\mathcal{A}}(\varnothing)$.

We claim that this homomorphism is the obvious one which sends e_i to e_i. This is because the projection $S_\varnothing/T \cong M(\mathcal{H}) \to M(\mathcal{H}_{\{i\}})$ can be covered by the T^I-equivariant projection $\Psi^{-1}(V^{\mathrm{hk}}) \cap (\mathbb{H}^\times)^I \to \mathbb{H}^\times$ onto the i^{th} factor. An easy computation shows that $H^\bullet_{T^I}(\mathbb{H}^\times) \cong \mathbb{R}[I]/\langle e_i^2 \rangle$, and the image of e_i gives a generator of the degree 2 cohomology of $\mathbb{H}^\times/T^I = \mathbb{H}^\times/U(1) \cong \mathbb{R}^3 \setminus \{0\}$.

Next, consider the ring $\widehat{\mathcal{A}}(E, \varnothing)$. It is the target of maps from the sources $S(\mathcal{H}) = \widehat{\mathcal{A}}(\varnothing)$ and $S(\mathcal{H}^E) \subset \widehat{\mathcal{A}}(E)$, and by our identification of the generators e_i, these maps induce a map

$$S(\mathcal{H}^E) \otimes_{\mathbb{R}[I \setminus E]} S(\mathcal{H}) \to \widehat{\mathcal{A}}(E, \varnothing). \tag{5.8}$$

We show that this map is an isomorphism. By Lemma 5.21, the ring on the left is

$$\mathbb{R}[I]/(J_{\mathcal{H}} + J_{\mathcal{H}^E} + Q_I) = \mathbb{R}[I]/(J_{\mathcal{H}_E} + J_{\mathcal{H}^E} + Q_I) \cong S(\mathcal{H}^E) \otimes_{\mathbb{R}} S(\mathcal{H}_E).$$

To understand the right hand side of (5.8), choose a splitting $T = T_E \times T^E$. Then T^E acts freely on $\mathfrak{M}_{\geq E} := \bigcup_{F \leq E} S_F$, so there is an equivalence of categories $D^b_T(\mathfrak{M}_{\geq E}) \simeq D^b_{T_E}(\mathfrak{M}_{\geq E}/T^E)$ which preserves cohomology and commutes with pullbacks and pushforwards (see [BerL, 2.6.2 and 3.4.1]). If we let $\hat{\jmath}_F \colon S_F/T^E \to \mathfrak{M}_{\geq E}/T^E$ denote the inclusion, then we see that $\widehat{\mathcal{A}}(E, \varnothing)$ is isomorphic to the T_E-equivariant cohomology of $B = \hat{\jmath}_E^*(\hat{\jmath}_\varnothing)_* \mathbb{R}_{S_\varnothing/T^E, T_E}$.

The group T_E acts trivially on S_E/T^E, and using Proposition 2.5 one can show that B is a locally constant sheaf with T_E-equivariant stalk cohomology which is isomorphic to $S(\mathcal{H}_E)$. On the other hand, $S_E/T^E \cong M(\mathcal{H}^E)$, so we have a spectral sequence with E_2 term $S(\mathcal{H}^E) \otimes_{\mathbb{R}} S(\mathcal{H}_E)$ converging to $\widehat{\mathcal{A}}(E, \varnothing)$. It collapses for parity reasons, so both sides of (5.8) have the same graded dimension.

[7]The generators of Q_I were inadvertently omitted from the statement of [dLS, 5.6].

Thus we only have to show that (5.8) is a surjection. This follows by the Leray-Hirsch theorem, since the image of e_i, $i \notin F$ give classes generating the action of the cohomology of the base $M(\mathcal{H}^E)$, while the image of $S(\mathcal{H}_E)$ gives classes which restrict to a vector space basis for $H^\bullet_{T_E}(B|_x)$ for a point $x \in S_E/T^E$. □

Corollary 5.23 *The hypertoric variety $\mathfrak{M}_\mathcal{H}$ satisfies the hypotheses of Theorem 5.19, thus commutative ring structures on* **IC** *correspond precisely to commutative $\widehat{\mathcal{A}}$-algebra structures on $\widehat{\mathcal{L}}$.*

Remark 5.24 We believe that the ring structure on intersection cohomology lifts to the derived category for arbitrary unimodular arrangements; more precisely, that Theorem 5.1 holds even in the non-central case. However, in the proof of Proposition 5.22 we needed to assume the arrangement is central in order to apply the results of [dLS]. For non-central arrangements it is still possible to compute the ring structure on the cohomology $H^\bullet(M(\mathcal{H}))$, using results of Deligne, Goresky, and MacPherson [DGM], but it is more difficult to write down an explicit presentation for this ring, so more care will be needed.

5.5 Generalized minimal extension sheaves for hypertoric varieties

In this section we show that $\widehat{\mathcal{L}} = \widehat{\mathrm{Loc}}\,\mathbf{IC}$ admits a unique commutative $\widehat{\mathcal{A}}$-algebra structure. By Remark 5.20, in order to define this algebra structure it is enough to define a sheaf $\widehat{\mathcal{R}}$ of commutative $\widehat{\mathcal{A}}$-algebras and prove that it is a GMES.

Recall the ring $R(\mathcal{H}) = \mathbb{R}[I]/J_\mathcal{H}$. Define the stalks of $\widehat{\mathcal{R}}$ by

$$\widehat{\mathcal{R}}(F) := R(\mathcal{H}) \otimes_{\mathbb{R}[I \smallsetminus F]} S(\mathcal{H}^F) \cong \mathbb{R}[I]/(J_\mathcal{H} + J_{\mathcal{H}^F} + Q_{I \smallsetminus F}),$$

and

$$\widehat{\mathcal{R}}(E,F) := R(\mathcal{H}) \otimes_{\mathbb{R}[I \smallsetminus F]} S(\mathcal{H}^F) \otimes_{\mathbb{R}[I \smallsetminus E]} S(\mathcal{H}^E) \cong \mathbb{R}[I]/(J_\mathcal{H} + J_{\mathcal{H}^F} + J_{\mathcal{H}^E} + Q_{I \smallsetminus F}),$$

and let the restriction maps be the obvious ones. It is clear that this makes $\widehat{\mathcal{R}}$ into a sheaf of commutative $\widehat{\mathcal{A}}$-algebras.

Not surprisingly, there is a close relation between $\widehat{\mathcal{R}}$ and the sheaf $\mathcal{R}$ which was defined in Section 3.3. Using Lemma 5.21, we have different expressions for the stalks of $\widehat{\mathcal{R}}$:

$$\widehat{\mathcal{R}}(F) = R(\mathcal{H}_F) \otimes_\mathbb{R} S(\mathcal{H}^F), \quad \widehat{\mathcal{R}}(E,F) = R(\mathcal{H}_F) \otimes_\mathbb{R} S(\mathcal{H}^F_E) \otimes_\mathbb{R} S(\mathcal{H}^E). \tag{5.9}$$

Let $\mathcal{I} \subset \widehat{\mathcal{A}}$ be the ideal sheaf which on a flat F or a pair (E,F) is generated by e_i, $i \notin F$, and consider the quotients $\widehat{\mathcal{A}}/\mathcal{I}$ and $\widehat{\mathcal{R}}/\widehat{\mathcal{R}}\mathcal{I} = \widehat{\mathcal{R}} \otimes_{\widehat{\mathcal{A}}} (\widehat{\mathcal{A}}/\mathcal{I})$. Their stalks are given by

$$(\widehat{\mathcal{A}}/\mathcal{I})(F) = (\widehat{\mathcal{A}}/\mathcal{I})(E,F) = A \otimes_{A^F} \mathbb{R} = \mathcal{A}(F)$$

and

$$(\widehat{\mathcal{R}}/\widehat{\mathcal{R}}\mathcal{I})(F) = (\widehat{\mathcal{R}}/\widehat{\mathcal{R}}\mathcal{I})(E,F) = R(\mathcal{H}_F) = \mathcal{R}(F),$$

and in fact we have $\widehat{\mathcal{A}}/\mathcal{I} = p^{-1}(\mathcal{A})$ and $\widehat{\mathcal{R}}/\widehat{\mathcal{R}}\mathcal{I} \cong p^{-1}(\mathcal{R})$, where as before p is the projection $p(F) = F$, $p((E,F)) = F$. Since the stalks of p are acyclic we have $p_*p^{-1}\mathcal{R} \cong \mathcal{R}$, giving an isomorphism

$$p_*(\widehat{\mathcal{R}} \otimes_{\widehat{\mathcal{A}}} p^{-1}(\mathcal{A})) \cong \mathcal{R}. \tag{5.10}$$

Example 5.25 Let $\mathcal{H}$ be the central arrangement with three hyperplanes in $V \cong \mathbb{R}^2$, indexed by $I = \{1, 2, 3\}$ and defined by covectors $w_i \in V^*$ satisfying the relation $w_1 + w_2 + w_3 = 0$. We have $\widehat{\mathcal{R}} = \widehat{\mathcal{A}}$ on all elements of $\widehat{L}$ except the maximal flat I. Therefore the stalks at these points are given by

- $\widehat{\mathcal{R}}(\varnothing) = \mathbb{R}[I]/\langle e_1e_2 + e_2e_3 + e_1e_3, e_1^2, e_2^2, e_3^2\rangle$
- $\widehat{\mathcal{R}}(\{1\}, \varnothing) = \mathbb{R}[I]/\langle e_2 + e_3, e_1^2, e_2^2, e_3^2\rangle$,
- $\widehat{\mathcal{R}}(\{1\}) = \widehat{\mathcal{R}}(I, \{1\}) = \mathbb{R}[I]/\langle e_2 + e_3, e_2^2, e_3^2\rangle$,

and similarly with $\{1\}$ replaced by $\{2\}$ and $\{3\}$. The space of sections of this sheaf on $\widehat{L} \setminus \{I\}$ injects into $\bigoplus_{i\in I} \widehat{\mathcal{R}}(I, \{i\})$, with image isomorphic to

$$\mathbb{R}[I]/\langle e_1e_2e_3, e_1e_2 + e_2e_3 + e_1e_3\rangle.$$

Then $R(\mathcal{H}) = \mathbb{R}[I]/\langle e_1e_2 + e_2e_3 + e_1e_3\rangle$ is the smallest free $\mathbb{R}[e_1 - e_2, e_1 - e_3]$-module that surjects onto this image.[8]

Proposition 5.26 *$\widehat{\mathcal{R}}$ is a GMES.*

Proof: It is clear that the sheaf $\widehat{\mathcal{R}}$ satisfies Conditions 1 and 3 of Definition 5.12. Since $\widehat{\mathcal{R}}(F) = R(\mathcal{H}_F) \otimes_{\mathbb{R}} S(\mathcal{H}^F)$, it is a free $\widehat{\mathcal{A}}(F) = \mathcal{A}(F) \otimes_{\mathbb{R}} S(\mathcal{H}^F)$-module.

Next we prove conditions 4 and 5. For a $\widehat{\mathcal{A}}$-module $\widehat{\mathcal{F}}$ and a flat F, define

$$\widehat{\mathcal{F}}_{\partial F} := \mathrm{Im}\left(\widehat{\mathcal{F}}(p^{-1}(\partial F)) \to \bigoplus_{E<F} \widehat{\mathcal{F}}(F, E)\right).$$

Then $\widehat{\mathcal{R}}(F)$ maps into $\widehat{\mathcal{R}}_{\partial F}$, because $\widehat{\mathcal{R}}$ is a quotient of the constant sheaf on $\widehat{L}$ with stalk $\mathbb{R}[I]$, so any element of $\widehat{\mathcal{F}}(F)$ extends to a global section of $\widehat{\mathcal{F}}$.

Take a flat $F \neq \varnothing$, and let $\widehat{\mathcal{L}} = \widehat{\mathrm{Loc}}\,\mathbf{IC}$. Consider the exact sequences

$$0 \to \mathrm{Ker}_{\widehat{\mathcal{R}}} \to \widehat{\mathcal{R}}(F) \to \widehat{\mathcal{R}}_{\partial F} \tag{5.11}$$

and

$$0 \to \mathrm{Ker}_{\widehat{\mathcal{L}}} \to \widehat{\mathcal{L}}(F) \to \widehat{\mathcal{L}}_{\partial F} \to 0. \tag{5.12}$$

The second sequence is right exact because $\widehat{\mathcal{L}}$ is a GMES, and so satisfies condition 4. We need to show that the first sequence is also right exact.

We can assume by induction that $\widehat{\mathcal{R}}$ satisfies conditions 4 and 5 for all flats $E < F$. This means that $\widehat{\mathcal{R}}$ restricts to a GMES on $\widehat{\partial F}$, and so $\widehat{\mathcal{R}}|_{\widehat{\partial F}} \cong \widehat{\mathcal{L}}|_{\widehat{\partial F}}$, by Proposition 5.14. Since

[8] We note that this is almost the same as the ring of global sections in Example 3.13. This is because the two arrangements are almost the same; they differ only in the coorientation of the third hyperplane, which is responsible for the sign that appears in Example 3.13.

both $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{L}}$ satisfy condition 3, this isomorphism extends to $p^{-1}(\partial F)$, and so $\widehat{\mathcal{R}}_{\partial F}$ and $\widehat{\mathcal{L}}_{\partial F}$ are isomorphic.

The middle terms $\widehat{\mathcal{R}}(F)$ and $\widehat{\mathcal{L}}(F)$ of (5.11) and (5.12) are also isomorphic, since both are isomorphic to

$$\overline{R(\mathcal{H}_F)} \otimes_{\mathbb{R}} \widehat{\mathcal{A}}(F) \cong IH^\bullet(\mathfrak{M}_{\mathcal{H}_F}) \otimes_{\mathbb{R}} \widehat{\mathcal{A}}(F).$$

Thus these modules are both generated in degrees less than $(1/2)\operatorname{codim}_{\mathbb{R}} S_F = 2\operatorname{rk} F$, and so $\widehat{\mathcal{L}}_{\partial F} \cong \widehat{\mathcal{R}}_{\partial F}$ is also generated in degrees $< 2\operatorname{rk} F$.

By Remark 5.16, $\operatorname{Ker}_{\widehat{\mathcal{L}}}$ is isomorphic to $H^\bullet_T(S; j^!_S\,\mathbf{IC})$, and so it vanishes in degrees $\leq 2\operatorname{rk} F$. It follows that $\widehat{\mathcal{L}}(F)$ and $\widehat{\mathcal{L}}_{\partial F}$ have the same dimension in these degrees, and hence so do $\widehat{\mathcal{R}}(F)$ and $\widehat{\mathcal{R}}_{\partial F}$. Our result will follow if we can show that $\operatorname{Ker}_{\widehat{\mathcal{R}}}$ also vanishes in these degrees, since then the right map of (5.11) must hit all the generators of $\widehat{\mathcal{R}}_{\partial F}$.

To see this vanishing, note that by (5.9) the quotient of $\widehat{\mathcal{R}}(F, E)$ by the generators e_i for $i \in F \setminus E$ is $R(\mathcal{H}_E) \otimes_{\mathbb{R}} S(\mathcal{H}^F)$. It follows that $\operatorname{Ker}_{\widehat{\mathcal{R}}}$ is contained in the kernel of the map

$$\widehat{\mathcal{R}}(F) \cong R(\mathcal{H}_F) \otimes_{\mathbb{R}} S(\mathcal{H}^F) \to \bigoplus_{E<F} R(\mathcal{H}_E) \otimes_{\mathbb{R}} S(\mathcal{H}^F).$$

But this is just $\mathcal{R}(F, \partial F) \otimes_{\mathbb{R}} S(\mathcal{H}^F)$, so the required vanishing follows from Proposition 3.5 and the first sentence of the proof of Corollary 3.7. □

Thus we get a canonical isomorphism $\widehat{\mathcal{R}} \cong \widehat{\mathcal{L}} = \widehat{\operatorname{Loc}}(\mathbf{IC})$, so we have induced a ring structure on $\widehat{\mathcal{L}}$, and hence $\mathbf{IC}$ becomes a ring object in $D^b_T(\mathfrak{M})$. The uniqueness of this ring structure follows from the following result.

Proposition 5.27 *The commutative $\widehat{\mathcal{A}}$-algebra structure on $\widehat{\mathcal{R}} \cong \widehat{\mathcal{L}}$ is unique.*

Proof: Let $m\colon \widehat{\mathcal{R}} \otimes_{\widehat{\mathcal{A}}} \widehat{\mathcal{R}} \to \widehat{\mathcal{R}}$ be the algebra structure on $\widehat{\mathcal{R}}$ that we have already defined, and let m' be another one. We show by induction on the rank of a flat F that $m = m'$ on F and (E, F) for any $E \geq F$. It is obvious that $m = m'$ for the minimal flat , since $\widehat{\mathcal{R}} = \widehat{\mathcal{A}}$.

If we know that $m = m'$ at a flat F, then the same is true for all (E, F) with $E \geq F$, since the map $\widehat{\mathcal{R}}(F) \otimes_{\widehat{\mathcal{A}}(F)} \widehat{\mathcal{A}}(E, F) \to \widehat{\mathcal{R}}(E, F)$ is a surjective ring homomorphism for either ring structure. So we can take $F \neq \varnothing$, and suppose inductively that $m = m'$ on (F, F') for all $F' \leq F$.

The degree restrictions on intersection cohomology stalks imply that $\widehat{\mathcal{R}}(F)$ is generated as an $\widehat{\mathcal{A}}(F)$-module in even degrees less than $2\operatorname{rk} F$. Similarly, we observed in Remark 5.16 that the kernel of $\widehat{\mathcal{R}}(F) \to \bigoplus_{F' \leq F} \widehat{\mathcal{R}}(F, F')$ is isomorphic to $H^\bullet_T(S; j^!_{S_F}\,\mathbf{IC})$, so by the degree restrictions on intersection cohomology costalks, it is generated as an $\widehat{\mathcal{A}}(F)$-module in even degrees greater than $2\operatorname{rk} F$. It follows that if $\widehat{\mathcal{R}}(F)$ has an algebra structure which is compatible with the restriction maps, then the multiplication by elements in degree 1 is determined by the the multiplication on $\widehat{\mathcal{R}}(F, F')$ for $F' \leq F$. But $\widehat{\mathcal{R}}(F)$ with the ring structure m is generated in degree 1, and so by associativity we must have $m = m'$ on all elements. □

5.6 Comparing the two localizations

To complete the proof of Theorem 5.1, we need to compare the two localization functors we have defined. As before, we are considering a hypertoric variety $\mathfrak{M} = \mathfrak{M}_{\mathcal{H}}$ defined by a central unimodular arrangement.

For arbitrary objects $B, C \in D^b_{T,\mathscr{S}}(\mathfrak{M})$ there is a natural map $\mathrm{Loc}(B) \otimes_{\mathcal{A}} \mathrm{Loc}(C) \to \mathrm{Loc}(B \otimes C)$, so our ring structure on $\mathbf{IC}$ induces a ring structure on the minimal extension sheaf $\mathcal{L} := \mathrm{Loc}(\mathbf{IC})$. The problem is to show that this agrees with the ring structure provided by the isomorphism $\mathcal{L} \cong \mathcal{R}$.

Let $\widehat{\mathcal{L}} = \widehat{\mathrm{Loc}}(\mathbf{IC})$, with the ring structure induced from our ring structure on $\mathbf{IC}$ (equivalently, it is induced from the isomorphism $\widehat{\mathcal{L}} \cong \widehat{\mathcal{R}}$). Recall that we have a map $p\colon \widehat{L} \to L$ given by $p((E,F)) = p(F) = F$. We define a map $\phi\colon p_*\widehat{\mathcal{L}} \to \mathcal{L}$ as follows. Let $U \subset L$ be an open set, and let $\mathfrak{M}_U = \bigcup_{F \in U} S_F$ be the corresponding open subset of $\mathfrak{M}$. The space of sections $p_*\widehat{\mathcal{L}}(U) = \widehat{\mathcal{L}}(p^{-1}(U))$ is canonically isomorphic to $IH^\bullet_T(\mathfrak{M}_U)$ by Theorem 5.11. This group then maps to $\mathcal{L}(U)$ by the natural map $\Gamma_{\mathbf{IC}(\mathfrak{M}_U)}$ introduced in Section 1.2. These are all maps of $A = H^\bullet_T(pt)$-modules, and they are compatible with restrictions of open sets, so this defines a map of $\mathcal{A}$-modules. It is also easy to check that it is a map of rings.

If $U = U_F$ is the smallest open set containing F, then the map $\Gamma_{\mathbf{IC}(\mathfrak{M}_U)}$ is just the restriction $IH^\bullet_T(\mathfrak{M}_U) \to H^\bullet_T(Tx; \mathbf{IC})$ to any orbit $Tx \subset S_F$, followed by the identification of $H^\bullet_T(Tx; \mathbf{IC})$ with $\mathcal{L}(F) = \mathcal{L}(U_F)$. Using Lemma 5.5(a), we see that if $s \in \widehat{\mathcal{L}}(U)$, then the section $\phi(s)$ sends F to the image of $s_F \in \widehat{\mathcal{L}}(F)$ in

$$\widehat{\mathcal{L}}(F) \otimes_{\widehat{\mathcal{A}}(F)} \mathcal{A}(F) = H^\bullet_T(S_F; \mathbf{IC}) \otimes_{H^\bullet_T(S_F)} H^\bullet_T(Tx) \cong H^\bullet_T(Tx; \mathbf{IC}) \cong \mathcal{L}(F).$$

In other words, we can factor ϕ as the composition

$$p_*\widehat{\mathcal{L}} \to p_*(\widehat{\mathcal{L}} \otimes_{\widehat{\mathcal{A}}} p^{-1}(\mathcal{A})) \to \mathcal{L},$$

where the second map is an isomorphism.

Applying the isomorphisms $\widehat{\mathcal{R}} \cong \widehat{\mathcal{L}}$, $\mathcal{R} \cong \mathcal{L}$, this second map becomes the map (5.10), since maps of minimal extension sheaves are rigid. Thus the ring structure on $\mathcal{L}$ induced by the one on $\mathbf{IC}$ is the same as the one induced by (5.10) and the isomorphism $\mathcal{R} \cong \mathcal{L}$, completing the proof of Theorem 5.1.

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