Quantizations of conical symplectic resolutions I: 
local and global structure

Tom Braden
Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003

Nicholas Proudfoot
Department of Mathematics, University of Oregon, Eugene, OR 97403

Ben Webster
Department of Mathematics, University of Virginia, Charlottesville, VA 22904

Abstract. We re-examine some topics in representation theory of Lie algebras and Springer 
theory in a more general context, viewing the universal enveloping algebra as an example 
of the section ring of a quantization of a conical symplectic resolution. While some 
modification from this classical context is necessary, many familiar features survive. These 
include a version of the Beilinson-Bernstein localization theorem, a theory of Harish-
Chandra bimodules and their relationship to convolution operators on cohomology, and a 
discrete group action on the derived category of representations, generalizing the braid 
group action on category $\mathcal{O}$ via twisting functors.

Our primary goal is to apply these results to other quantized symplectic resolutions, 
including quiver varieties and hypertoric varieties. This provides a new context for 
known results about Lie algebras, Cherednik algebras, finite W-algebras, and hypertoric 
enveloping algebras, while also pointing to the study of new algebras arising from more 
general resolutions.

1 Introduction

The dazzling success of algebraic geometry... has so much reoriented the field 
that one particular protagonist has suggested, no doubt with much justification, 
that enveloping algebras should now be relegated to a subdivision of the theory of 
rings of differential operators.

—Anthony Joseph, On the classification of primitive ideals in the enveloping 
 algebra of a semisimple Lie algebra

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3Supported by NSA grant H98230-10-1-0199. Email: b.webster@neu.edu. Phone: 617-852-1387. Fax: 617-373-5658.
In this paper, we argue against the relegation suggested above, in favor of a different geometric context. While viewing universal enveloping algebras as differential operators is unquestionably a powerful technique, the differential operators on flag varieties are odd men out in the world of differential operators as a whole. For example, the only known examples of projective varieties that are D-affine are homogeneous spaces for semi-simple complex Lie groups, and it is conjectured that no other examples exist. On the other hand, in this paper we consider a world where this special case is very much at home: quantizations of symplectic resolutions of affine singularities.

Differential operators on a smooth projective variety $X$ form a deformation quantization of the cotangent bundle $T^*X$. If $X$ is a homogeneous space for a semi-simple complex Lie group $G$, its cotangent bundle is a resolution of the closure of a nilpotent orbit in $g^*$ (or an affine variety finite over this one). If $X$ is the flag variety, this is known as the Springer resolution. This is yet another sense in which these spaces are misfits; homogeneous spaces for semi-simple complex Lie groups are conjecturally the only examples of projective varieties whose cotangent bundles resolve affine singularities. For most projective varieties $X$, $T^*X$ does not have enough global functions.

There are, however, many other examples of symplectic algebraic varieties that resolve affine cones. While the Springer resolution is the most famous, other examples include the minimal resolution of a Kleinian singularity, the Hilbert scheme of points on such a resolution, Nakajima quiver varieties, and hypertoric varieties. One can study deformation quantizations of these varieties, and many of them have the same affinity property enjoyed by the Springer resolution. This paper is a study of these deformation quantizations and their representation theory.

Several examples have been studied extensively by other authors. Universal enveloping algebras have been considered from an enormous number of angles for decades, and other examples such as spherical Cherednik algebras and finite W-algebras have been active fields of research for many years. The hypertoric case has recently been studied by Bellamy and Kuwabara [BK] and by the authors of this paper, jointly with Licata [BLPW12]. On the other hand, very few works attempt to view all these examples in a single coherent theory. Kashiwara and Rouquier began to develop such a theory [KR08], and our paper might be regarded as a continuation of their work. A recent preprint of McGerty and Nevins [MNa] addresses similar issues, with results that are complementary to ours.

In Section 2 we discuss the algebraic geometry of conical symplectic resolutions; this is essentially all material already in the literature, but we collect it here for the convenience of the reader. Particularly important for us are deformations which appear in the work of Kaledin and Verbitsky; these show that any symplectic resolution flatly deforms to a smooth affine variety, which is key to many properties of its quantization. One ingredient we will use systematically is the conical structure: a choice of $\mathbb{C}^*$-action which makes the base into a cone.
and acts with positive weight on the symplectic form.

In Section 3, we discuss equivariant quantizations of a conical symplectic resolution $M$, which are classified by $H^2(M; \mathbb{C})$ [BK04, Los]. We prove some basic results about the ring $A$ of $S$-invariant global sections, a filtered algebra whose associated graded is isomorphic to $\mathbb{C}[M]$. We also study the behavior of quantizations under (quantum) Hamiltonian reduction, proving a quantum version of the Duistermaat-Heckmann theorem (Proposition 3.15).

In Section 4, we introduce the appropriate category $D$-mod of modules over a quantization $D$, which one may regard as the quantum analogue of the category of coherent sheaves (in particular, there is a finiteness assumption built into the definition). In the case where $M$ is a cotangent bundle, we show that this category is equivalent to the category of finitely generated twisted $D$-modules on the base, where the twist is determined by the period of the quantization. The rest of the section is dedicated to the study of the sections and localization functors that relate the category of modules over a quantization to the category of modules over the section ring $A$. We establish in Theorem 4.16 that these functors induce derived equivalences for generic periods.

Theorem A Let $M$ be a conical symplectic resolution, and fix two classes $\eta, \lambda \in H^2(M; \mathbb{C})$ such that $\eta$ is the Chern class of an ample line bundle, or the strict transform of an ample line bundle on any other conical symplectic resolution of $M_0$. For all but finitely many complex numbers $k$, the quantization of $M$ with period $\lambda + k\eta$ is derived affine; that is, the derived functors of global sections and localization are inverse equivalences.

In order to obtain an equivalence of abelian (rather than derived) categories that works for all (rather than only generic) periods, we replace the section ring $A$ with a $\mathbb{Z}$-algebra, which mimics in a non-commutative setting the homogeneous coordinate ring of a projective variety. Given a quantized symplectic resolution along with a very ample line bundle, we construct a $\mathbb{Z}$-algebra $Z$ and prove the following result (Theorem 5.8).

Theorem B Let $M$ be a conical symplectic resolution, let $L$ be a very ample line bundle on $M$, and let $Z$ be the associated $\mathbb{Z}$-algebra. Then the category $D$-mod is equivalent to the category of finitely generated modules over $Z$ modulo the subcategory of bounded modules.

Theorem B has three nice consequences. First, we use it to prove the following abelian analogue of Theorem A (Corollary 5.13).

Corollary B.1 Let $M$ be a conical symplectic resolution, and fix two classes $\eta, \lambda \in H^2(M; \mathbb{C})$ such that $\eta$ is the Chern class of an ample line bundle. For all but finitely many positive integers $k$, the quantization of $M$ with period $\lambda + k\eta$ is affine; that is, the (abelian) functors of global sections and localization are inverse equivalences.
Next, we prove a categorical version of Kirwan surjectivity, relating the category of equivariant modules on a quantization to the category of modules on the Hamiltonian reduction. We consider a restriction functor defined by Kashiwara and Rouquier, and we use our \( \mathbb{Z} \)-algebra formalism to construct left and right adjoints, thus proving that the restriction functor is essentially surjective (Theorem 5.23). In particular, this result establishes that our category \( D \)-mod is the same as the analogous category considered by McGerty and Nevins (Remark 5.24).

**Corollary B.2** If \( \mathcal{M} \) is obtained via symplectic reduction from an action of a reductive group \( G \) on \( X \), then every object of \( D \)-mod extends to a twisted \( G \)-equivariant module over a quantization of \( X \).

Finally, we use Theorem B to prove a version of Serre’s GAGA theorem [Ser56]. More precisely, we consider the analytic quantization \( D^{an} \) with the same period as \( D \), define the appropriate module category \( D^{an} \)-mod, and prove that it is equivalent to \( D \)-mod (Theorem 5.25). The existing literature is fairly evenly divided between working in the algebraic and analytic categories, and this corollary is an indispensable tool that allows us to import previous results from both sides.

**Corollary B.3** If \( \mathcal{M} \) is a conical symplectic resolution, then the analytification functor from \( D \)-mod to \( D^{an} \)-mod is an equivalence of categories.

Let \( \mathcal{M}_0 := \text{Spec } \mathbb{C}[\mathcal{M}] \) be the cone resolved by \( \mathcal{M} \), and consider the **Steinberg variety** \( \mathcal{Z} := \mathcal{M} \times \mathcal{M}_0 \). The cohomology \( H^{2 \text{dim } \mathcal{M}}_3(\mathcal{Z} \times \mathcal{M}) \) with supports in \( \mathcal{Z} \), which by Poincaré duality can be identified with the Borel-Moore homology group \( H^{BM}_{2 \text{dim } \mathcal{M}}(\mathcal{Z}) \), has a natural algebra structure via convolution [CG97, §2.7]. Furthermore, if \( \mathcal{L} \subset \mathcal{M} \) is a Lagrangian subscheme that is equal to the preimage of its image in \( \mathcal{L}_0 \subset \mathcal{M}_0 \), then the convolution algebra acts on \( H^{\text{dim } \mathcal{M}}_2(\mathcal{L}) \). In the special case where \( \mathcal{M} = T^*(G/B) \) and \( \mathcal{L} \) is the conormal variety to the Schubert stratification of \( G/B \), the convolution algebra is isomorphic to the group algebra of the Weyl group, and \( H^{\text{dim } \mathcal{M}}_2(\mathcal{L}) \) is isomorphic to the regular representation. More generally, there is a natural algebra homomorphism from the group algebra \( \mathbb{C}[W] \) of the Namikawa Weyl group of \( \mathcal{M} \) to the convolution algebra \( H^{\text{dim } \mathcal{M}}_3(\mathcal{M} \times \mathcal{M}) \).

Section 6 is devoted to categorifying the picture described in the paragraph above. The convolution algebra is replaced by the monoidal category of Harish-Chandra bimodules, which comes in both an algebraic and a geometric version. The module \( H^{\text{dim } \mathcal{M}}_2(\mathcal{L}) \) is replaced by a subcategory \( \mathcal{C}^{\mathcal{L}} \subset D \)-mod (respectively \( \mathcal{C}^{\mathcal{L}_0} \subset A \)-mod) which is a module category for the category of geometric (respectively algebraic) Harish-Chandra bimodules. Following Kashiwara and Schapira [KS], we define the characteristic cycle of a geometric Harish-Chandra bimodule, which lies in \( \mathcal{Z} \), and the characteristic cycle of an object of \( \mathcal{C}^{\mathcal{L}} \), which lies in \( \mathcal{L} \). Using the machinery developed in [KS], we prove that these cycles are compatible with convolution.
Theorem C The characteristic cycle map intertwines convolution of geometric Harish-Chandra bimodules with convolution in the Borel-Moore homology of the Steinberg variety (Proposition 6.14); it also intertwines the action of Harish-Chandra bimodules on $\mathcal{C}_L$ with the action of $H^2_{\dim \mathcal{M}}(\mathcal{M} \times \mathcal{M})$ on $H^\dim \mathcal{M}(\mathcal{M})$ (Proposition 6.15).

There is particularly nice collection of algebraic Harish-Chandra bimodules which appear naturally from changing the period of the quantization. Let $A_\lambda$ be the section ring of the quantization with period $\lambda \in H^2(\mathcal{M}; \mathbb{C})$. Derived tensor products with these special bimodules give derived equivalences between the derived categories of modules over $A_\lambda$ for various different $\lambda$. These equivalences are far from being unique; instead, they induce a large group of autoequivalences of $D(A_\lambda\text{-mod})$ for each fixed $\lambda$, called twisting functors. There is a hyperplane arrangement in $H^2(\mathcal{M}; \mathbb{R})$ whose chambers are the Mori chambers of $\mathcal{M}$; let $E \subset H^2(\mathcal{M}; \mathbb{C})$ be the complement of the complexification of this arrangement. The Namikawa Weyl group $W$ acts on $H^2(\mathcal{M}; \mathbb{C})$ preserving $E$.

Theorem D There is a weak action of $\pi_1(E/W, [\lambda])$ on $D(A_\lambda\text{-mod})$ by twisting functors (Theorem 6.34); this action preserves the subcategory $D(C_0^\lambda)$ (Remark 6.36). The subgroup $\pi_1(E, \lambda)$ preserves the characteristic cycle of a module, thus $W \cong \pi_1(E/W, [\lambda])/\pi_1(E, \lambda)$ acts on $H^\dim_2(\mathcal{M})$ (Proposition 6.38). This action agrees with the action induced by the natural map from $\mathbb{C}[W]$ to the convolution algebra (Remark 6.39).

In the case where $\mathcal{M}$ is the Springer resolution for $G$, the space $E$ is the complement of the complexified Coxeter arrangement, $W$ is the classical Weyl group, and $\pi_1(E/W)$ is the generalized braid group. If $\mathbf{E} \subset \mathcal{M}$ is taken to be the conormal variety to the Schubert stratification and the period of the quantization is regular, then $C^\lambda_0$ is equivalent to a regular block of category $\mathcal{O}$ (Example 6.11). In this case, the action of the generalized braid group coincides with Arkhipov’s twisting action (Proposition 6.37), which categorifies the regular representation of $W$.

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2 Conical symplectic resolutions

Let $\mathcal{M}$ be a smooth, symplectic, complex algebraic variety. By this we mean that $\mathcal{M}$ is equipped with a closed, nondegenerate, algebraic 2-form $\omega$. Suppose further that $\mathcal{M}$ is equipped with an action of the multiplicative group $\mathbb{S} := \mathbb{C}^\times$ such that $s^* \omega = s^n \omega$ for some integer $n \geq 1$. 5
We also assume that $S$ acts on the coordinate ring $\mathbb{C}[\mathcal{M}]$ with only non-negative weights, and that the trivial weight space $\mathbb{C}[\mathcal{M}]^S$ is 1-dimensional, consisting only of the constant functions. Geometrically, this means that the affinization $\mathcal{M}_0 := \text{Spec} \mathbb{C}[\mathcal{M}]$ is a cone, and the $S$-action contracts $\mathcal{M}_0$ to the cone point $o \in \mathcal{M}_0$. Finally, we assume that the canonical map $\nu : \mathcal{M} \to \mathcal{M}_0$ is a projective resolution of singularities. (That is, it must be projective and an isomorphism over the smooth locus of $\mathcal{M}_0$.) We will refer to this collection of data as a **conical symplectic resolution of weight $n$**.

Examples of conical symplectic resolutions include the following:

- $\mathcal{M}$ is a crepant resolution of $\mathcal{M}_0 = \mathbb{C}^2/\Gamma$, where $\Gamma$ is a finite subgroup of $\text{SL}(2; \mathbb{C})$. The action of $S$ is induced by the inverse of the diagonal action on $\mathbb{C}^2$, and $n = 2$.

- $\mathcal{M}$ is the Hilbert scheme of a fixed number of points on the crepant resolution of $\mathbb{C}^2/\Gamma$, and $\mathcal{M}_0$ is the symmetric variety of unordered collections of points on the singular space. Once again, $S$ acts by the inverse diagonal action on $\mathbb{C}^2$, and $n = 2$.

- $\mathcal{M} = T^*(G/P)$ for a reductive algebraic group $G$ and a parabolic subgroup $P$, and $\mathcal{M}_0$ is the affinization of this variety. (If $G = \text{SL}(r; \mathbb{C})$, then $\mathcal{M}_0$ is isomorphic to the closure of a nilpotent orbit in the Lie algebra of $G$.) The action of $S$ is the inverse scaling action on the cotangent fibers, and $n = 1$.

- $\mathcal{M}$ is a hypertoric variety associated to a simple, unimodular, hyperplane arrangement in a rational vector space [BD00, Pro08], and $\mathcal{M}_0$ is the hypertoric variety associated to the centralization of this arrangement. If the arrangement is coloop-free, then it possible to define an $S$-action with $n = 1$ [HP04]; it is always possible to define an action with $n = 2$ [BK, BLPW12].

- $\mathcal{M}$ and $\mathcal{M}_0$ are Nakajima quiver varieties [Nak94, Nak98]. If the quiver is acyclic, then there is a natural action with $n = 1$ [Nak94 §5]; it is always possible to define an action with $n = 2$ [Nak01 §2.7].

- $\mathcal{M}_0$ is a transverse slice to one Schubert variety $\text{Gr}_\mu$ in an affine Grassmannian inside another $\text{Gr}_\lambda$. When $\lambda$ is a sum of a uminsculre coweights, this variety hasa natural conical symplectic resolution constructed from a convolution variety; in most other cases, it seems to possess no such resolution. This example is discussed in greater generality in [KWWY].

**Remark 2.1** The fifth class of examples overlaps significantly with each of the first four. The first two examples are special cases of quiver varieties, where the underlying graph of the quiver is the extended Dynkin diagram corresponding to $\Gamma$. When the group $G$ of the third example is $\text{SL}(r; \mathbb{C})$, then $T^*(G/P)$ is a quiver variety. Finally, a hypertoric variety associated to a cographical arrangement is a quiver variety.
Example 2.2 Almost all of the examples above arise as symplectic quotients of vector spaces. This applies to the first, second, fourth, and fifth classes of examples, as well as the third class when $G = \text{SL}(r; \mathbb{C})$. More precisely, let $G$ be a reductive algebraic group and $V$ a faithful linear representation of $G$. Then $G$ acts on the cotangent bundle $T^*V \cong V \times V^*$ with moment map
\[
\mu : V \times V^* \to \mathfrak{g}^*
\]
given by the formula $\mu(z, w)(x) := w(x \cdot z)$ for all $x \in \mathfrak{g}$, $z \in V$, and $w \in V^*$. Choose a character $\theta$ of $G$, and let $\mathcal{M}$ be the associated GIT quotient of $\mu^{-1}(0)$. If the center of $G$ is large enough and $\theta$ is generic, then $G$ will act locally freely on the semistable locus of $T^*V$ and $\mathcal{M}$ will be an orbifold; by [Bea00, 2.4], this orbifold will have symplectic singularities in the sense of that article.

We make the further assumption that the action is free, which implies that $\mathcal{M}$ is a manifold. The variety $\mathcal{M}$ is a conical symplectic resolution of $\mathcal{M}_0$, the categorical quotient of $\mu^{-1}(0)$ with no stability condition imposed. It inherits a conical action of $S$ of weight 2 from the inverse scaling action on $V \times V^*$. If $V$ has no $G$-invariant functions, then we may take $S$ to act only on $V^*$ and obtain a conical action of weight 1.

Remark 2.3 All of these examples admit complete hyperkähler metrics, and in fact we know of no examples that do not admit complete hyperkähler metrics. (Such examples do exist if we drop the hypothesis that $\mathcal{M}$ is projective over $\mathcal{M}_0$; these examples will appear in subsequent work by the second author and Arbo.) The unit circle in $S$ acts by hyperkähler isometries, but is Hamiltonian only with respect to the real symplectic form. Our assumptions about the $S$-weights of $C[\mathcal{M}]$ translate to the statement that the real moment map for the circle action is proper and bounded below.

Proposition 2.4 For all $i > 0$, $H^i(\mathcal{M}; \mathcal{S}_\mathcal{M}) = 0$, where $\mathcal{S}$ is the structure sheaf of $\mathcal{M}$.

Proof: This follows from the Grauert-Riemenschneider theorem; see, for example, [Kal, 2.1]. □

Proposition 2.5 All odd cohomology groups of $\mathcal{M}$ vanish, and for all non-negative integers $p$ we have $H^{2p}(\mathcal{M}; \mathbb{C}) = H^{p,p}(\mathcal{M}; \mathbb{C})$. In particular, the class of the symplectic form, which lies in $H^{2,0}(\mathcal{M}; \mathbb{C})$, is trivial.

Proof: The analogous result with $\mathcal{M}$ replaced by a fiber of $\nu$ is proven in [Kal09, 1.9], thus it suffices to prove that $\nu^{-1}(0)$ is homotopy equivalent to $\mathcal{M}$. To see this, let $p: \mathcal{M}_0 \to \mathbb{R}$ be a real algebraic function which takes non-negative values and which is $S$-equivariant for an

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2Throughout this paper we will use the symbol $\mathcal{S}$ for the structure sheaf of a variety. We avoid the usual symbol $\mathcal{O}$ because this symbol will needed for the analogue of BGG category $\mathcal{O}$ in the sequel to this paper [BLPW].
action of the form $z \cdot t = |z|^k \cdot t$ of $\mathbb{S}$ on $\mathbb{R}$, where $k$ is some positive integer. The argument from [Dur83, 1.6] shows that the inclusions $\nu^{-1}(0) \hookrightarrow (p \circ \nu)^{-1}[0, t] \hookrightarrow \mathcal{M}$ induce isomorphisms of homotopy groups, and so are homotopy equivalences.

Remark 2.6 The subvariety $\nu^{-1}(0) \subset \mathcal{M}$ is often called the **core** or **compact core**, see for example [AB02, §4] or [Pro04, §2.2]. If $\mathcal{M}$ is the cotangent bundle of a projective variety $X$, then the core of $\mathcal{M}$ is simply the zero section. If $\mathcal{M}$ is a crepant resolution of $\mathbb{C}^2/\Gamma$, then the core of $\mathcal{M}$ is a union of projective lines in the shape of the Dynkin diagram for $\Gamma$. If $\mathcal{M}$ is the Hilbert scheme of points on such a resolution, then the core of $\mathcal{M}$ consists of configurations supported on the core of the resolution. If $\mathcal{M}$ is the hypertoric variety associated to a real hyperplane arrangement, then the core of $\mathcal{M}$ is a union of toric varieties corresponding to the bounded chambers of the arrangement [BD00, 6.5].

2.1 Deformations

We next collect some results of Namikawa and Kaledin on deformations of conical symplectic resolutions. The following proposition is due to Namikawa (see Lemma 12, Proposition 13, and Lemma 22 of [Nam08]).

**Proposition 2.7 (Namikawa)** The variety $\mathcal{M}$ has a universal Poisson deformation $\pi: \mathcal{M} \to H^2(\mathcal{M}; \mathbb{C})$ which is flat. The variety $\mathcal{M}$ admits an action of $\mathbb{S}$ extending the action on $\mathcal{M} \cong \pi^{-1}(0)$, and $\pi$ is $\mathbb{S}$-equivariant with respect to the weight $-n$ action on $H^2(\mathcal{M}; \mathbb{C})$.

**Remark 2.8** A formal version of this result appears in the work of Kaledin and Verbitsky [KV02]; the work of Kaledin on twistor families contains a very similar result, but not quite in the form we need.

**Remark 2.9** One interesting consequence of Proposition 2.7 is that the inclusion of any fiber $\pi^{-1}(\lambda) \hookrightarrow \mathcal{M}$ is a homotopy equivalence, and thus we have a canonical isomorphism between the cohomology of any pair of fibers.

**Example 2.10** Suppose that $\mathcal{M}$ arises from the quotient construction of Example 2.2. Let $\chi(\mathfrak{g})$ denote the vector space of characters $\mathfrak{g} \to \mathbb{C}$, and consider the Kirwan map $K: \chi(\mathfrak{g}) \to H^2(\mathcal{M}; \mathbb{C})$ that takes an integral character to the Euler class of the induced line bundle on $\mathcal{M}$. If the Kirwan map is an isomorphism (this is known when $\mathcal{M}$ is a hypertoric variety, and conjectured in all cases), then $\mathcal{M}$ is isomorphic to the GIT quotient of $\mu^{-1}((\mathfrak{g}^*)^G)$, with the map to $H^2(\mathcal{M}; \mathbb{C}) \cong \chi(\mathfrak{g}) \cong (\mathfrak{g}^*)^G$ given by $\mu$.

Given any class $\eta \in H^2(\mathcal{M}; \mathbb{C})$, let $\mathcal{M}_\eta := \mathcal{M} \times_{H^2(\mathcal{M}; \mathbb{C})} \mathbb{A}^1$, where $\mathbb{A}^1$ maps to $H^2(\mathcal{M}; \mathbb{C})$ via the linear map that takes 1 to $\eta$. Of particular interest is the case where $\eta$ is the Euler class.
of a line bundle $\mathcal{L}$ on $\mathcal{M}$. In this case, the following result follows from the work of Kaledin [Kal06 1.4-1.6].

**Proposition 2.11 (Kaledin)** There exists a unique $\mathcal{S}$-equivariant Poisson line bundle $\mathcal{L}$ on $\mathcal{M}$ extending the bundle $\mathcal{L}$ on $\mathfrak{M}$ such that the Poisson action of the coordinate function $t \in \mathbb{C}[\mathbb{A}^1]$ on the space of sections of $\mathcal{L}$ is the identity.

**Remark 2.12** Kaledin refers to the pair $(\mathcal{M}_\eta, \mathcal{L})$ as a twistor family. The second half of the proposition can be stated more geometrically as the condition that the complement $\mathcal{L}^\times$ of the zero section in the total space of $\mathcal{L}$ (the relative spectrum of the algebra sheaf $\bigoplus_{m \in \mathbb{Z}} \mathcal{L}^m$) carries a symplectic structure coinducing the Poisson structure on $\mathcal{M}$ such that the Hamiltonian vector field $\{t, -\}$ is the infinitesimal rotation of the fibers. In particular, $\mathfrak{M}$ is the symplectic reduction of $\mathcal{L}^\times$ by this Hamiltonian vector field.

Kaledin also tells us that $\mathcal{M}_\eta$ is symplectic over $\mathbb{A}^1$, and he computes the class of the relative symplectic form as follows [Kal06 1.7].

**Proposition 2.13 (Kaledin)** The Poisson structure on $\mathcal{M}_\eta$ is nondegenerate over $\mathbb{A}^1$, and the relative symplectic form $\omega_{\mathcal{M}_\eta} \in \Omega^2(\mathcal{M}_\eta/\mathbb{A}^1)$ satisfies

$$[\omega_{\mathcal{M}_\eta}] = t \eta \in H^2_{DR}(\mathcal{M}_\eta/\mathbb{A}^1) \cong H^2(\mathfrak{M}; \mathbb{C})[t].$$

**Remark 2.14** Proposition [2.13] may be easily extended to say that $\mathcal{M}$ has a nondegenerate Poisson structure over $H^2(\mathfrak{M}; \mathbb{C})$ with relative symplectic form

$$[\omega_{\mathcal{M}}] = I \in H^2_{DR}(\mathcal{M}/H^2(\mathfrak{M}; \mathbb{C})), $$

where we identify the latter cohomology group with the space of polynomial maps from $H^2(\mathfrak{M}; \mathbb{C})$ to itself, and $I$ is the identity map.

Note that the $\mathcal{S}$-action may be used to identify all of the nonzero fibers of $\mathfrak{M}_\eta$ with a single symplectic variety $\mathcal{M}_\eta(\infty) := (\mathcal{M}_\eta \setminus \mathfrak{M}) / \mathcal{S}$. The following result of Kaledin [Kal08 2.5] will be crucial to our proof of Proposition 5.12.

**Proposition 2.15 (Kaledin)** If $\mathcal{L}$ is ample, then $\mathcal{M}_\eta(\infty)$ is affine.

### 2.2 The Weyl group

Next, we put some results of Namikawa [Nam10] into a form which is convenient for our purposes. Let $\{\Sigma_j\}$ be the codimension 2 connected components of the smooth part of the singular locus of $\mathfrak{M}_0$. At any point $\sigma_j \in \Sigma_j$, there exists a normal slice to $\Sigma_j$ at $\sigma_j$ which is isomorphic to a Kleinian singularity, thus the preimage $\nu^{-1}(\sigma_j) \subset \mathfrak{M}$ is a union of projective lines in the
shape of a simply-laced finite-type Dynkin diagram $D_j$. The monodromy representation of the fundamental group $\pi_1(\Sigma_j)$ defines an action on $D_j$ by diagram automorphisms. Let $W_j$ be the centralizer of $\pi_1(\Sigma_j)$ in the Coxeter group associated to $D_j$, and let $W := \prod W_j$. We will call $W$ the Weyl group of $\mathcal{M}$ (see Remark 2.16 for motivation). Namikawa constructs an action of $W$ on $H^2(\mathcal{M}; \mathbb{R})$; he proves that the natural restriction map

$$H^2(\mathcal{M}; \mathbb{R}) \rightarrow \bigoplus_j H^2(\nu^{-1}(\sigma_j); \mathbb{R})^{\pi_1(\Sigma_j)}$$

is $W$-equivariant and that $W$ acts trivially on the kernel $\text{[Nam10, 1.1]}$.

**Remark 2.16** Let $G$ be the reductive algebraic group associated to a simply-laced finite-type Dynkin diagram $D$, and let $B$ be a Borel subgroup. If $\mathcal{M} = T^*(G/B)$, then $\mathcal{M}_0$ is isomorphic to the nilpotent cone in $\mathfrak{g} := \text{Lie}(G)$. The singular locus of $\mathcal{M}_0$ is irreducible, and its smooth locus is called the subregular nilpotent orbit. The normal slice to the subregular orbit is isomorphic to the Kleinian singularity associated to $D$ and $W$ is isomorphic to the Weyl group of $G$. The action of $W$ on $H^2(\mathcal{M}; \mathbb{C})$ is isomorphic to the action on the dual of a Cartan subalgebra of $\mathfrak{g}$ and the restriction map (1) is an isomorphism.

Let $\mathcal{N} := \text{Spec} \mathbb{C}[\mathcal{M}]$. Then the map $\pi : \mathcal{M} \rightarrow H^2(\mathcal{M}; \mathbb{C})$ factors canonically through $\mathcal{N}$. Namikawa [Nam11, Nam10, Nam] proves that the action of $W$ on $H^2(\mathcal{M}; \mathbb{C})$ lifts to a symplectic action on $\mathcal{N}$, and that the quotient map $\mathcal{N}/W \rightarrow H^2(\mathcal{M}; \mathbb{C})/W$ is the universal Poisson deformation of the central fiber $\mathcal{M}_0$.

**Remark 2.17** The quotient $H^2(\mathcal{M}; \mathbb{C})/W$ is itself a vector space, which may be identified by a theorem of Namikawa [Nam10, 1.1] with the Poisson cohomology group $H^p(\mathcal{M}_0; \mathbb{C})$ as defined in [Nam08, §2].

### 2.3 Birational geometry

Let $P := \text{Pic}(\mathcal{M})$ be the Picard group of $\mathcal{M}$. Proposition 2.5 tells us that

$$P_{\mathbb{R}} := P \otimes_{\mathbb{Z}} \mathbb{R} \cong H^2(\mathcal{M}; \mathbb{R});$$

in particular, $P$ has finite rank. A class $\eta \in P$ is called movable if the associated line bundle is globally generated away from a codimension 2 subvariety of $\mathcal{M}$. Let $\text{Mov} \subset P_{\mathbb{R}}$ be the movable cone (the convex hull of the images of movable classes), and let $\overline{\text{Mov}}$ be its closure.

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5 This statement is equivalent to Namikawa’s statement that the map $\iota$, which he defines in the proof of his theorem, is an isomorphism.

6 It would be natural to use the notation $\mathcal{M}_0$ rather than $\mathcal{N}$, but unfortunately that notation has already been used in the previous section to mean something else.
Proposition 2.18 The cone $\text{Mov} \subset P_R$ is a fundamental domain for $W$.

Proof: Consider the restriction map (1). Since $W$ acts trivially on the kernel, any fundamental domain for the action on the target pulls back to a fundamental domain for the action on the source. The space $H^2(\nu^{-1}(\sigma_j); \mathbb{R})^{\pi_1(\Sigma_j)}$ may be identified $W_j$-equivariantly with the real part of the dual of the Cartan subalgebra of the Lie algebra determined by the Dynkin diagram $D_j$. The standard fundamental domain is the positive Weyl chamber, which may be characterized as the set of classes that are non-negative on the fundamental classes of the components of $\nu^{-1}(\sigma_i)$.

We have thus reduced the proposition to showing that a class $\eta \in P$ is movable if and only if $\eta \cdot E \geq 0$ for every curve $E \subset M$ such that $E$ is a component of $\nu^{-1}(\sigma_j)$ for some $j$. Suppose first that $\eta \cdot E < 0$ for some such curve $E$. Since $E \cong \mathbb{P}^1$, this implies that every section of the line bundle associated to $\eta$ vanishes on $E$, and therefore on the component of $\nu^{-1}(\Sigma_j)$ containing $E$. Since this component has codimension 1 in $M$, $\eta$ cannot be movable. On the other hand, suppose that $\eta \cdot E \geq 0$ for every such curve. This implies that the associated line bundle is globally generated over $\nu^{-1}(\Sigma_j)$ for every $j$. It is obviously globally generated over the preimage of the smooth locus of $M_0$, since $M_0$ is affine. It is therefore globally generated over an open set whose complement has codimension 2, thus $\eta$ is movable. □

We will wish to consider not just a single conical symplectic resolution, but rather a collection of varieties $M_1, \ldots, M_\ell$, all conical symplectic resolution of the same cone $M_0$; for any two of these, there is a birational map $f_{ij} : M_i \dasharrow M_j$, given by composing the resolution of $M_0$ by $M_i$ with the inverse of the resolution by $M_j$.

Proposition 2.19 Each $M_i$ contains an open subvariety $U_i$ with $\text{codim}(M_i \setminus U_i) \geq 2$ such that $f_{ij}$ induces an isomorphism $U_i \cong U_j$ for all $j$, and thus a canonical isomorphism between Picard groups of the different resolutions.

Proof: Since the spaces in question are symplectic and therefore Calabi-Yau, there exist open subsets $U^i_j \subset M_i$ and $U^j_i \subset M_j$ with complements of codimension $\geq 2$ such that $f_{ij}$ induces an isomorphism from $U^j_i$ to $U^i_j$; see, for example [Kaw02, 4.2]. Briefly, one can take any resolution $Q \rightarrow M_0$ (no longer symplectic!) which factors through $M_i$ and $M_j$ and pull out all irreducible components of the canonical divisor of $Q$; the remainder of $Q$ maps isomorphically to subsets of $M_i$ and $M_j$ with complements of codimension $\geq 2$; there is a canonical largest such set, so we can take $U^j_i$ to be that one. We then let $U_i := \bigcap_{j=1}^{\ell} U^j_i$. □

Note that any class $\eta \in P$ which is movable for $M_i$ is also movable for $M_i$, thus we have a well-defined movable cone $\text{Mov} \subset P_R$. The following result of Namikawa [Nam] can be roughly summarized by the statement that $M$ is a relative Mori dream space over $M_0$ [AW, 2.4].
Theorem 2.20 (Namikawa) There are finitely many isomorphism classes of conical symplectic resolutions of $\mathcal{M}_0$. Furthermore, there exists a finite collection $\mathcal{H}$ of hyperplanes in $P_\mathbb{R}$, preserved by the action of $W$, with the following properties:

- For each conical symplectic resolution $\mathcal{M}$, the ample cone of $\mathcal{M}$ is a chamber of $\mathcal{H}$ (and different resolutions have different ample cones).
- The union of the closures of these ample cones is equal to $\overline{\text{Mov}}$.
- The union $\bigcup_{H \in \mathcal{H}} H \subset P_\mathbb{C} \cong H^2(\mathcal{M}; \mathbb{C})$ is precisely equal to the locus over which the map $\mathcal{M} \to \mathcal{N}$ fails to be an isomorphism. Equivalently, it is the locus over which the fibers are not affine.

Remark 2.21 Note that, by Proposition 2.18 and Theorem 2.20, the chambers of $\mathcal{H}$ are in bijection with the set of pairs $(\mathcal{M}, w)$, where $\mathcal{M}$ is a conical symplectic resolution of $\mathcal{M}_0$ and $w$ is an element of $W$. This bijection sends the pair $(\mathcal{M}, w)$ to the $w$ translate of the ample cone of $\mathcal{M}$.

Remark 2.22 If $\mathcal{M}$ is a quotient as in Example 2.2 and the Kirwan map of Example 2.10 is an isomorphism in degree 2, then the chambers of $\mathcal{H}$ are exactly the top dimensional cones in the GIT fan in $\chi(G)_\mathbb{R} \cong P_\mathbb{R}$.

3 Quantizations

Let $\mathcal{X}$ be a Noetherian nondegenerate Poisson scheme over a base $S$, and let $\pi: \mathcal{X} \to S$ be the projection. A quantization of $\mathcal{X}$ consists of

- sheaf $\mathcal{Q}$ of flat $\pi^{-1}\mathcal{G}_S[[h]]$-algebras on $\mathcal{X}$, complete in the $h$-adic topology
- an isomorphism from $\mathcal{Q}/h\mathcal{Q}$ to the structure sheaf $\mathcal{G}_X$ of $\mathcal{X}$

satisfying the condition that, if $f$ and $g$ are functions over some open set and $\tilde{f}$ and $\tilde{g}$ are lifts to $\mathcal{Q}$, the image in $\mathcal{G}_X \cong \mathcal{Q}/h\mathcal{Q} \cong h\mathcal{Q}/h^2\mathcal{Q}$ of the element $[\tilde{f}, \tilde{g}] \in h\mathcal{Q}$ is equal to the Poisson bracket $\{f, g\}$. The cases that will be of primary interest to us are when

- $\mathcal{X} = \mathcal{M}$ and $S$ is a point
- $\mathcal{X} = \mathcal{M}_\eta$ and $S = \mathbb{A}^1$
- $\mathcal{X} = \mathcal{M}$ and $S = H^2(\mathcal{M}; \mathbb{C})$. 

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3.1 The period map

Bezrukavnikov and Kaledin [BK04, 1.8] show that for any symplectic variety $X$ smooth over $S$ with the property that $H^i(X; \mathcal{O}_X) = 0$ for $i > 0$, and thus for any conical symplectic resolution, the set of quantizations of $X$ is in natural bijection via the period map with the vector space $[\omega_X] + h \cdot H^2_{DR}(X/S; \mathbb{C})[[h]]$.

More concretely, by Propositions 2.5 and 2.13 and Remark 2.14

- the period map for $\mathcal{M}$ takes values in $h \cdot H^2(\mathcal{M}; \mathbb{C})[[h]]$
- the period map for $\mathcal{M}_\eta$ takes values in $t\eta + h \cdot H^2(\mathcal{M}; \mathbb{C})[[h]]$
- the period map for $\mathcal{M}$ takes values in $I + h \cdot (H^2(\mathcal{M}; \mathbb{C}) \otimes \mathbb{C}[H^2(\mathcal{M}; \mathbb{C})])[[h]]$.

The (unique) quantization with period $[\omega_X]$ is called the canonical quantization of $X$.

Let $\mathcal{Q}$ be a quantization of $\mathcal{M}_\eta$. There is an obvious way to recover a quantization of $\mathcal{M}$ from $\mathcal{Q}$: if we divide by the ideal sheaf of $\pi^{-1}(0)$, we obtain a sheaf supported on $\pi^{-1}(0) \cong \mathcal{M}$, and this sheaf is clearly a quantization. However, this is not the only quotient of $\mathcal{Q}$ which is supported on $\pi^{-1}(0)$. Fix an element $P(h) \in h \cdot \mathbb{C}[[h]]$. The map from $\mathbb{C}[t]$ to $\mathbb{C}[[h]]$ taking $t$ to $P(h)$ induces a map from $\Delta := \text{Spec} \mathbb{C}[[h]]$ to $\mathbb{A}^1$ sending the closed point to 0, and therefore a section $\sigma_P$ of the projection $\mathbb{A}^1 \times \Delta \to \Delta$ which sends the closed point of $\Delta$ to 0.

Dividing $\mathcal{Q}$ by the ideal sheaf in $\mathcal{M}[[[h]]$ of the image of $\sigma_P$ also gives a quantization of $\mathcal{M}$. Following Bezrukavnikov and Kaledin, we denote this quantization by $\sigma_P^* \mathcal{Q}$. Note that the first construction in this paragraph corresponds to the choice $P = 0$.

More generally, for any quantization $\mathcal{Q}$ of $X/S$, let $\sigma: \Delta \to \Delta \times S$ be any section of the projection $\Delta \times S \to \Delta$. If $\ast$ is the unique closed point of $\Delta$ and $\sigma(\ast) = (\ast, s)$, then we may define $\sigma^* \mathcal{Q}$ to be the quotient of $\mathcal{Q}$ by the ideal sheaf of this section, thought of as a sheaf on $\pi^{-1}(s)$.

Let $\mathcal{Q}_R$ be the quantization of $X$ with period given by $R(h) \in H^2_{DR}(X/S; \mathbb{C})[[h]]$. If $X = \mathcal{M}_\eta$, we can think of this as a two variable function $R(t, h) \in t\eta + h \cdot H^2(\mathcal{M}; \mathbb{C})[[t]][[h]]$.

The following proposition is an easy modification of [BK04, 6.4]; it follows immediately from the naturality of periods under pullback.

Proposition 3.1 (Bezrukavnikov and Kaledin) The period of the quantization $\sigma^* \mathcal{Q}_R$ is $\sigma^* R(h) \in H^2(\pi^{-1}(s); \mathbb{C})[[h]]$. In particular, if $X = \mathcal{M}_\eta$ and $S = \mathbb{A}^1$, then $\sigma_P^* \mathcal{Q}_R$ has period $R(P(h), h) \in h \cdot H^2(\mathcal{M}; \mathbb{C})[[h]]$.

---

Following the conventions of [BK04], we will mean here the $h$-adic completion of $H^2_{DR}(X/S; \mathbb{C}) \otimes \mathbb{C}[[h]]$. This applies whenever we use the notation $V[[h]]$ for some vector space $V$. 

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Let us collect one more fact about quantizations which will be important for us. If \( Q \) is a quantization of \( \mathcal{X}/S \), we let \( Q^{\text{op}} \) be the opposite algebra of \( Q \), thought of as a \( \mathbb{C}[[h]] \)-algebra with the action twisted by the automorphism \( h \mapsto -h \); this convention is necessary to assure that \( Q^{\text{op}} \) again quantizes the same Poisson structure.

**Proposition 3.2** If \( P(h) \in [\omega_\mathcal{X}] + h \cdot H^2_{\text{DR}}(\mathcal{X}/S; \mathbb{C})[[h]] \) is the period of \( Q \), then the period of \( Q^{\text{op}} \) is \( P(-h) \).

**Proof:** A proof of this fact is given in the proof of [Los, 2.3.2], but the result is not stated as a theorem. As defined in [BK04, 4.1], the period map is the localization of a universal class \( c \in H^2((\text{Aut } D, \text{Der } D), h\mathbb{C}[[h]]) \) in the cohomology of the Harish-Chandra pair \((\text{Aut } D, \text{Der } D)\), where \( D \) is the Weyl algebra. The existence of a particular anti-automorphism sending \( h \mapsto -h \) and \( c(h) \mapsto c(-h) \) given in [Los] shows that the period transforms the same way.

**Remark 3.3** Not every symplectic variety (in the sense of Beauville [Bea00]) admits a symplectic resolution; for example, closures of non-Richardson nilpotent orbits do not [Fu03]. On the other hand, every symplectic variety has a crepant partial resolution \( \mathcal{M} \) which is terminal and \( \mathbb{Q} \)-factorial; this is again a symplectic variety, since it is dominated by some resolution of \( \mathcal{M}_0 \). The fact that this variety is \( \mathbb{Q} \)-factorial means that it cannot be resolved further without introducing discrepancy: a crepant partial resolution of \( \mathcal{M} \) would have to be isomorphic to \( \mathcal{M} \) in codimension 1 so their group of Weil divisors would be the same; thus an ample line bundle on the resolution would have to correspond to a Weil divisor on \( \mathcal{M} \), some power of which is a Cartier divisor, showing that the resolution is in fact \( \mathcal{M} \).

While the theory of quantizations we have discussed thus far cannot be directly applied to \( \mathcal{M} \), it can be applied to the smooth locus \( \mathcal{M}_0 \). More generally, let \( \mathcal{X}/S \) be a convex symplectic variety with terminal singularities [Nam08, §1], and \( \hat{\mathcal{X}} \) its smooth locus. As noted by Namikawa in the proof of [Nam08, Lemma 12], \( H^1(\hat{\mathcal{X}}; \mathcal{O}_{\hat{\mathcal{X}}}) = H^2(\hat{\mathcal{X}}; \mathcal{O}_{\hat{\mathcal{X}}}) = 0 \), so we can apply the results of [BK04]. In the terminology of that paper, \( \hat{\mathcal{X}} \) is admissible. By [BK04, 1.8], the quantizations of \( \hat{\mathcal{X}} \) are in bijection with \( [\omega_{\hat{\mathcal{X}}}] + h \cdot H^2_{\text{DR}}(\hat{\mathcal{X}}/S; \mathbb{C})[[h]] \). Let \( i: \hat{\mathcal{X}} \to \mathcal{X} \) be the inclusion map.

**Proposition 3.4** If \( \hat{Q} \) is a quantization of \( \hat{\mathcal{X}} \), then \( i_* \hat{Q} \) is a quantization of \( \mathcal{X} \). If \( Q \) is a quantization of \( \mathcal{X} \), then \( i^{-1} Q \) is a quantization of \( \hat{\mathcal{X}} \). These two operations induce inverse bijections between isomorphism classes of quantizations of \( \hat{\mathcal{X}} \) and \( \mathcal{X} \).

**Proof:** The fact that \( i_* \hat{Q} \) is a quantization follows from normality of symplectic varieties; the fact that \( i^{-1} Q \) is a quantization is trivial, as is the isomorphism \( i^{-1} i_* \hat{Q} \cong Q \). In the other direction, the natural map \( i_* i^{-1} Q \to Q \) is an isomorphism mod \( h \), and thus is an isomorphism by Nakayama’s lemma.
In most sections of this paper (with the exception of Section 6.2), we could allow our conical symplectic resolutions to be \( \mathbb{Q} \)-terminal rather than smooth. For ease of exposition, however, we will continue to assume smoothness.

### 3.2 \( \mathbb{S} \)-structures

Suppose that \( \mathbb{S} \) acts compatibly on \( \mathcal{X} \) and \( S \), and that \( s^*\omega_\mathcal{X} = s^n\omega_\mathcal{X} \) for some positive integer \( n \).

In this section we define the notion of an \( \mathbb{S} \)-equivariant quantization of \( \mathcal{X}/S \), and we consider the question of which quantizations carry \( \mathbb{S} \)-structures.

Let \( a: \mathbb{S} \times \mathcal{X} \to \mathcal{X} \) be the action map, let \( p: \mathbb{S} \times \mathcal{X} \to \mathcal{X} \) be the projection onto \( \mathcal{X} \), and let \( e: \mathbb{S} \times \mathcal{X} \to \mathbb{S} \) be the projection onto \( \mathbb{S} \). If \( Q \) is a quantization of \((\mathcal{X}, \omega_\mathcal{X})\), then the naive pullback \( a^*Q := a^{-1}Q \otimes_{\mathbb{C}[[h]]} e^{-1}\mathcal{G}_S[[h]] \) is a quantization of \( \mathcal{X} \times \mathbb{S} \) over \( \mathbb{S} \) with the relative symplectic form \( a^*\omega_\mathcal{X} = z^n p^*\omega_\mathcal{X} \), where \( z \) is the coordinate function on \( \mathbb{S} \). Since forms are contravariant and bivectors covariant, the corresponding Poisson brackets are related by \( \{-,-\}_a = z^{-n}\{-,-\}_p \). As long as the Poisson bracket on \( \mathcal{X} \) is nontrivial, the sheaves \( a^*Q \) and \( p^*Q \) are quantizations of different Poisson brackets on \( \mathcal{X} \times \mathbb{S} \), thus they are never isomorphic.

This difference between the two Poisson brackets can be resolved by twisting the action of \( h \). More precisely, let \( a^*_{tw}Q := a^{-1}Q \otimes_{\mathbb{C}[[h]]} e^{-1}\mathcal{G}_S[[h]] \), where this time the action of \( \mathbb{C}[[h]] \) on \( e^{-1}\mathcal{G}_S[[h]] \) is given by sending \( h \) to \( z^n h \). Put differently, \( a^*_{tw}Q \) and \( a^*Q \) are isomorphic as sheaves of vector spaces, but the endomorphism given by multiplication by \( h \) in \( a^*_{tw}Q \) corresponds to the endomorphism given by multiplication by \( z^{-n}h \) in \( a^*Q \). Then \( a^*_{tw}Q \) is a quantization of the Poisson bracket \( z^n\{-,-\}_a = \{-,-\}_p \), that is, corresponding to the relative symplectic form \( p^*\omega_\mathcal{X} \).

An \( \mathbb{S} \)-structure on \( Q \) is an isomorphism \( a^*_{tw}Q \cong p^*Q \) as \( (\text{id}_\mathbb{S} \times \pi)^{-1}\mathcal{G}_{\mathbb{S} \times S}[[h]] \)-algebras, satisfying the natural cocycle condition. That is, the above isomorphism induces an isomorphism \( s^*Q \cong Q \) for every \( s \in \mathbb{S} \), and we require that for any three elements of \( \mathbb{S} \) with \( s \cdot s' \cdot s'' = 1 \), the composition of the three isomorphisms is the identity. In \[Los\], this is called a “grading” on the quantization. As a general principle, quantizations have \( \mathbb{S} \)-structures whenever their period does not obstruct this possibility. More precisely, Losev \[Los\ 2.3.3\] proves the following result.

**Proposition 3.5 (Losev)** Suppose that \( H^1(\mathcal{X}; \mathcal{G}_\mathcal{X}) = H^2(\mathcal{X}; \mathcal{G}_\mathcal{X}) = 0 \)\(^a\) A quantization of \( \mathcal{X} \) admits an \( \mathbb{S} \)-structure if and only if its period lies in the vector space \([\omega_\mathcal{X}] + h \cdot H^2_{DR}(\mathcal{X}/S; \mathcal{C}) \subset [\omega_\mathcal{X}] + h \cdot H^2_{DR}(\mathcal{X}/S; \mathcal{C})[[h]]\), in which case its \( \mathbb{S} \)-structure is unique.

\(^a\)Losev assumes that \( n = 2 \), but his proof never depends on this fact.

\(^b\)We note that this condition is satisfied by \( \mathfrak{M} \) by Propositions 2.4. It is also easy to verify for \( \mathfrak{M}_q \) and \( \mathfrak{M} \).
3.3 The section ring

As in Section 3.2 let \( X \) be a nondegenerate \( S \)-equivariant Poisson scheme over a base \( S \), where \( S \) acts on the relative symplectic form with weight \( n \), and let \( Q \) be an \( S \)-equivariant quantization. Define

\[
D(0) := Q[h^{1/n}], \quad D := Q[h^{-1/n}], \quad \text{and} \quad D(m) := h^{-m/n}D(0) \subset \mathcal{D} \text{ for all } m \in \mathbb{Z}.
\]

We will frequently abuse notation by referring to \( \mathcal{D} \) as a quantization of \( X \).

Let \( A := \Gamma_S(\mathcal{D}) \) be the ring of \( S \)-invariant sections of \( \mathcal{D} \). This ring inherits a \( \mathbb{Z} \)-filtration

\[
\ldots \subset A(-1) \subset A(0) \subset A(1) \subset \ldots \subset A
\]

given by putting

\[
A(m) := \Gamma_S(\mathcal{D}(m)).
\]

The associated graded of \( A \) may be canonically identified with \( \mathbb{C}[X] \) as a \( \mathbb{Z} \)-graded ring via the maps

\[
A(m) = \Gamma_S(\mathcal{D}(m)) \xrightarrow{h^{m/n}} \Gamma(\mathcal{D}(0)) \rightarrow \Gamma(\mathcal{D}(0)/\mathcal{D}(-1)) \cong \Gamma(S_X) = \mathbb{C}[X].
\]

Many of the examples of conical symplectic resolutions we gave at the beginning of Section 2 admit quantizations for which the ring \( A \) is of independent interest. (In all of these examples \( S \) is a point.)

- Let \( \Gamma \subset \text{SL}_2(\mathbb{C}) \) be a finite subgroup. Any quantization of the Hilbert scheme of \( m \) points on a crepant resolution of \( \mathbb{C}^2/\Gamma \) has its invariant section ring \( A \) isomorphic to a spherical symplectic reflection algebra for the wreath product \( S_m \wr \Gamma \), with parameters corresponding to the period of the quantization [EGGO07, 1.4.4], [Gor06, 1.4].

- Let \( G \) be a reductive Lie group and \( B \subset G \) a Borel subgroup. Then each quantization of \( T^*(G/B) \) has its invariant section ring \( A \) isomorphic to a central quotient of the universal enveloping algebra \( U(\mathfrak{g}) \). All central quotients arise this way, and two quantizations give the same central quotient if their periods are related by the action of the Weyl group [BB81, Lemma 3].

- Any quantization of a resolution of a Slodowy slice to a nilpotent orbit in \( \mathfrak{g} \) has its invariant section ring \( A \) isomorphic to a central quotient of a finite \( W \)-algebra. Again, all central quotients arise this way, and two quantizations give the same central quotient if their periods are related by the action of the Weyl group [Pre02, 6.4].

- Any quantization of a hypertoric variety has its invariant section ring \( A \) isomorphic to a central quotient of the hypertoric enveloping algebra. Once more, all central quotients
arise this way, and two quantizations give isomorphic central quotients if their periods are related by the action of the Weyl group [BK §5], [BLPW12 5.9].

• In [KWWY], it is conjectured that the algebra arising from the slices in the affine Grassmannian can be described as a quotient of a shifted Yangian, a variant of the usual Yangian of Drinfeld.

Consider the universal Poisson deformation \( \pi : \mathcal{M} \to H^2(\mathfrak{m}; \mathbb{C}) \) of \( \mathfrak{m} \). Let \( \mathcal{D} \) be the canonical quantization of \( \mathcal{M} \), and let \( \mathcal{A} := \Gamma_\mathbb{S}(\mathcal{D}) \) be its invariant section algebra. The \( \pi^{-1}\mathcal{O}_{H^2(\mathfrak{m}; \mathbb{C})} \)-structure on \( \mathcal{D} \) induces a map

\[
c : \mathbb{C}[H^2(\mathfrak{m}; \mathbb{C})] \to \Gamma(\mathcal{M}; \mathcal{D})
\]

which is \( \mathbb{S} \)-equivariant for the weight \( n \) action on \( \mathbb{C}[H^2(\mathfrak{m}; \mathbb{C})] \). In particular, if \( x \in H^2(\mathfrak{m}; \mathbb{C})^* \) is a linear function on \( H^2(\mathfrak{m}; \mathbb{C}) \), we have that \( h^{-1}c(x) \in \mathcal{A} \).

Let \( \lambda \in H^2(\mathfrak{m}; \mathbb{C}) \) be the period of \( \mathcal{D} \). By Proposition 3.1, \( \mathcal{D} = \sigma^*_\lambda \mathcal{D} \), and this induces a restriction map from \( \mathcal{A} \) to \( \mathcal{A} \).

**Proposition 3.6** The map from \( \mathcal{A} \) to \( \mathcal{A} \) is surjective with kernel generated by \( h^{-1}c(x) - \lambda(x) \) for all \( x \in H^2(\mathfrak{m}; \mathbb{C})^* \).

**Proof:** Let \( C_\lambda \) be the evaluation module at \( \lambda \) of \( \mathbb{C}[H^2(\mathfrak{m}; \mathbb{C})] \). The sheaf \( \mathcal{D} \) can be rewritten as the cohomology of the tensor product of \( \mathcal{D} \) with the Koszul resolution of \( C_\lambda \). Thus, the sheaf cohomology of \( \mathcal{D} \) is the hypercohomology of this complex. Filtering this complex by degrees in the Koszul resolution, we obtain the spectral sequence

\[
\text{Tor}^i_{\mathbb{C}[H^2(\mathfrak{m}; \mathbb{C})]}(H^j(\mathcal{D})^\mathbb{S}, C_\lambda) \Rightarrow H^{j-i}(\mathcal{D})^\mathbb{S}
\]

converging to the cohomology of \( \mathcal{D} \). Since \( \mathcal{D} \) has trivial higher cohomology, this spectral sequence collapses immediately, and we obtain the desired isomorphism. \( \square \)

**Lemma 3.7** Let \( \mathcal{X} \) be a smooth symplectic variety over a smooth base \( S \). Let \( i : U \hookrightarrow \mathcal{X} \) be an open inclusion, and let \( c \) be the codimension of the complement of \( U \).

• If \( c \geq 2 \), then for any quantization \( Q \) of \( \mathcal{X} \), the restriction \( i^*Q \) to \( U \) is a quantization of \( U \) with \( \Gamma(U; i^*Q) \cong \Gamma(\mathcal{X}; Q) \).

• If \( c \geq 3 \), then for any quantization \( Q' \) of \( U \), the pushforward \( i_*Q' \) is a quantization of \( \mathcal{X} \) with \( \Gamma(U; Q') \cong \Gamma(\mathcal{X}; i_*Q') \).

**Proof:** Let \( j : \mathcal{X} \setminus U \to \mathcal{X} \) be the inclusion. As usual for complementary closed and open embeddings, we have an exact triangle \( j_*j^!\mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{X} \to i_*i^*\mathcal{O}_\mathcal{X} \to j_*j^!\mathcal{O}_\mathcal{X}[1] \). The induced
long exact sequence takes the form of a short exact sequence

\[ 0 \rightarrow S \rightarrow i_* \mathcal{G}_U \rightarrow j_* R^1 j^! S \rightarrow 0 \]

along with isomorphisms \( R^k i_* \mathcal{G}_U \cong j_* R^{k+1} j^! (S_X) \) for all \( k > 0 \). The local cohomology sheaf \( R^k j^! S_X \) vanishes for all \( k < c \), so we may conclude that \( i_* \mathcal{G}_U \cong S_X \) if \( c \geq 2 \), and \( R^1 i_* \mathcal{G}_U = 0 \) if \( c \geq 3 \).

Assume that \( c \geq 2 \), and consider a quantization \( Q \) on \( X \). It is clear that \( i^* Q \) is a quantization of \( U \), so we need only show that the sections are unchanged. For each \( m \geq 0 \), the natural map \( Q/h^m Q \rightarrow i^* (Q/h^m Q) \) is an isomorphism; this follows from induction and the five-lemma applied to the diagram:

\[
\begin{array}{ccc}
& hQ/h^m Q & \\
\downarrow & & \downarrow \\
i^* (hQ/h^m Q) & \rightarrow & i^* (Q/h^m Q) \\
\downarrow & & \downarrow \\
& i^* (S_X) & \\
\end{array}
\]

Since \( U \) is open, \( i_* i^* \) commutes with projective limits, so we have an isomorphism \( i_* i^* Q \cong Q \). The isomorphism of sections of \( Q \) and \( i^* Q \) now follows by the functoriality of push-forward.

Now assume that \( c \geq 3 \), and let \( Q' \) be a quantization on \( U \). The flatness of \( i_* Q' \) is automatic, so we need only show that \( i_* Q'/i_* (hQ') \cong S_X \). The short exact sequence

\[ hQ'/h^m Q' \rightarrow Q'/h^m Q' \rightarrow \mathcal{G}_U \]

similarly shows inductively that \( R^1 i_* (Q'/h^m Q') = 0 \) for all \( m \). An argument as in [KR08, 2.12], using the Mittag-Leffler condition, shows that thus \( R^1 i_* Q' = 0 \).

Consider the long exact sequence

\[ 0 \rightarrow i_*(hQ') \rightarrow i_* Q' \rightarrow S_X \rightarrow R^1 i_* (hQ') \rightarrow \cdots \]

Since \( R^1 i_* (hQ') \cong R^1 i_* Q' = 0 \), we get an isomorphism \( i_* Q'/i_* (hQ') \cong S_X \), and so the \( \mathbb{C}[[h]] \)-module \( i_* Q' \) is a quantization.

In the case of a conical symplectic resolution, it is interesting to note that the ring \( A \) depends only on the cone \( M_0 \), and not on the choice of resolution. More precisely, let \( \mathcal{M} \) and \( \mathcal{M}' \) be two conical symplectic resolutions of the same affine cone. By Proposition 2.19, the groups \( H^2(\mathcal{M}; \mathbb{C}) \) and \( H^2(\mathcal{M}'; \mathbb{C}) \) are canonically isomorphic. Let \( \mathcal{D} \) and \( \mathcal{D}' \) be quantizations of \( \mathcal{M} \) and \( \mathcal{M}' \) with the same period, and \( \mathcal{D} \) and \( \mathcal{D}' \) the corresponding quantizations of the universal quantizations \( \mathcal{M} \) and \( \mathcal{M}' \).
Proposition 3.8 There is a canonical isomorphism between the section rings $\mathcal{A} := \Gamma_{\mathcal{S}}(\mathcal{M}; \mathcal{D})$ and $\mathcal{A}' := \Gamma_{\mathcal{S}}(\mathcal{M}'; \mathcal{D}')$.

**Proof:** We have a canonical rational map $\mathcal{M} \dashrightarrow \mathcal{M}'$. This induces an isomorphism between the fiber over a generic point in $H^2(\mathcal{M}; \mathbb{C}) \cong H^2(\mathcal{M}'; \mathbb{C})$, and gives a pair of crepant resolutions of each fiber. Thus, applying Proposition 2.19 to each fiber, we find that the exceptional locus of this map is codimension 2 in each fiber. Combining this with the fact that the generic fiber avoids the exceptional locus, we see that it has codimension 3. Let $U \subset \mathcal{M}, U' \subset \mathcal{M}'$ be the complements to the exceptional loci, so that $\mathcal{M} \dashrightarrow \mathcal{M}'$ induces an isomorphism $U \cong U'$.

Let $i: U \to \mathcal{M}$ and $i': U' \to \mathcal{M}'$ be the inclusions of these sets. By Lemma 3.7, $\mathcal{D}'' := i_* i^* \mathcal{D}$ is a quantization of $\mathcal{M}'$ with section ring $\Gamma_{\mathcal{S}}(\mathcal{M}'; \mathcal{D}'') \cong \Gamma_{\mathcal{S}}(U; \mathcal{D}) \cong \Gamma_{\mathcal{S}}(\mathcal{M}; i_* \mathcal{D}) \cong \mathcal{A}$. Since $(i')^* \mathcal{D}'' \cong i^* \mathcal{D}$ and $(i')^* \mathcal{D}'$ have the same period (by definition), the quantizations $\mathcal{D}'$ and $\mathcal{D}''$ must also have the same period and thus are isomorphic. Thus, we have that $\mathcal{A} \cong \Gamma_{\mathcal{S}}(\mathcal{M}'; \mathcal{D}'') \cong \Gamma_{\mathcal{S}}(\mathcal{M}'; \mathcal{D}') \cong \mathcal{A}'$. \[\square\]

Propositions 3.6 and 3.8 have the following corollary.

**Corollary 3.9** There is a canonical isomorphism between the section rings $A := \Gamma_{\mathcal{S}}(\mathcal{M}; \mathcal{D})$ and $A' := \Gamma_{\mathcal{S}}(\mathcal{M}'; \mathcal{D}')$.

We may now use Proposition 3.6 to show that the ring $A$ does not change when the period of $\mathcal{D}$ changes by an element of the Weyl group; this unifies the isomorphisms mentioned in three of the four examples above. For any $\lambda \in H^2(\mathcal{M}; \mathbb{C})$, let $A_\lambda$ be the invariant section algebra of the quantization with period $\lambda$.

**Proposition 3.10** For any $\lambda \in H^2(\mathcal{M}; \mathbb{C})$ and $w \in W$, we have an isomorphism $A_\lambda \cong A_{w \cdot \lambda}$. Furthermore, these isomorphisms may be chosen to be compatible with multiplication in the Weyl group.

**Proof:** As in Section 2.1, let $\mathcal{N} := \text{Spec} \mathbb{C}[\mathcal{M}]$ be the affinization of the universal deformation of $\mathcal{M}$, and let $\mathcal{M} \subset \mathcal{M}$ be the locus on which the map to $\mathcal{N}$ is a local isomorphism. Since this map is a crepant resolution of singularities, it induces an isomorphism from $\mathcal{M}$ to the smooth locus of $\mathcal{N}$. Thus, $\mathcal{M}$ inherits a $W$-action from $\mathcal{N}$ and the canonical quantization $\mathcal{D}$ of $\mathcal{M}$ restricted to $\mathcal{M}$ is also $W$-equivariant. Note that $\mathcal{A} := \Gamma_{\mathcal{S}}(\mathcal{M}; \mathcal{D})$ is isomorphic to $\Gamma_{\mathcal{S}}(\mathcal{M}; \mathcal{D})$ by Lemma 3.7, since the codimension of the complement of $\mathcal{M}$ is at least 2. Thus $\mathcal{A}$ carries a natural $W$-action. The proposition now follows from Proposition 3.6 and the $W$-equivariance of $h^{-1}c$. \[\square\]
3.4 Quantum Hamiltonian reduction

Let $G$ be a connected reductive algebraic group over $S$, and let $\mathfrak{X}$ be a smooth Noetherian symplectic variety equipped with an action of $G \times S$, such that the symplectic form has $S$-weight $n > 0$. We will assume that the action of $G$ is Hamiltonian with flat moment map $\mu : \mathfrak{X} \rightarrow \mathfrak{g}^*$, and that $\mu$ is $S$-equivariant with respect to the weight $n$ scalar action on $\mathfrak{g}^*$. A Hamiltonian $G$-action on the pair $(\mathfrak{X}, Q)$ consists of

- an action of $G$ on $\mathfrak{X}$ as above
- a $G$-structure on $Q$
- a $G$-equivariant filtered $\mathbb{C}[S]$-algebra homomorphism $\eta : U(\mathfrak{g}) \rightarrow \Gamma_S(\mathcal{D}(n)) \subset A$

such that for all $x \in \mathfrak{g}$, the adjoint action of $\eta(x)$ on $Q$ agrees with the action of $x$ induced by the $G$-structure on $Q$. The map $\eta$ is called a quantized moment map because the associated graded

$$\text{gr } \eta : \mathbb{C}[\mathfrak{g}^*] \cong \text{gr } U(\mathfrak{g}) \rightarrow \text{gr } A \cong \mathbb{C}[\mathfrak{X}]$$

induces a $G \times S$-equivariant classical moment map $\mu : \mathfrak{X} \rightarrow \mathfrak{g}^*$, where $S$ acts on $\mathfrak{g}^*$ with weight $-n$. We note that for any $x \in \mathfrak{g}$, we will have

$$\eta(x) \in \Gamma_S(h^{-1}Q) \subset \Gamma_S(\mathcal{D}(n)) = A(n) \subset A.$$

The following proposition says that the condition of admitting a quantized moment map is no stronger than the condition of admitting a classical moment map. Recall that we use $\chi(\mathfrak{g})$ to denote the vector space of characters $\mathfrak{g} \rightarrow \mathbb{C}$.

**Proposition 3.11** Suppose that the action of $G$ on $\mathfrak{X}$ admits a $G \times S$-equivariant classical moment map $\mu : \mathfrak{X} \rightarrow \mathfrak{g}^*$. Then the action on $(\mathfrak{X}, Q)$ admits a quantized moment map that induces $\mu$ in the manner described above. Furthermore, if $\mathfrak{X}$ has no non-constant functions of non-positive $S$-weight, then the set of quantized moment maps is a torsor for $\chi(\mathfrak{g})$.

**Proof:** Consider the short exact sequence of Lie algebras

$$0 \rightarrow \Gamma_S(\mathfrak{X}; Q) \rightarrow \Gamma_S(\mathfrak{X}; h^{-1}Q) \rightarrow \mathbb{C}[\mathfrak{X}]_n \rightarrow 0,$$

where $\mathbb{C}[\mathfrak{X}]_n$ is the space of functions with $S$-weight $n$. The middle term is a Lie algebra because sections of $Q$ commute modulo $h$, and the right-hand term is a Lie algebra because the Poisson bracket has weight $-n$. The moment map $\mu$ induces a map of Lie algebras $\mu^* : \mathfrak{g} \rightarrow \mathbb{C}[\mathfrak{X}]_n$, and a quantized moment map is nothing but a $G$-equivariant lift of $\mu^*$ to $\Gamma_S(\mathfrak{X}; h^{-1}Q)$.

---

10We filter $U(\mathfrak{g})$ so that the associated graded $\mathbb{C}[\mathfrak{g}^*]$ has $\mathfrak{g}$ sitting in degree $n$. 

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We may assume that the action of $G$ is effective, so that $\mathfrak{g}$ is a Lie sub-algebra of $\mathbb{C}[\mathfrak{X}]_n$. Let $\tilde{\mathfrak{g}}$ denote the preimage of $\mathfrak{g}$ in $\Gamma_S(\mathfrak{X}; h^{-1} \mathcal{Q})$; $\tilde{\mathfrak{g}}$ is a $G$-equivariant pro-unipotent extension of $\mathfrak{g}$. Since $G$ is reductive, this extension splits, and we obtain the desired lift.

If $\mathfrak{X}$ has no non-constant functions of non-positive $S$-weight, then $\tilde{\mathfrak{g}}$ is a 1-dimensional central extension of $\mathfrak{g}$, and the set of splittings is a torsor for $\chi(\mathfrak{g})$. In this case $\mathfrak{X}$ is connected, so $\mu$ is the unique $G \times S$-equivariant moment map, and all quantized moment maps induce $\mu$. □

Let $\eta: U(\mathfrak{g}) \to A$ be a quantized moment map and let $\mu: \mathfrak{X} \to \mathfrak{g}^*$ be the associated classical moment map. Let $\mathfrak{U} \subset \mathfrak{X}$ be the semistable locus for a $G$-equivariant ample line bundle $\mathcal{L}$ on $\mathfrak{X}$. We will assume that the action of $G$ on $\mathfrak{U}$ is free; in particular, semistability and stability coincide. Let $\mathfrak{Y} := \mu^{-1}(0) \cap \mathfrak{U}$, let $\mathfrak{X}_{red} := \mathfrak{Y}/G$ with its induced relative symplectic form and $S$-action, and let $\psi: \mathfrak{Y} \to \mathfrak{X}_{red}$ be the natural projection.

For any $\xi \in \chi(\mathfrak{g})$, let

$$R_\xi := \mathfrak{Q}_\mathfrak{U}/\mathfrak{Q}_\mathfrak{U} \cdot \langle h\eta(x) - h\xi(x) \mid x \in \mathfrak{g} \rangle,$$

$$E_\xi(0) := D_\mathfrak{U}(0)/D_\mathfrak{U}(-n) \cdot \langle \eta(x) - \xi(x) \mid x \in \mathfrak{g} \rangle,$$

$$E_\xi := D_\mathfrak{U}/D_\mathfrak{U} \cdot \langle \eta(x) - \xi(x) \mid x \in \mathfrak{g} \rangle.$$

These are all sheaves on $\mathfrak{U}$ with support $\mathfrak{Y}$, which we use to define sheaves of algebras on $\mathfrak{X}_{red}$ as follows:

$$\mathfrak{Q}_{red} := \psi_* \mathcal{E}nd_{\mathfrak{Q}_\mathfrak{U}}(R_\xi)^{op},$$

$$\mathfrak{D}_{red}(0) := \psi_* \mathcal{E}nd_{\mathfrak{D}(0)_{\mathfrak{U}}}(E_\xi(0))^{op},$$

$$\mathfrak{D}_{red} := \psi_* \mathcal{E}nd_{\mathfrak{D}_\mathfrak{U}}(E_\xi)^{op}.$$

Kashiwara and Rouquier [KR08, 2.8(i)] show that the first sheaf is an $S$-equivariant quantization of $\mathfrak{X}_{red}$ of weight $n$, and the second and third are related to the first in the usual way. Kashiwara and Rouquier work in the classical topology, but their argument works equally well in the Zariski topology.

**Remark 3.12** Kashiwara and Rouquier also take the fixed points of $G$. Since we have assumed that $G$ is connected, this is redundant; the pushforward is automatically invariant under $\mathfrak{g}$. Of course, a reader interested in quotients by disconnected groups can apply our results to the connected component of the identity, and then consider the residual action of the component group.

We observe that this geometric operation of symplectic reduction is closely related to an algebraic one introduced by Crawley-Boevey, Etingof, and Ginzburg [CBEG07]. Let
\( Y_\xi := A/A \cdot \langle \eta(x) - \xi(x) \mid x \in \mathfrak{g} \rangle. \)

**Proposition 3.13** If \( A = \Gamma_S(D) \) and \( A_{\text{red}} = \Gamma_S(D_{\text{red}}) \), then \( A_{\text{red}} \cong \text{End}_A(Y_\xi) \).

**Proof:** The natural map from the endomorphism ring to \( A_{\text{red}} \) has as its associated graded the natural map from \( \mathbb{C}[\mu^{-1}(0)]^G \) to the ring of functions on \( \mathfrak{X}_{\text{red}} \) \cite[2.4]{Hol99}. Since this map is an isomorphism, so is the original one. \( \square \)

Next we compute the period of \( Q_{\text{red}} \) in terms of the parameter \( \xi \); this will prove to be an important technical tool that is needed for the proofs of Proposition \ref{prop:4.3} and Lemma \ref{lem:4.14}.

For simplicity, we assume that \( \mathfrak{X} \) is symplectic over \( \text{Spec} \mathbb{C} \) (rather than over an arbitrary base), that \( Q \) is the canonical quantization of \( \mathfrak{X} \), and that the action of \( G \) on \( \mathfrak{X} \) admits a \( G \times S \)-equivariant classical moment map \( \mu: \mathfrak{X} \to \mathfrak{g}^* \). A quantized moment map \( \eta: U(\mathfrak{g}) \to A \) is called **canonical** if, when \( \xi = 0 \), \( Q_{\text{red}} \) is the canonical quantization of \( \mathfrak{X}_{\text{red}} \).

**Lemma 3.14** If \( \mathfrak{X} \) has no non-constant functions of non-positive \( S \)-weight, then it has a unique canonical quantized moment map.

**Proof:** By Proposition \ref{prop:3.11} the set of quantized moment maps is a torsor for \( \chi(\mathfrak{g}) \). Since \( Q \) is the canonical quantization, we know that \( Q \cong Q^{\text{op}} \), and any choice of such an isomorphism (that is, any algebra anti-automorphism \( \phi \) of \( Q \)) sends a quantized moment map to minus a quantized moment map. Thus, \( -\phi \) preserves the set of quantized moment maps, and is an anti-automorphism of \( \chi(\mathfrak{g}) \)-torsors, so it fixes a unique point.

Recall that

\[
Q_{\text{red}} = \psi_\ast \mathcal{E}nd_{Q_\mathfrak{u}} \left( Q_\mathfrak{u}/Q_\mathfrak{u} \cdot \langle h\eta(x) - h\xi(x) \mid x \in \mathfrak{g} \rangle \right)^{\text{op}}.
\]

Its opposite ring is obtained as the analogous reduction of the opposite ring

\[
Q_{\text{red}}^{\text{op}} \cong \psi_\ast \mathcal{E}nd_{Q_\mathfrak{u}} \left( Q_\mathfrak{u}^{\text{op}}/Q_\mathfrak{u}^{\text{op}} \cdot \langle -h\eta(x) + h\xi(x) \mid x \in \mathfrak{g} \rangle \right)^{\text{op}}.
\]

Twisting the action of \( Q \) by the action of \( \phi \), this sheaf is also isomorphic to

\[
\psi_\ast \mathcal{E}nd_{Q_\mathfrak{u}} \left( Q_\mathfrak{u}/Q_\mathfrak{u} \cdot \langle -h\phi(\eta(x)) + h\xi(x) \mid x \in \mathfrak{g} \rangle \right)^{\text{op}}.
\]

Thus, if we choose \( \eta \) to be the fixed point of \( -\phi \) and take \( \xi = 0 \), the quantization \( Q_{\text{red}} \) is isomorphic to its own opposite, and therefore to the canonical quantization. \( \square \)

The following proposition is implicit in the principal results of \cite{Los}, but does not seem to be explicitly stated in the generality that we need. Our proof is similar to the proof of \cite[5.3.1]{Los}.
Proposition 3.15 If $\eta$ is a canonical quantized moment map and $\xi \in \chi(\mathfrak{g})$ is arbitrary, then the period of $Q_{\text{red}}$ is equal to $[\omega_{\text{red}}] + hK(\xi)$, where $K: \chi(\mathfrak{g}) \to H^2(\mathfrak{x}_{\text{red}}; \mathbb{C})$ is the Kirwan map.

Proof: Consider the inclusion $\chi(\mathfrak{g}) \cong (\mathfrak{g}^*)^G \subset \mathfrak{g}^*$, and let $\mathfrak{P} := (\mathfrak{U} \cap \mu^{-1}(\chi(\mathfrak{g}))) / G$, which is equipped with a natural map $\gamma: \mathfrak{P} \to \chi(\mathfrak{g})$. Since $G$ acts freely on $\mathfrak{U}$, $\gamma$ is a submersion and $\mathfrak{P}$ is a flat deformation of $\mathfrak{x}_{\text{red}} = \gamma^{-1}(0)$, symplectic over the base $\chi(\mathfrak{g})$. The quantization

$$\hat{Q}_{\text{red}} \cong \gamma_* \mathcal{E}nd\left( \mathcal{O}_\mathfrak{U} / \mathcal{O}_\mathfrak{U} \cdot \langle h\eta(x) \mid x \in [\mathfrak{g}, \mathfrak{g}] \rangle \right)^G$$

of $\mathfrak{P}$ is self opposite, and thus canonical, so its period is equal to the class of the relative symplectic form $\omega_{\mathfrak{P}} \in \Omega^2(\mathfrak{P}/\chi(\mathfrak{g}))$. The quotient

$$Q_{\text{red}} = \hat{Q}_{\text{red}} / \hat{Q}_{\text{red}} \cdot \langle h\eta(x) - h\xi(x) \mid x \in \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \rangle,$$

which is supported on $\mathfrak{x}_{\text{red}}$, can be thought of as the pullback of $\hat{Q}_{\text{red}}$ by the map $s: \Delta \to \Delta \times \chi(\mathfrak{g})$ which is the identity on $\Delta$ and has the property that $s^*(x) = h \cdot \xi(x)$ for any element $x \in \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong \chi(\mathfrak{g})^*$. By Proposition 3.1, this quantization of $\mathfrak{x}_{\text{red}}$ has period $s^* \omega_{\mathfrak{P}}$. The usual Duistermaat-Heckman theorem implies that $s^* \omega_{\mathfrak{P}} = [\omega_{\text{red}}] + hK(\xi)$. \hfill $\Box$

4 Modules over quantizations

Let $\mathcal{Q}$ be an $\mathcal{S}$-equivariant quantization of $\mathfrak{x}$, and consider the sheaves $\mathcal{D}$ and $\mathcal{D}(m)$ defined in the beginning of Section 3.3. An $h$-adically complete module over $\mathcal{Q}$ (respectively $\mathcal{D}(0)$) is called coherent if it is locally a quotient of a sheaf which is free of finite rank. By Nakayama’s lemma, this is equivalent to the property that one obtains a coherent sheaf by setting $h$ (respectively $h^{1/n}$) to zero.

Remark 4.1 Some other sources on modules over deformation quantizations contain an a priori stronger notion of “coherent” as in defined in [KS, §1.1]. However, since $\mathfrak{x}$ (and thus $\mathcal{D}$) is Noetherian, [KS, 1.2.5] shows that this notion coincides with the one we have given above. In general, we simplify many issues around finiteness by assuming that the modules we consider are coherent. Removing this condition would complicate matters substantially.

A $\mathcal{S}$-equivariant $\mathcal{D}$-module is a $\mathcal{D}$-module equipped with an $\mathcal{S}$-structure in the sense of Section 3.2 compatible with the $\mathcal{S}$-structure on $\mathcal{D}$. More precisely, it is a $\mathcal{D}$-module $\mathcal{N}$ along with an isomorphism $a_{\text{tw}}^* \mathcal{N} \cong p^* \mathcal{N}$ satisfying the natural cocycle condition, such that the
The following diagram commutes.

\[
\begin{array}{ccc}
\mathfrak{a}^\ast \mathcal{D} \otimes \mathfrak{a}^\ast \mathcal{N} & \to & \mathfrak{a}^\ast \mathcal{N} \\
\cong & & \cong \\
p^\ast \mathcal{D} \otimes p^\ast \mathcal{N} & \to & p^\ast \mathcal{N}
\end{array}
\]

An \(\mathbb{S}\)-equivariant \(\mathcal{D}\)-module \(\mathcal{N}\) is called \textbf{good} if it admits a coherent \(\mathbb{S}\)-equivariant \(\mathcal{D}(0)\)-lattice \(\mathcal{N}(0)\). Let \(\mathcal{D}-\text{Mod}\) be the category of arbitrary \(\mathbb{S}\)-equivariant modules over \(\mathcal{D}\), and let \(\mathcal{D}-\text{mod} \subset \mathcal{D}-\text{Mod}\) be the full subcategory consisting of good modules. Note that the choice of lattice is not part of the data of an object of \(\mathcal{D}\)-mod. The reason for this is that we want an abelian category, which would fail if we worked with lattices: the quotient of a lattice by a sublattice is only a lattice after killing torsion.

We will be interested in the unbounded derived category \(D(\mathcal{D}-\text{Mod})\) along with the bounded derived category \(D^b(\mathcal{D}-\text{mod})\). Note that \(D^b(\mathcal{D}-\text{mod})\) is equivalent to the full subcategory of \(D(\mathcal{D}-\text{Mod})\) consisting of objects whose cohomology is both good and bounded.

**Remark 4.2** If \(\mathfrak{X} = \mathfrak{M}\) is a conical symplectic resolution, there are heuristic reasons to treat \(\mathcal{D}-\text{mod}\) as an algebraic version of the Fukaya category of \(\mathfrak{M}\) twisted by the B-field defined by \(e^{2\pi i \lambda} \in H^2(\mathfrak{M}; \mathbb{C}\times)\), where \(\hbar \lambda\) is the period of \(\mathcal{D}\). The firmest justification at moment lies in the physical theory of A-branes, which the Fukaya category is an attempt to formalize. Kapustin and Witten [KW07] suggest that on a hyperkähler manifold, there are objects in an enlargement of the Fukaya category which correspond not just to Lagrangian submanifolds, but higher dimensional coisotropic submanifolds. In particular, there is a object in this category supported on all of \(\mathfrak{M}\) called the \textbf{canonical coisotropic brane}. Following the prescription of Kapustin and Witten further shows that \(\mathcal{D}\) is isomorphic to the sheaf of endomorphisms of this object. Nadler and Zaslow [NZ09] prove a related result in which \(\mathfrak{M}\) is replaced by the cotangent bundle of an arbitrary real analytic manifold.

### 4.1 Cotangent bundles

Let us consider the special case of quantizations of \(\mathfrak{X} = T^\ast X\) for some smooth variety \(X\), where \(\mathbb{S}\) acts by inverse scaling of the cotangent fibers. In order to simplify our discussion, we will assume in this section that \(H^2(\mathfrak{X}) = H^{1,1}(\mathfrak{X})\). It’s worth noting that this case has been considered many times before in different contexts, but for the sake of completeness, we wish to show in detail how it fits in our schema.

A \textbf{Picard Lie algebroid} \(\mathcal{P}\) on \(X\) is an extension in the abelian category of Lie algebroids of the tangent sheaf \(T_X\), with its tautological Lie algebroid structure, by the structure sheaf \(\mathcal{O}_X\), with the trivial Lie algebroid structure. Such an extension in the category of coherent
sheaves is classified by
\[
\text{Ext}^1(\mathcal{T}_X, \mathcal{S}_X) \cong H^1(X; \mathcal{T}_X) \cong H^{1,1}(X; \mathbb{C}) \cong H^{1,1}(\mathcal{X}; \mathbb{C}) \cong H^2(\mathcal{X}; \mathbb{C}).
\]
Since we have that
\[
H^0(X; \lambda^2 \mathcal{T}_X^*) \cong H^{0,2}(X; \mathbb{C}) \cong H^{0,2}(\mathcal{X}; \mathbb{C}) = 0,
\]
there is a unique Picard Lie algebroid \( P_\lambda \) on \( X \) for each \( \lambda \in H^2(\mathcal{X}; \mathbb{C}) \).

Let \( U_\lambda \) be the universal enveloping algebra of \( P_\lambda \) modulo the ideal that identifies the constant function \( 1 \in \mathcal{S}_X \) with the unit of the algebra. If \( \lambda \) is the image of the Euler class of a line bundle \( \mathcal{L} \) on \( X \), then \( U_\lambda \) is isomorphic to the sheaf of differential operators on \( \mathcal{L} \). More generally, \( U_\lambda \) is referred to as the sheaf of \( \lambda \)-twisted differential operators on \( X \). A coherent sheaf of \( U_\lambda \)-modules is called a \( \lambda \)-twisted D-module on \( X \). The sheaf \( U_\lambda \) has an order filtration, and any coherent sheaf of \( U_\lambda \)-modules admits a compatible filtration.

However, \( U_\lambda \) is a sheaf on \( X \), and we wish to find one on \( T^*X \). This requires the technique of \textit{microlocalization} (see, for example, [Kas03, AVdBVO89] for more detailed discussion of this technique). The associated graded of \( U_\lambda \) with respect to the order filtration is isomorphic to \( \text{Sym}_{\mathcal{S}_X} \mathcal{T}_X \); put differently, if
\[
R_\lambda := \left\{ \sum u_i h^i \in U_\lambda[h] \mid u_i \text{ has order } \leq i \right\}
\]
is the Rees algebra of the order filtration on \( \lambda \), then \( R_\lambda/hR_\lambda \cong \text{Sym}_{\mathcal{S}_X} \mathcal{T}_X \). Given an open subset \( U \subset T^*X \), we obtain a multiplicative subset \( S_U \subset \text{Sym}_{\mathcal{S}_X} \mathcal{T}_X(\pi(U)) \) consisting of functions on \( \pi^{-1}(\pi(U)) \) which are invertible on \( U \).

We can give a non-commutative version of this construction using an associated multiplicative system in \( R_\lambda(\pi(U)) \). Let
\[
S'_U = \{ r \in R_\lambda(\pi(U)) \mid \exists m \text{ such that } r \in h^m R_\lambda \text{ with } \overline{h^{-m}r} \in S_U \}.
\]
This is a multiplicative system because \( \text{Sym}_{\mathcal{S}_X} \mathcal{T}_X \) is a sheaf of domains. Furthermore, since \([r, R_\lambda] \subset hR_\lambda\), the operation of bracket with any algebra element is topologically nilpotent (the successive powers converge to 0 in the \( h \)-adic topology). Thus, in any quotient \( R_\lambda/h^m R_\lambda \), the reduction of this set \( S'_U \) satisfies the Ore condition, and we can define the localization of \( R_\lambda \) by \( S'_U \) as the inverse limit \( R(\lambda)(U) := \varprojlim (R_\lambda/h^m R_\lambda)_{S'_U} \). This defines an \( \mathcal{S} \)-equivariant sheaf of rings \( R_\lambda \) on \( \text{Spec}(\text{Sym}_{\mathcal{S}_X} \mathcal{T}_X) \cong T^*X = \mathcal{X} \), which is free over \( \mathbb{C}[[h]] \) and satisfies \( R/hR \cong \mathcal{S}_{T^*X} \), and is therefore a quantization of \( \mathcal{X} \).

**Proposition 4.3** The period of \( R_\lambda \) is \( h(\lambda - \varpi/2) \), where \( \varpi = c_1(T^*X) \in H^2(X; \mathbb{C}) \cong H^2(\mathcal{X}; \mathbb{C}) \) is the canonical class.
**Proof:** We begin by choosing line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_k$ on $X$ and complex numbers $\zeta_1, \ldots, \zeta_k$ such that $\lambda = \sum_{i=1}^k \zeta_i c_1(\mathcal{L}_i)$. Let $Y$ be the total space of $\oplus \mathcal{L}_i$ and let $T := (\mathbb{C}^*)^k$ act on $Y$ by scaling the fibers of the individual lines. Let $S$ act on $T^*Y$ via the inverse scaling action on the fibers, and let $\tilde{R}$ be the $T \times S$-equivariant quantization of $T^*Y$ obtained by microlocalizing the sheaf of (untwisted) differential operators on $Y$. The action of $T$ on $(T^*Y, \tilde{R})$ admits a quantized moment map

$$\varphi: U(t) \rightarrow \Gamma_S(\tilde{R}) \cong \Gamma(Y, D_Y)$$

given on $t$ by the equation

$$\varphi(a_1, \ldots, a_k) = \sum_{i=1}^k a_i t_i \frac{\partial}{\partial t_i},$$

where $t_i$ is any coordinate on the fiber of $\mathcal{L}_i$ (the operator $t_i \frac{\partial}{\partial t_i}$ is independent of this choice). If we take $\zeta := (\zeta_1, \ldots, \zeta_k) \in \mathbb{C}^k \cong \chi(t)$, then symplectic reduction of $(T^*Y, \tilde{R})$ at the parameter $\zeta$ yields the pair $(X, R_\lambda)$, as noted by Beilinson and Bernstein in [BB93, §2.5].

First, consider the special case where $k = 1$ and $\mathcal{L}_1 = \omega_X^{-1}$, the anti-canonical bundle of $X$. Then $Y$ is Calabi-Yau, $\tilde{R}$ is the canonical quantization, and thus we can apply Proposition 3.15. In order to do this, we must find a quantized moment map with self-opposite reduction. By [BB93, §2.5], the reduction by $\varphi$ at the parameter $\xi \in \mathbb{C} \cong \chi(t)$ is isomorphic to the sheaf of differential operators on $X$ twisted by $-\xi \omega \in H^2(X; \mathbb{C})$, and this sheaf is self-opposite when $\xi = -1/2$. This implies that

$$\eta(a) := a \left( 1 \frac{\partial}{\partial t_1} + \frac{1}{2} \right)$$

is a canonical quantized moment map. By Proposition 3.15, the reduction by $\eta$ at the parameter $\xi$ has period equal to $-\hbar \xi \omega$, and is isomorphic to the sheaf of differential operators twisted by $(-\xi + 1/2) \omega$, confirming the result for multiples of the canonical class.

Now, assume that $\mathcal{L}_1 = \omega_X^{-1}$, which we can always arrange. If $\sigma: T^*Y \rightarrow T^*Y/G$ is the projection, then $\sigma_* \tilde{R}^T$ is an $S$-equivariant quantization of the relative Poisson scheme $T^*Y/G \rightarrow t^*$, and thus has period $[\omega_{T^*Y/G}] + \hbar \epsilon$ for some $\epsilon \in H^2(X; \mathbb{C})$. If $s: \Delta \rightarrow \Delta \times t^*$ is the section corresponding to $\zeta_1 = -1/2$ and $\zeta_i = 0$ for $i > 1$, then we arrive at the conclusion that $s_* \sigma_* \tilde{R}^T \cong R_{-\omega/2}$, which already know has period 0. Thus, we must have $s^*([\omega_{T^*Y/G}] + \hbar \epsilon) = h(\omega/2 + \epsilon) = 0$, so $\epsilon = -\omega/2$.

For arbitrary $\zeta$, we have a section $s_\zeta: \Delta \rightarrow \Delta \times t^*$, and

$$s_\zeta^*([\omega_{T^*Y/G}] - h\omega/2) = h \left( \sum_{i=1}^n \zeta_i c_1(\mathcal{L}_i) - \omega/2 \right) = h(\lambda - \omega/2).$$

Thus $R_\lambda \cong D_{\lambda - \omega/2}$ as desired. \qed
There is a natural $\mathbb{S}$-equivariant map $p^{-1}U_\lambda \to \mathcal{R}_\lambda$, where $p : \mathfrak{X} \to X$ is the projection and $\mathbb{S}$ acts trivially on $p^{-1}U_\lambda$. For any $\lambda$-twisted D-module $\mathcal{N}$ on $X$, the microlocalization of $\mathcal{N}$ is defined to be the $\mathcal{R}_\lambda$-module $\mathcal{R}_\lambda \otimes_{p^{-1}U_\lambda} \mathcal{N}$. Proposition 4.4, which is well-known to the experts, may be regarded as a non-commutative version of the equivalence between coherent sheaves on $\mathfrak{X}$ and sheaves of coherent $\text{Sym}_{\mathbb{S}_X} T_X$-modules on $X$.

**Proposition 4.4** Microlocalization defines an equivalence of categories from the category of finitely generated $\lambda$-twisted D-modules on $X$ to $\mathcal{R}_\lambda$-mod.

**Proof:** The adjoint equivalence is $\mathcal{J} \mapsto (p_* \mathcal{J})^\mathbb{S}$; we need only check this on the algebras themselves. It is clear that the microlocalization of $U_\lambda$ is $\mathcal{R}_\lambda$. On the other hand, we have a map $U_\lambda \to (p_* \mathcal{R}_\lambda)^\mathbb{S}$ which is injective, and whose surjectivity is easily verified by passing to the associated graded. \qed

**Remark 4.5** While the cotangent bundles of smooth projective varieties provide a large supply of conical symplectic varieties, these varieties very rarely are conical symplectic resolutions. In general they do not have enough global functions to be resolutions of their affinizations. For example, consider the case of a curve:

- If $X = \mathbb{P}^1$, $T^*X$ is a resolution of a singular quadric.
- If $X$ is elliptic, $T^*X \cong X \times \mathbb{A}^1$, so the affinization of $T^*X$ is isomorphic to $\mathbb{A}^1$.
- If $X$ has genus greater than 1, then $T^*X$ is a line bundle of positive degree, and thus has no nonconstant global functions.

**Example 4.6** One class of projective varieties whose cotangent bundles are conical symplectic resolutions are varieties of the form $X = G/P$, where $G$ is a reductive algebraic group and $P \subset G$ is a parabolic subgroup. Philosophically, the reason is that $X$ has a lot of vector fields (induced by the action of $\mathfrak{g}$), therefore its cotangent bundle has a lot of functions. It is conjectured (see for example [Kal09, 1.3]) that these are the only such projective varieties.

If $P$ is a Borel subgroup, then $T^*X$ is the Springer resolution of the nilpotent cone in $\mathfrak{g}$. More generally, the moment map $\mu : T^*X \to \mathfrak{g}^* \cong \mathfrak{g}$ is always generically finite, and its image is the closure $\bar{O}_P = G \cdot \mathfrak{p}^\perp$ of the Richardson orbit $O_P$ associated with $P$. If $G = \text{SL}(r; \mathbb{C})$, or if $O_P$ is simply-connected, then $\mu$ is generically one to one, and $T^*X$ is a symplectic resolution of $\bar{O}_P$ [Hes78, 1.3]. In other cases, it is still a symplectic resolution of its affinization, but this affinization may be a finite cover of $\bar{O}_P$. 

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4.2 Localization

In this section we fix a quantization $\mathcal{D}$ of a conical symplectic resolution $\mathcal{M}$, and we let $A := \Gamma_S(\mathcal{D})$ be its section algebra. Let $A\text{-Mod}$ be the category of arbitrary $A$-modules, and let $A\text{-mod}$ be the full subcategory of finitely generated modules. As in the case of $\mathcal{D}$-modules, we will be interested in the unbounded derived category $\mathcal{D}(A\text{-Mod})$ and the bounded derived category $\mathcal{D}^b(A\text{-mod})$, which is equivalent to the full subcategory of $D(A\text{-Mod})$ consisting of objects whose cohomology is both bounded and finitely generated.

We have a functor

$$\Gamma_S: \mathcal{D}\text{-mod} \to A\text{-mod}$$

given by taking $S$-invariant global sections. The left adjoint functor

$$\text{Loc}: A\text{-mod} \to \mathcal{D}\text{-mod}$$

is defined by putting $\text{Loc}(N) := D \otimes_A N$, with the $S$-action induced from the action on $\mathcal{D}$. To see that $\text{Loc}(N)$ is indeed an object of $\mathcal{D}$-mod, let $Q \subset N$ be a finite generating set and define a filtration of $N$ by putting $N(m) := A(m) \cdot Q$. We define the Rees algebra $R(A)$ to be the $h$-adic completion of

$$A(0)[[h^{1/n}]] + h^{1/n} A(1)[[h^{1/n}]] + h^{2/n} A(2)[[h^{1/n}]] + \ldots \subset A[[h^{1/n}]]$$

and the Rees module $R(N)$ to be the $h$-adic completion of

$$N(0)[[h^{1/n}]] + h^{1/n} N(1)[[h^{1/n}]] + h^{2/n} N(2)[[h^{1/n}]] + \ldots \subset N[[h^{1/n}]].$$

Note that $R(N)$ is a module over $R(A) \cong \Gamma(\mathcal{D}(0))$, and $\mathcal{D}(0) \otimes_{R(A)} R(N)$ is a coherent lattice in $\text{Loc}(N)$.

**Remark 4.7** If $N$ is an object of $A$-mod, we have shown that $\text{Loc}(N)$ always admits a coherent lattice, but the construction of that lattice depends on a choice of filtration of $N$. Conversely, any coherent lattice $N(0)$ for an object $N$ of $\mathcal{D}$-mod induces a filtration of $N := \Gamma_S(N)$ by putting $N(m) := \Gamma_S(h^{-m/n}N(0)[[h^{1/n}]]).

If $\Gamma_S$ and $\text{Loc}$ are biadjoint equivalences of categories, we will say that localization holds for $\mathcal{D}$ or that localization holds at $\lambda$, where $[\omega_X] + h\lambda$ is the period of $\mathcal{D}$. Localization is known to hold for certain parameters in many special cases, including quantizations of the Hilbert scheme of points in the plane [KR08 4.9], the cotangent bundle of $G/P$ [BB81], resolutions of Slodowy slices [Gin 3.3.6] & [DK 7.4], and hypertoric varieties [BK 5.8]. We conjecture that every conical symplectic resolution admits many quantizations for which localization holds.
Conjecture 4.8 Let \( \Lambda \subset H^2(\mathcal{M}; \mathbb{C}) \) be the set of periods of quantizations for which localization holds. There exists

- a finite list of effective classes \( x_1, \ldots, x_r \in H_2(\mathcal{M}; \mathbb{Z}) \)
- a finite list of rational numbers \( a_i \in \mathbb{Q} \)

such that \( \Lambda = H^2(\mathcal{M}; \mathbb{C}) \setminus \bigcup_{i=1}^{r} D_i \), where

\[
D_i := \{ \lambda \in H^2(\mathcal{M}; \mathbb{C}) \mid \langle x_i, \lambda \rangle - a_i \in \mathbb{Z} \leq 0 \}.
\]

Remark 4.9 The classes \( x_1, \ldots, x_r \) should exactly correspond to the effective curve classes in “generic non-affine deformations” of \( \mathcal{M} \) in the sense of [BMO 1.15]. These classes play an important role in the formula for quantum cohomology of the Springer resolution [BMO 1.1], and conjecturally of any conical symplectic resolution.

Though we cannot prove Conjecture 4.8, we will establish asymptotic results both in the derived (Theorem 4.16) and abelian (Corollary 5.13) settings.

4.3 Derived localization

We next wish to consider the derived functors \( \mathbb{R} \Gamma_S \) and \( \mathbb{L} \text{Loc} \) relating the triangulated categories \( D(A\text{-Mod}) \) and \( D(D\text{-Mod}) \). Note that these derived functors are well-defined by [Spa88 Th. A]. If they are biadjoint equivalences, we say that derived localization holds for \( D \) or that derived localization holds at \( \lambda \), where \([\omega \chi] + h\lambda \) is the period of \( D \).

Lemma 4.10 For any good \( \mathcal{S} \)-equivariant module \( \mathcal{N} \), the map

\[
\mathbb{R}^k \Gamma(\mathcal{M}; \mathcal{N}(0)) \to \lim_{\to} \mathbb{R}^k \Gamma(\mathcal{M}; \mathcal{N}(0)/\mathcal{N}(\text{nm}))
\]

is an isomorphism for all \( k \).

Proof: Throughout, let \( G^k(m) := \mathbb{R}^k \Gamma(\mathcal{M}; \mathcal{N}(\text{nm})) \) and \( G^k(m|p) := \mathbb{R}^k \Gamma(\mathcal{M}; \mathcal{N}(\text{nm})/\mathcal{N}(\text{np})) \) for \( p \geq m \). The cohomology \( G^k(0) \) is a finitely generated \( \mathcal{S} \)-equivariant \( R(A) \)-module, since we have an injective map \( G^k(0)/hG^k(0) \to G^k(0|-1) = \mathbb{R}^k \Gamma(\mathcal{M}; \mathcal{N}(0)/\mathcal{N}(\text{nm})) \). The latter is the cohomology of a coherent sheaf, and thus finitely generated over \( \mathbb{C}[\mathcal{M}] \). Nakayama’s lemma thus shows that \( G^k(0) \) is finitely generated as well.

Thus, \( G^k(0) \) is a quotient of a finite rank free module \( R(A)^{\oplus n} \) by a submodule \( K \). Consider the short exact sequence of projective systems

\[
0 \to K/(K \cap h^m R(A)^{\oplus n}) \to (R(A)/h^m R(A))^{\oplus n} \to G^k(0)/h^m G^k(0) \to 0.
\]
Since the kernel satisfies Mittag-Leffler, we obtain an isomorphism
\[ G^k(0) \cong R(A)^{\oplus n}/K \cong (\varprojlim (R(A)/h^m R(A))^{\oplus n})/(\varprojlim K/(K \cap h^m R(A)^{\oplus n})) \cong \varprojlim G^k(0)/h^m G^k(0). \]

Note that the long exact sequence associated to the short exact sequence of projective systems
\[ h^m G^k(0) \to G^k(0) \to G^k(0)/h^m G^k(0) \]
进一步表明，第一个导数的functor of \( h^m \) vanishes on the left hand system:
\[ \varprojlim h^m G^k(0) \cong (\varprojlim G^k(0)/h^m G^k(0))/G^k(0) = 0. \]

All higher derived functors vanish, since this holds for any projective system over \( \mathbb{Z}_{\geq 0} \) in the category of modules over a ring. Now, we consider the long exact sequence
\[ \cdots \to G^{k-1}(0|m) \to G^k(m) \to G^k(0) \to G^k(0|m) \to G^{k+1}(m) \to \cdots. \tag{3} \]

This breaks into a series of short exact sequences
\[ 0 \to \text{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) \to G^k(m) \to G^k(0) \to G^k(0)/h^m G^k(0) \to 0. \]

The submodule of all \( h \)-torion elements in \( G^k(0) \) is finitely generated, so it is killed by \( h^M \) for some \( M \). For \( m > M \), the group \( \text{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) \) stabilizes, and the induced map in the projective system is multiplication by \( h \). This projective system satisfies the property that the image of \( \text{Tor}^1(\mathbb{C}[h]/(h^{m+M}), G^k(M + m)) \) in \( \text{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) \) is trivial, so the projective system has \( \varprojlim \text{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) \cong \varprojlim \text{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) \) = 0 by Mittag-Leffler again. The short exact sequence
\[ 0 \to \text{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) \to G^k(m) \to h^m G^k(0) \to 0 \]
shows that \( \varprojlim G^k(m) = \varprojlim G^k(m) = 0 \) as well.

Since \( G^k(0|m) \) is the extension of two projective systems with higher derived limits vanishing, the higher projective limits of \( G^k(0|m) \) vanish as well. The long exact sequence (3) thus remains exact when we take the projective limit, since the higher derived functors of all its terms vanish. Therefore, we obtain the desired isomorphism \( G^k(0) \cong \varprojlim G^k(0|m) \). \qed

**Proposition 4.11** The functor \( \mathbb{R}\Gamma_S \) induces a functor \( D^b(\mathcal{D}\text{-mod}) \to D^b(\mathcal{A}\text{-mod}) \).

**Proof:** Since any complex of \( A \)-modules with cohomology that is finitely generated and bounded is quasi-isomorphic to a bounded complex, we need only prove that \( \mathbb{R}\Gamma_S \) applied to any good \( \mathcal{D} \)-module \( \mathcal{N} \) has finitely generated cohomology in finitely many degrees. By Lemma 4.10, we need only check this for \( G^k(0|m) \) (using the notation of the lemma). Since
\( \mathcal{N}(0)/\mathcal{N}(-nm) \) is just an iterated extension of \( \mathcal{N}(0)/\mathcal{N}(-1) \) it suffices to show the same for \( H^i(M; \mathcal{N}(0)/\mathcal{N}(-1)) \). Since \( \mathcal{M} \) is projective over \( \mathcal{M}_0 \), this group is finitely generated over \( \mathbb{C}[\mathcal{M}] \) and can only be non-zero if \( 0 \leq i \leq \dim \mathcal{M} \).

Remark 4.12 By work of McGerty and Nevins [MNa, 1.1], the functor \( \mathbb{L} \text{Loc} \) will only be bounded if it is a derived equivalence; more generally, it may be defined on the bounded above derived category.

The following result of Kaledin [Kal08, §3.1] gives a sufficient condition for derived localization to hold. Let \( D^\text{op} \) be the opposite ring of \( D \), and let \( A^\text{op} \) be its section algebra. Consider the sheaf of algebras \( D \boxtimes_{\mathbb{C}((h))} D^\text{op} \) on \( X \times X \), which has section algebra \( A \otimes A^\text{op} \). Let \( D_\text{diag} \) be the \( D \boxtimes_{\mathbb{C}((h))} D^\text{op} \)-module obtained by pushing \( D \) forward along the diagonal inclusion from \( X \) to \( X \times X \), and let \( A_\text{diag} \) be the algebra \( A \), regarded as a module over \( A \otimes A^\text{op} \).

Theorem 4.13 (Kaledin) Suppose that the higher cohomology groups of \( D \) vanish. Then derived localization holds if and only if the natural map \( \mathbb{L} \text{Loc}(A_\text{diag}) \to D_\text{diag} \) is a quasi-isomorphism.

We will use Kaledin’s result to prove Theorem A from the introduction. To do this, we first need to establish a technical result. Let \( D \) be any \( S \)-equivariant quantization of \( \mathcal{M}_\eta \).

Lemma 4.14 Assume that \( \mathcal{N} \) is a \( D \)-module supported on a Lagrangian subvariety of \( \mathcal{M} \subset \mathcal{M}_\eta \) for \( \eta \in H^2(\mathcal{M}; \mathbb{Z}) \). Let \( t \) be the coordinate on \( A^1 \). The action of \( h^{-1}t \in \mathcal{A} \) on \( \mathcal{N} \) satisfies a finite degree, complex coefficient polynomial equation \( p(h^{-1}t) = 0 \).

Proof: Let \( \mathcal{L} \) be the twistor line bundle on \( \mathcal{M}_\eta \), i.e. the line bundle satisfying the statement of Proposition 2.11 and let \( u: \text{Tot}(\mathcal{L}^\times) \to \mathcal{M}_\eta \) be the projection. Then the total space \( \text{Tot}(\mathcal{L}^\times) \) is symplectic, and the fiberwise \( \mathbb{C}^* \)-action is Hamiltonian with moment map \( t \), where \( t \) is the coordinate on \( A^1 \), and the map \( u \) coincides the Poisson structure on \( \mathcal{M}_\eta \).

Since the quantization \( D \) is \( S \)-equivariant, its period will be of the form \( [\omega_{\mathcal{M}_\eta}] + h\lambda = t\eta + h\lambda \) for some \( \lambda \in H^2(\mathcal{M}_\eta; \mathbb{C}) \). Let \( \mathcal{W} \) be the quantization of \( \text{Tot}(\mathcal{L}^\times) \) with period \( u^*\lambda \). As noted by Bezrukavnikov and Kaledin [BK04, 6.2], the algebra \( \mathcal{W} \) carries a \( \mathbb{C}^* \)-equivariant structure for the fiberwise action, commuting with the \( S \)-equivariant structure. By Proposition 3.11 \( \mathcal{W} \) has a quantized moment map for the \( \mathbb{C}^* \)-action; choose one, and let \( \tau \in \Gamma_\mathcal{S}(\mathcal{W}[h^{-1/\epsilon}]) \) be the image of the generator \( y \) of \( \text{Lie}(\mathbb{C}^*) \). By the definition of a non-commutative moment map, \( y - \tau \) commutes with the action of any \( \mathbb{C}^* \)-invariant section of \( \mathcal{W} \) on any \( \mathbb{C}^* \)-equivariant module over this algebra.

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11By Proposition 2.4, this condition is satisfied by any conical symplectic resolution.

12Kaledin’s version of this theorem uses bounded above derived categories, but the proof is the same in either case.
As noted in the proof of Proposition \ref{prop:twistor}, the invariant pushforward \((u_*\mathcal{W}|_{h^{-1/n}})^{C^*}\) is the quantization of \(\mathcal{M}_\eta\) with period \(h\lambda + t\eta\), and is therefore isomorphic to our given quantization \(\mathcal{D}\). Thus, we have an equivalence between good \(\mathcal{S}\)-equivariant \(\mathcal{D}\)-modules and \(\mathcal{S} \times C^*\)-equivariant \(\mathcal{W}|_{h^{-1/n}}\)-modules, induced by the adjoint functors \(u^*\) and \(u_*^{C^*}\).

Recall that we are given a \(\mathcal{D}\)-module \(N\) on \(\mathcal{M}_\eta\) supported on a Lagrangian subvariety of \(\mathcal{M}\). Thus, \(u^*N\) is supported on the preimage of \(\text{Supp}(N)\) which is Lagrangian (so this sheaf is holonomic). By a finiteness theorem of Kashiwara and Schapira \cite[7.1.10]{KS}, the self \(\text{Ext}\)-sheaf of \((u^*N)^{an}\) is perverse, and in particular, its endomorphism algebra commuting with \(\mathcal{S}\) is finite dimensional over \(C\). By Theorem \ref{thm:main}, the same holds for \(u^*N\). As with any element of any finite dimensional algebra over \(C\), the endomorphism \(y - \tau\) has a minimal polynomial with complex coefficients such that \(p(y - \tau) = 0\). Since the structure map \(\pi^{-1}\mathcal{G}_k^1 \to \mathcal{D}\) is given by \(t \mapsto h\tau\), we thus have that the action of \(\tau = h^{-1}t\) on the reduction \(N = (u_*u^*N)^{C^*}\) satisfies the same polynomial equation. \hfill \square

**Remark 4.15** Lemma \ref{lem:invariant} almost certainly holds for general \(\eta \in H^2(\mathcal{M}; \mathbb{C})\) rather than just classes in the image of integral cohomology; however the proof uses the line bundle \(L\) in a very strong way. Proving the general case will require understanding the theory of twistor deformations of line bundles over gerbes.

**Theorem 4.16** Fix a class \(\eta \in H^2(\mathcal{M}; \mathbb{Z})\) such that \(\mathcal{M}_\eta(\infty)\) is affine\footnote{See Proposition \ref{prop:affine} and the preceding paragraph for a discussion of \(\mathcal{M}_\eta(\infty)\).} Derived localization holds at \(\lambda + k\eta\) for all but finitely many \(k \in \mathbb{C}\).

**Proof:** Let \(\mathcal{D}_k\) be the quantization with period \(h(\lambda + k\eta)\). By Theorem \ref{thm:quantization} we need to show that the map \(\text{LLoc}((A_k)_{\text{diag}}) \to (\mathcal{D}_k)_{\text{diag}}\) is an isomorphism for all but finitely many \(k\); let \(\mathcal{P}_k\) denote the cone of this map. Let \(\mathcal{D}\) be the quantization of \(\mathcal{M}_\eta\) with period \(t\eta + h\lambda\), and let \(\sigma_k: \Delta \to \mathbb{A}^1 \times \Delta\) be the map associated to the polynomial \(hk\) as in Section 3.1 Proposition \ref{prop:cone}. It tells us that \(\sigma_k^*\mathcal{D} \cong \mathcal{D}_k\), which implies that the morphism \(\text{LLoc}((A_k)_{\text{diag}}) \to (\mathcal{D}_k)_{\text{diag}}\) on \(\mathcal{M} \times \mathcal{M}\) is the pullback of the morphism \(\phi: \text{LLoc}(\mathcal{A}_{\text{diag}}) \to \mathcal{D}_{\text{diag}}\) on \(\mathcal{M}_\eta \times_{\mathbb{A}^1} \mathcal{M}_{\eta}\). It follows that \(\mathcal{P}_k \cong \sigma_k^*\mathcal{P}\) where \(\mathcal{P}\) is the cone of \(\phi\).

We now apply Lemma \ref{lem:invariant} to the symplectic resolution \(\mathcal{M} \times \mathcal{M}\), with cohomology class \((\eta, -\eta)\); the associated twistor deformation is \(\mathcal{M}_\eta \times_{\mathbb{A}^1} \mathcal{M}_{-\eta}\). The sheaf we will apply it to is \(\mathcal{H}^j(\mathcal{P})\). This is supported on the preimage of \(0 \in \mathbb{A}^1\), since all fibers over non-zero points of \(\mathbb{A}^1\) are affine varieties, where obviously the map of interest is an isomorphism. In fact, it’s supported on the the Steinberg variety \(\mathcal{M} \times_{\mathcal{M}_0} \mathcal{M} \subset \mathcal{M} \times \mathcal{M}\) which is Lagrangian, since any symplectic resolution is semi-small.

Thus the hypotheses of Lemma \ref{lem:invariant} are satisfied, so if \(k\) is not a root of the polynomial \(p, h^{-1}t - k\) acts invertibly on \(\mathcal{H}^j(\mathcal{P})\), so it has trivial specialization at \(k\), and we have
\[ \sigma_k^* \mathcal{H}^j(\mathcal{P}) = 0. \] Thus, for any integer \( m \), we can find a polynomial (the product of those for each individual homological degree) where \( \mathcal{H}^j(\mathcal{P}_k) \) is trivial for \( j \geq -m \).

By [Kal08, 3.3], \( \mathcal{P}_k \) is trivial if and only if it has trivial homology in degrees above \( -\ell \) where \( \ell \) is the global dimension of \( \mathcal{D}_k \mathcal{C}(\mathcal{h}) \mathcal{D}_k^{op} \)-mod (which is finite since the same is true for \( \mathcal{G}_{\mathfrak{M} \times \mathfrak{M}} \)). By the argument above, this happens at all \( k \) other than the roots of a polynomial with complex coefficients, and thus for all but finitely many \( k \).

McGerty and Nevins [MNa, §7.5] have also studied the question of when derived localization holds, obtaining the following results.

**Theorem 4.17 (McGerty and Nevins)**

(i) If \( A \) has finite global dimension, then derived localization holds; the converse of this result is clear, since the sheaf \( \mathcal{D} \) always has finite global dimension since \( \mathfrak{M} \) is smooth.

(ii) If \( \mathfrak{M} \) and \( \mathfrak{M}' \) are conical symplectic resolutions of the same cone and derived localization holds at \( \lambda \) on \( \mathfrak{M} \), then derived localization holds at \( \lambda \) on \( \mathfrak{M}' \).

Since the left and right global dimensions of a Noetherian ring coincide, we have the following corollary.

**Corollary 4.18** Derived localization holds at \( \lambda \) if and only if it holds at \( -\lambda \).

Note that this will often not be the case for abelian localization, as the case of \( T^* \mathbb{P}^1 \) shows.

### 5 \( \mathbb{Z} \)-algebras

A \( \mathbb{Z} \)-algebra is an algebraic structure that mimics the homogeneous coordinate ring of a projective variety in a noncommutative setting. More precisely, it is an \( \mathbb{N} \times \mathbb{N} \)-graded vector space

\[ Z = \bigoplus_{k \geq m \geq 0} kZ_m \]

with a product that satisfies the condition \( kZ_\ell \cdot Z_m \subseteq kZ_m \) for all \( k \geq \ell \geq m \) and \( kZ_\ell \cdot Z_m = 0 \) if \( \ell \neq \ell' \). While \( Z \) itself will usually not have a unit, each algebra \( kZ_k \) is required to be unital; we will also always assume that \( kZ_k \) is Noetherian, that \( kZ_m \) is finitely generated as a left \( kZ_k \)-module and as a right \( mZ_m \)-module, and that there exists a natural number \( r \) such that \( Z \) is generated as an algebra by those \( kZ_m \) with \( k - m \leq r \). A left module over the \( \mathbb{Z} \)-algebra \( Z \) is an \( \mathbb{N} \)-graded vector space

\[ N = \bigoplus_{m \geq 0} mN \]
with an action of $Z$ such that $k Z_m \cdot m N \subset k N$ for all $k \geq m$ and $k Z_m \cdot m' N = 0$ if $m \neq m'$. It is called **bounded** if $m N = 0$ for $m \gg 0$.

**Remark 5.1** Some authors also discuss **torsion** modules, which are those isomorphic to a direct limit of bounded modules. We will only be interested in finitely generated $Z$-modules, and in this setting the conditions of being bounded and being torsion are equivalent.

In this section we assume that $X$ is projective over a complex affine variety, which is equivalent to saying that it is the projective spectrum of a finitely generated graded $C$-algebra. We also assume that $S$ is affine. Our goal is to define a $Z$-algebra whose localized module category is equivalent to $D$-mod.

### 5.1 Quantizations of line bundles

In addition to modules over quantizations, we will also need to consider bimodules over pairs of quantizations. Let $Q$ and $Q'$ be $S$-equivariant quantizations. A $Q' - Q$ bimodule is a sheaf of modules over the sheaf of algebras $Q' \otimes_{\mathbb{C}[\hbar]} Q'^{\text{op}}$. Such a bimodule is called **coherent** if it is a quotient of a bimodule which is locally free of finite rank. The most important examples will be quantizations of line bundles.

Let $L$ be an $S$-equivariant line bundle on $X$ and let $\eta \in H^2_{DR}(X/S; \mathbb{C})$ be the image of the Euler class of $L$. Fix an $S$-equivariant quantization $Q_0$ with period $[\omega_X] + h \lambda$, and for any integer $k$, let $Q_k$ be the quantization with period $[\omega_X] + h(\lambda + k\eta)$.

**Proposition 5.2** For every pair of integers $k$ and $m$, there exists a coherent $S$-equivariant $Q_k - Q_m$ bimodule $\mathcal{I}_m$ with an isomorphism $\mathcal{I}_m/(h \cdot k \mathcal{I}_m) \cong L^{k-m}$. This bimodule is unique up to canonical isomorphism, and tensor product with $\mathcal{I}_m$ defines an equivalence of categories from $Q_m$-mod to $Q_k$-mod.

**Proof:** By the usual sheaf theory, the locally free $S$-equivariant modules of rank 1 over $Q_k$ are in bijection with $H^1_2(X; Q^*_k)$. We have a surjective map of sheaves of groups $Q^*_k \rightarrow \mathcal{G}_X^*$. The kernel of this map is $1 + h Q_k$. As a sheaf of groups, this possesses a filtration by the subgroups $1 + h^n Q_k$, with successive quotients isomorphic to the structure sheaf $\mathcal{G}_X$ considered as a sheaf of abelian groups since

$$(1 + h^n a)(1 + h^n b) \equiv 1 + h^n (a + b) \pmod{h^{n+1}}.$$  

Since $\mathcal{G}_X$ has vanishing higher cohomology, an argument as in [KR08, 2.12] shows the inverse limit $1 + h Q_k$ has vanishing higher cohomology as well. By the Hochschild-Serre spectral sequence

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14In [Kal06], Kaledin uses **algebraically convex** to describe this condition, but in other papers this term allows the map to only be proper; we should emphasize here that projectivity, not merely properness, is essential.

15This is an isomorphism of $\mathcal{G}_X$ and $\mathcal{G}_X$ bimodules, where the two actions of $\mathcal{G}_X$ on $\mathcal{L}^{k-m}$ are the same.
sequence he have vanishing higher equivariant cohomology. In particular, we have an induced isomorphism
\[ H^1_\delta(X; \mathcal{Q}_k^\times) \cong H^1_\delta(X; \mathcal{G}_X^\times). \]

The line bundle \( \mathcal{L}^{k-m} \) is classified by an element \([\mathcal{L}^{k-m}]\) of \( H^1_\delta(X; \mathcal{G}_X^\times) \), and we define \( \kappa T_m \) to be the locally free rank 1 left \( \mathcal{Q}_k \)-module given by the corresponding element \([\kappa T_m]\) of \( H^1_\delta(X; \mathcal{Q}_k^\times) \).

The structure maps of \( \kappa T_m/(\hbar \cdot \kappa T_m) \) as a \( \mathcal{G}_X \)-module are just the reduction mod \( \hbar \) of the structure maps of \( \kappa T_m \), which tells us that \( \kappa T_m/(\hbar \cdot \kappa T_m) \cong \mathcal{L}^{k-m} \).

Now consider the sheaf of \( \pi^{-1}\mathfrak{S}_S[[\hbar]] \)-algebras \( \text{End}_{\mathcal{Q}_k}(\kappa T_m)^{\text{op}} \). This sheaf is an \( \mathfrak{S} \)-equivariant quantization of \( X \), and it is obtained from \( \mathcal{Q}_k \) by twisting the transition functions by the 1-cocycle representing \( \kappa T_m \). We want to show that this quantization is isomorphic to \( \mathcal{Q}_m \).

We first note that we can reduce to the case where \( S \) is a point, since no two sections of \( H^2_{DR}(\mathfrak{X}/S; \mathbb{C}) \) agree at infinitely many points on \( S \).

Furthermore, it is enough to check this fact modulo \( \hbar^2 \), since the isomorphism type of this reduction determines \( \mathcal{Q}_m \) amongst \( \mathfrak{S} \)-equivariant quantizations. As shown in the proof of [BK04, 1.8], the set of quantizations up to first order is a torsor over \( H^1_{DR}(\mathfrak{X}; \mathfrak{H}) \) where \( \mathfrak{H} \) is, in the language of [BK04], the localization \( \text{Loc}(\mathcal{M}_s, \mathcal{H}) \) of the module \( \mathcal{H} \) of Hamiltonian vector fields on the formal disk for the Harish-Chandra torsor.

It is helpful to think about the classical rather than Zariski topology in order to understand this action. As we discuss in Section 5.5, associated to \( \mathcal{Q} \), there is a quantization of the structure sheaf of the complex manifold \( \mathfrak{M}^{an} \), which we denote \( \mathcal{Q}^{an} \), and analytic versions of all the sheaves we have considered. Since the higher pushforwards \( R^m \pi_* \mathcal{G}_X \) or \( R^n \pi_* \mathcal{G}^{an}_X \) vanish, we have an isomorphisms of groups
\[ H^1_{DR}(\mathfrak{X}; \mathfrak{H}) \cong H^2_{DR}(\mathfrak{X}; \mathbb{C}) \quad H^1_{DR}(\mathfrak{X}; \mathfrak{H}^{an}) \cong H^2_{DR}(\mathfrak{X}^{an}; \mathbb{C}) \]

via the boundary map \( \delta \) for the short exact sequence of sheaves
\[ 0 \to \mathcal{G}_X \to J_\infty \mathcal{G}_X \xrightarrow{H} \mathfrak{H} \to 0, \]

(or its counterpart in the classical topology). By a classical result of Grothendieck, algebraic and analytic de Rham cohomology of the structure sheaf agree, so the same holds for \( H^1_{DR}(\mathfrak{X}; \mathfrak{H}) \cong H^1_{DR}(\mathfrak{X}^{an}; \mathfrak{H}^{an}) \).

The classical topology has the advantage that the de Rham cohomology of \( \mathcal{G}^{an}_X \) and \( \mathfrak{H}^{an} \) agree with the usual sheaf cohomology of their flat sections, which are locally constant functions and Hamiltonian vector fields \( \mathfrak{H}^{an} \) respectively; thus we can think of an element of \( H^1_{DR}(\mathfrak{X}^{an}; \mathfrak{H}^{an}) \cong H^1(\mathfrak{H}^{an}) \) as a 1-cocycle in Hamiltonian vector fields. In the torsor action, a 1-cocycle acts on a first order quantization \( \mathcal{Q}^{an}/\hbar^2 \mathcal{Q}^{an} \) by twisting it via the action of \( \mathfrak{H}^{an} \) on \( \mathcal{Q}^{an}/\hbar^2 \mathcal{Q}^{an} \) by \( X \cdot a = a + \hbar X(\bar{a}) \) where \( X(\bar{a}) \) denotes the usual action of a vector field on the
function \( \bar{a} \), which is the image of \( a \) in \( \mathcal{Q}^{an}/h\mathcal{Q}^{an} \cong \mathcal{G}^{an}_X \). The period mod \( h^2 \) changes by the image under the boundary map \( h \delta \).

Now, we have a map of abelian groups \( \beta: (\mathcal{Q}^{an}_k)^\times \to \mathcal{H}^{an} \) uniquely determined by \( a^{-1}qa = q + h\beta(a)(\bar{q}) \), which thus matches actions on \( \mathcal{Q}^{an}_k/h^2\mathcal{Q}^{an}_k \). Thus, when we twist by a 1-cocycle in \( (\mathcal{Q}^{an}_k)^\times \), this is the same as twisting by its image under \( \beta \). That is, the period mod \( h^2 \) of \( \text{End}_{\mathcal{Q}^{an}_k}(\mathcal{X}^\delta_{\mathcal{T}m^{an}})^{\text{op}} \) is \( \omega_X + h(\lambda + k\eta + \delta \circ \beta_*(\mathcal{L}^{k-m})) \) where \( \beta_* \) is the induced map \( H^1(\mathcal{Q}^\times_k) \to H^1((\mathcal{Q}^{an}_k)^\times) \to H^1(\mathcal{H}^{an}) \cong H^1_{DR}(\mathcal{H}) \) induced by the map \( \beta: (\mathcal{Q}^{an}_k)^\times \to \mathcal{H}^{an} \).

Now, we calculate that

\[
aqa^{-1} = q - a^{-1}[a,q] \equiv q - h\{\log \bar{a}, q\} \pmod{h^2}
\]

so \( \beta(a) = -H(\log \bar{a}) = -H(\bar{a})/a \). Thus, we wish to understand the map induced on the composition \( \delta \circ \beta \) in first cohomology. Consider the diagram of sheaves in the analytic topology (we leave off superscripts to avoid clutter) with short exact rows, along with the relevant piece of the associated long exact sequences:

\[
\begin{array}{cccccc}
\mathbb{Z}_X & \to & \mathcal{G}_X & \to & \mathcal{G}_X & \to \mathbb{H}^1(X, \mathbb{G}_X) \\
\downarrow & & \downarrow \exp & & \downarrow -H \circ \log & \downarrow c_1 \\
\pi^{-1}\mathcal{S}_S & \to & \mathcal{G}_X & \to & \mathcal{H} & \to \mathbb{H}^1(X, \mathbb{G}_X) \\
\downarrow & & -H \circ \log & & \downarrow -1 & \downarrow -1 \\
\pi^{-1}\mathcal{S}_S & \to & \mathcal{G}_X & \to & \mathcal{H} & \to \mathbb{H}^2(X, \mathcal{S}^\times) \\
\end{array}
\]

This shows that

\[
\delta \circ \beta([\mathcal{L}^{k-m}]) = -c_1(\mathcal{L}^{k-m}) = (m - k)\eta,
\]

so by our previous calculation \( \text{End}_{\mathcal{Q}_k}(\mathcal{T}_m)^{\text{op}} \) and \( \mathcal{Q}_m \) have identical periods and thus are isomorphic as \( \pi^{-1}\mathcal{S}_S[[h]] \)-algebras.

Also, we wish to show that on \( \mathcal{T}_m/(h \cdot \mathcal{T}_m) \), the quotient \( \mathcal{Q}_m/h\mathcal{Q}_m \cong \mathcal{G}_X \) acts by the usual module structure on \( \mathcal{L}^{k-m} \). This is a local question, so we may assume that the line bundle \( \mathcal{L} \) is trivial, in which case, \( \mathcal{Q}_k \cong \mathcal{T}_m \cong \mathcal{Q}_m \) with the left and right actions just being left and right multiplication, which both coincide with the usual \( \mathcal{G}_X \)-action after killing \( h \).

Of course, \( \mathcal{T}_m \otimes_{\mathcal{Q}_m} \mathcal{T}_k \) is a quantization of \( \mathcal{L}^{k-m} \otimes_{\mathcal{S}_m} \mathcal{L}^{m-k} \cong \mathcal{S}_m, \) so by uniqueness, \( \mathcal{T}_m \otimes_{\mathcal{Q}_m} \mathcal{T}_k \cong \mathcal{Q}_k \), and tensor product is indeed an equivalence. \( \square \)

**Remark 5.3** In the next proposition and later in Section \( \square \) we will want to vary the periods
of the quantizations in more than one-dimensional families, so we will use an alternate notation and label the quantizations and bimodules by elements of $H^2_{DR}(X/S; \mathbb{C})$ instead of integers. In other words, the quantization $Q_k$ will be written $Q_{\lambda + k\eta}$ and the bimodule denoted $\mathcal{T}_0$ in the notation of Proposition 5.2 will be written $\lambda + \eta \mathcal{T}_\lambda$.

We conclude this section by studying quantizations of line bundles in the context of Hamiltonian reduction. For any $\xi \in \chi(G)$, let $\mathcal{L}_\xi$ be the line bundle on $X_{\text{red}}$ descending from the trivial bundle on $X$ with $G$-structure given by $\xi$. Fix a quantized moment map $\eta$ for the action of $G$ on $(X, \mathcal{O})$ and a pair of elements $\xi, \xi' \in \chi(\mathfrak{g})$, and let $Q_{\text{red}} = Q_{K(\xi)}$ and $Q'_{\text{red}} = Q_{K(\xi')}$ be the corresponding reductions. Consider the $Q'_{\text{red}} - Q_{\text{red}}$ bimodule

$$\xi \mathcal{S}_\xi := \psi_*(\text{Hom}_{Q_0}(\mathcal{R}_\xi, \mathcal{R}_\xi)).$$

**Proposition 5.4** If $\xi' - \xi$ does not integrate to a character of $G$, then $\xi \mathcal{S}_\xi$ is trivial. If it does, then it is isomorphic to the quantization $K(\xi') \mathcal{T}_{K(\xi)}$ of $\mathcal{L}_{\xi' - \xi}$.

**Proof:** First, note that the sheaf $\mathcal{R}_\xi$ inherits a left $\mathfrak{g}$-module structure via the action of left multiplication by $\eta(x) - \xi(x)$; furthermore $\mathcal{R}_\xi/h\mathcal{R}_\xi \cong \mathfrak{g}_{\text{red}}^{-1}(\chi(\mathfrak{g}))$, with the induced $\mathfrak{g}$-action coinciding with the natural one on $\mathfrak{g}_{\text{red}}^{-1}(\chi(\mathfrak{g}))$. In particular, it integrates to the group $G$.

The sheaf $\text{End}(\mathcal{R}_\xi)^{\text{op}}$ is naturally isomorphic to the $\mathfrak{g}$-invariant subsheaf of $\mathcal{R}_\xi$ via the map that takes an endomorphism over any open set to the image of $\bar{1} \in \mathcal{R}_\xi(\mathfrak{U})$. Similarly, a map $\mathcal{R}_{\xi'} \to \mathcal{R}_\xi$ must take $\bar{1} \in \mathcal{R}_{\xi'}(\mathfrak{U})$ to a section $r$ killed by $\eta(x) - \xi'(x)$, that is, one on which the $\mathfrak{g}$-action is of the form

$$x \cdot r = (\eta(x) - \xi(x))r = (\xi'(x) - \xi(x))r.$$

Since this action must integrate to an action of the group $G$, there can be no such maps if $\xi' - \xi$ does not integrate. If it does, then the pushforward $\xi \mathcal{S}_\xi$ is a quantization of the line bundle $\mathcal{L}_{\xi' - \xi}$ and thus isomorphic to $K(\xi') \mathcal{T}_{K(\xi)}$. \qed

### 5.2 The quantum homogeneous coordinate ring of $X$

Fix an $S$-equivariant quantization $Q$ of $X$ with period $[\omega_X] + h\lambda \in H^2_{DR}(X/S; \mathbb{C})[[h]]$ and an $S$-equivariant line bundle $\mathcal{L}$ on $X$ that is very ample relative to the affinization of $X$. To these data we will associate a $\mathbb{Z}$-algebra $Z = Z(X, Q, \mathcal{L})$. Let $\eta \in H^2_{DR}(X/S; \mathbb{C})$ be the Euler class of $\mathcal{L}$, let $Q_k$ be the quantization with period $[\omega_X] + h(\lambda + k\eta)$, let $\mathcal{D} := Q[h^{-1/n}]$ and $\mathcal{D}_k := Q_k[h^{-1/n}]$, and let $k\mathcal{T}_m$ be the $Q_k - Q_m$ bimodule that quantizes the line bundle $\mathcal{L}^{k-m}$.

**Definition 5.5** Let $k\mathcal{T}_m' := k\mathcal{T}_m[h^{-1/n}]$ be the $\mathcal{D}_k - \mathcal{D}_m$ bimodule associated to the $Q_k - Q_m$ bimodule $k\mathcal{T}_m$.
Definition 5.6 Let \( kZ_m := \Gamma_S(kT'_m) \) with products induced by the canonical isomorphisms \( kT'_\ell \otimes_D \ell T_m \cong kT_m \). We call \( Z \) the quantum homogeneous coordinate ring of \( X \).

We filter the sheaf \( kT'_m \) by setting \( kT'_m(0) = kT_m[h^{1/n}] \) and \( kT'_m(\ell) = h^{1/n}kT'_m(0) \), and give \( kZ_m \) the induced filtration; it is compatible with the multiplication, so it makes \( Z \) into a filtered \( \mathbb{Z} \)-algebra. The Noetherian and finite generation properties of \( Z \) that we require follow directly from the corresponding properties of the associated graded, which in turn follow because \( \mathcal{L} \) is very ample over an affine base.

Note that the associated graded of \( kZ_m \) is isomorphic to \( \Gamma(X; \mathcal{L}^{k-m}) \), and for any \( \mathbb{Z} \)-module \( N \) with a compatible filtration, the associated graded of \( N \) is a module over the section ring \( R(\mathcal{L}) := \bigoplus_{k \geq 0} \Gamma(X; \mathcal{L}^k) \).

A filtration of \( N \) is called good if its associated graded is a finitely generated module over \( R(\mathcal{L}) \). Then \( N \) has a good filtration if and only if it is finitely generated over \( Z \).

For any natural number \( p \), let \( Z[p] \) be the \( \mathbb{Z} \)-algebra defined by putting \( kZ[p]_m := k_pZ_{m+p} \). It is clear that \( Z[p] \) is isomorphic to the \( \mathbb{Z} \)-algebra \( Z(X, \mathbb{Q}_p, \mathcal{L}) \). For any \( \mathbb{Z} \)-module \( N \), we define a \( Z[p] \)-module \( N[p] \) by \( N[p]_k = N_{k+p} \).

Let \( \mathbb{Z} \)-mod be the category of finitely generated modules over \( \mathbb{Z} \), and let \( \mathbb{Z} \)-mod\_bd be the full subcategory of \( \mathbb{Z} \)-mod consisting of bounded modules. We define the functors

\[
\Gamma^Z_S : \mathcal{D} \text{-mod} \to \mathbb{Z} \text{-mod} \quad \text{and} \quad \text{Loc}^Z : \mathbb{Z} \text{-mod} \to \mathcal{D} \text{-mod}
\]

by putting

\[
\Gamma^Z_S(N) := \bigoplus_{k \geq 0} \Gamma_S(kT'_0 \otimes_D N) \quad \text{and} \quad \text{Loc}^Z(N) := \left( \bigoplus_{k \geq 0} 0T'_k \right) \otimes_Z N.
\]

Lemma 5.7 If \( N \) is finitely generated over \( \mathbb{Z} \), then \( \text{Loc}^Z(N) \) is finitely generated over \( \mathcal{D} \).

Proof: There is some integer \( K \) such that \( \bigoplus_{k=0}^K N_k \) generates \( N \). Thus, we have that \( \text{Loc}^Z(N) \) is a quotient of \( \bigoplus_{k=0}^K 0T'_k \otimes_k Z_k N_k \). Since the latter module is clearly finitely generated, the former is as well. \( \square \)

A coherent lattice \( N(0) \) in \( N \) induces a filtration on \( \Gamma^Z_S(N) \), which is good because we have an injection

\[
\text{gr} \Gamma^Z_S(N) \hookrightarrow \bigoplus_{0 \leq m} \Gamma(X; \overline{N} \otimes \mathcal{L}^{\otimes m}),
\]

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where we put $\overline{\mathcal{N}} := \mathcal{N}(0)/\mathcal{N}(−1)$. Conversely, a good filtration on a $\mathbb{Z}$-module $\mathcal{N}$ induces a lattice in $\text{Loc}^\mathbb{Z}(\mathcal{N})$, which is coherent because we have

$$\text{Loc}(\text{gr } N) \cong \overline{\text{Loc}^\mathbb{Z}(\mathcal{N})},$$

where $\text{Loc}$ is the usual functor sending a graded module over $R(L)$ to a coherent sheaf on $X$.

The functor $\text{Loc}^\mathbb{Z}$ is left-adjoint to $\Gamma^\mathbb{Z}_S$; let $\iota_N: \mathcal{N} \to \Gamma^\mathbb{Z}_S(\text{Loc}^\mathbb{Z}(\mathcal{N}))$ and $\epsilon_N: \text{Loc}^\mathbb{Z}(\Gamma^\mathbb{Z}_S(\mathcal{N})) \to \mathcal{N}$ be the unit and co-unit of the adjunction. The following theorem justifies our name for $\mathbb{Z}$.

**Theorem 5.8** The co-unit $\epsilon_N$ is always an isomorphism and the unit $\iota_N$ is an isomorphism in sufficiently high degree. Furthermore, $\text{Loc}^\mathbb{Z}$ kills all bounded modules, thus $\Gamma^\mathbb{Z}_S$ and $\text{Loc}^\mathbb{Z}$ are biadjoint equivalences between $\mathcal{D}$-mod and the quotient of $\mathbb{Z}$-mod by $\mathbb{Z}$-mod$_{bd}$.

**Remark 5.9** We note that this theorem is quite close in flavor to several others in the theory of $\mathbb{Z}$-algebras, such as [SvdB01, 11.1.1], but these typically assume finiteness hypotheses that are too strong for our situation.

**Remark 5.10** If we dropped the assumption that $S$ is affine, we would expect to be able to prove a Theorem similar to Theorem 5.8 in which the $\mathbb{Z}$-algebra is replaced by a sheaf of $\mathbb{Z}$-algebras over $S$.

**Proof of Theorem 5.8** The induced map

$$\overline{\epsilon_N}: \overline{\text{Loc}^\mathbb{Z}(\Gamma^\mathbb{Z}_S(\mathcal{N}))} \to \overline{\mathcal{N}}$$

is an isomorphism by the usual localization theorem for sheaves on a projective (over affine) variety. By Nakayama’s lemma, $\epsilon_N$ is an isomorphism as well. Similarly, the map

$$\overline{\text{gr}(\iota_N)}: \overline{\mathcal{N}} \to \overline{\text{Loc}^\mathbb{Z}(\Gamma^\mathbb{Z}_S(\mathcal{N}))}$$

is an isomorphism in high degree, thus the same is true for $\iota_N$. If $N$ is bounded, then $\overline{\text{Loc}^\mathbb{Z}(N)} \cong \overline{\text{Loc}(\text{gr } N)}$ is the zero sheaf, thus $\overline{\text{Loc}^\mathbb{Z}(\mathcal{N})} = 0$, as well.

### 5.3 $\mathbb{Z}$-algebras and abelian localization

We call a bimodule between two rings **Morita** if it induces a Morita equivalence between the two rings. We call a $\mathbb{Z}$-algebra $\mathbb{Z}$ **Morita** if for all $k \geq m \geq 0$ the bimodule $k \mathbb{Z}_m$ is Morita and the natural map

$$k \mathbb{Z}_{k-1} \otimes k_{k-2} \mathbb{Z} \otimes \cdots \otimes m+1 \mathbb{Z}_m \to k \mathbb{Z}_m$$

(4)
is an isomorphism. In the terminology of [GS05 §5.4], this means that $Z$ is isomorphic to the Morita $Z$-algebra attached to the bimodules $m_{+1}Z_m$.

**Proposition 5.11** The $Z$-algebra constructed in Section 5 is Morita if and only if, for all $k \geq 0$, localization holds for $D_k$.

**Proof:** Consider the functor $\gamma(M) = \bigoplus_k kZ_0 \otimes_A M$ from finitely generated modules over $A = qZ_0$ to $Z$-mod/$Z$-mod$_{bd}$. Let $\beta$ denote the adjoint to this functor; one description of $\beta$ is that $\beta(jN) = 0Z_j \otimes_A jN$ for $j \gg 0$. There is a natural transformation $\beta(\Gamma_S^Z(M)) \to \Gamma_S(M)$, induced by the natural transformation $\text{Loc}(M) \to \text{Loc}^Z(\{kZ_0 \otimes_A M\})$. The latter natural transformation has inverse given by the multiplication map of sections $\text{Loc}(\{0Z_0 \otimes A M\}) \to D_0$, tensored with $M$ over $A$. Thus the former natural transformation is an isomorphism as well.

In particular, if we assume that $Z$ is Morita then Gordon and Stafford [GS05 §5.5] show that $\gamma$ and $\beta$ are equivalences. Thus combining this result with Theorem 5.8, we see that $\Gamma_S = \beta \circ \Gamma_S^Z$ is the composition of two equivalences, and thus an equivalence itself and localization holds for $D$. Furthermore, if $Z$ is Morita, then $Z[k]$ is Morita for all $k \geq 0$, so localization holds for $D_k$ for all $k \geq 0$.

Conversely, suppose that localization holds for $D_k$ for all $k \geq 0$. We have a natural isomorphism of functors

$$k_{+1}Z_k \otimes - \cong \Gamma_S(k_{+1}T'_k \otimes \text{Loc}(-))$$

from $A_k$-mod to $A_{k+1}$-mod. Since the right hand side is an equivalence, so is the left hand side; this proves that the bimodule $k_{+1}Z_k$ is Morita for all $k \geq 0$. Similarly, this implies that

$$k_{+1}Z_k \otimes kZ_m \cong \Gamma_S(k_{+1}T'_k \otimes \text{Loc}(kZ_m)) \cong \Gamma_S(k_{+1}T'_m) \cong k_{+1}Z_m.$$

By induction, this implies that the map (4) is an isomorphism. Thus, $Z$ is Morita. \hfill \Box

For the remainder of the section, suppose that $X = M$ is a conical symplectic resolution.

**Proposition 5.12** There is a positive integer $p$ such that $Z[p]$ is Morita.

**Proof:** Let $kZ_{\ell} := \Gamma_S(kT'_\ell)$ for all $k, \ell \in \mathbb{Z}$. (Previously we had only made this definition when $k \geq \ell$.) We will first show that there exists a $p$ such that the multiplication map

$$kZ_{k+1} \otimes A_{k+1} k_{+1}Z_k \to A_k$$

is surjective for all $k \geq p$; this will establish that $kZ_{k+1}$ is Morita for all $k \geq p$.

As in Section 2.1, we let $\pi : M_\eta \to \mathbb{A}^1$ be the twistor family of $M$, and let $\mathcal{L}$ be the line bundle on $M_\eta$ extending $L$ as in Proposition 2.11. Let $D_k$ be the $S$-equivariant quantization of $M_\eta$ with period $[\omega, M_\eta] + h(\lambda + k\eta)$. Let $0\mathcal{I}_{\pm 1}$ be the $D_0 - D_{\pm 1}$ bimodule quantizing $\mathcal{L}_{\pm 1}$,
and let $\mathcal{T}_0$ be the $\mathcal{L}_1 - \mathcal{L}_0$ bimodule quantizing $\mathcal{L}_1$. Using notation similar to that of Proposition 3.1, we have

$$k\mathcal{T}_{k\pm 1} \cong \sigma_k^* (0\mathcal{T}_{\pm 1}) \quad \text{and} \quad k\mathcal{T}_k \cong \sigma_k^* (\mathcal{T}_0).$$

Thus, taking invariant sections $k\mathcal{Z}_k = \Gamma_S (k\mathcal{T}_k)$, we have that

$$k\mathcal{Z}_{k\pm 1} \cong \sigma_k^* (0\mathcal{Z}_{\pm 1}) \quad \text{and} \quad k\mathcal{Z}_k \cong \sigma_k^* (\mathcal{Z}_0).$$

Of course, after taking invariant sections, $\sigma_k$ simply means imposing the equation $h^{-1}t = k$.

Consider the tensor product map

$$0\mathcal{Z}_{\pm 1} \otimes \mathcal{A}_{\pm 1} \to \mathcal{A}_0. \quad (6)$$

Let $E_{\pm}$ be the cokernel of this map, which is a bimodule over $\mathcal{A}_0$, and let $\text{Loc}(E_{\pm})$ be the localization of this bimodule. This module is supported on the preimage of $0 \in \mathcal{A}_1$, since the fibers over other points of $\mathcal{A}_1$ are affine (Proposition 2.15), and the corresponding map is an isomorphism after taking associated graded. It’s also supported on the Steinberg $\mathfrak{M} \times_{\mathcal{M}_0} \mathfrak{M}$, since the left and right actions of $\mathcal{A}_0$ on the associated graded agree.

Thus we can apply Lemma 4.14: it follows that the action of $h^{-1}t$ on $\text{Loc}(E_{\pm})$ must satisfy a polynomial equation with complex coefficients $p(h^{-1}t) = 0$. In particular, if $k$ is not a root of $p$, $h^{-1}t - k$ acts invertibly on the sheaf, so it also acts invertibly on $E_{\pm}$. That is, after specializing at $t = hk$, we have that the map of (5) is surjective.

It remains only to establish that (4) is an isomorphism for all $k > m \geq p$, possibly after enlarging $p$. If we fix $k - m$, then we can proceed exactly as in the first part of the proof to find a $p$ such that (4) is an isomorphism whenever $m \geq p$. The only potential problem is that we will be forced to make $p$ bigger and bigger as $k - m$ grows. The reason that this will not happen is that $Z$ is Noetherian. More precisely, fix an integer $r$ such that $R(\mathcal{L})$ is generated in degrees less than or equal to $r$; it follows that $Z$ is generated by $kZ_m$ for $k - m \leq r$. Now enlarge $p$ so that the map (4) is an isomorphism whenever $m + r \geq k > m \geq p$. This immediately implies that (4) is surjective whenever $k > m \geq p$; we claim that it is in fact an isomorphism in this range.
To see this, let $k > m \geq p$ be given, and consider the composition

$$kZ_{k-1} \otimes k_{k-1}Z_{k-2} \otimes \cdots \otimes m_{m+1}Z_m \otimes m_{m+1}Z_{m+1} \otimes \cdots \otimes k_{k-2}Z_{k-1} \otimes k_{k-1}Z_k$$

$$\downarrow$$

$$kZ_m \otimes m_{m+1}Z_m \otimes \cdots \otimes k_{k-2}Z_{k-1} \otimes k_{k-1}Z_k$$

$$\downarrow$$

$$kZ_k = A_k.$$  

(7)

(8)

Since (5) is an isomorphism, so is the composition of (7) and (8). It follows that (7) is injective. Since (7) is the tensor product of (4) with the Morita bimodule $m_{m+1}Z_m \otimes \cdots \otimes k_{k-2}Z_{k-1} \otimes k_{k-1}Z_k$, (4) must also be injective, and therefore an isomorphism. □

Propositions 5.11 and 5.12 immediately yield the following corollary.

**Corollary 5.13** There is an integer $p$ such that localization holds for $D_k$ for all $k \geq p$.

**Remark 5.14** Corollary 5.13 is precisely the first statement of Corollary B.1 from the introduction for very ample line bundles. If $\eta$ is only ample, then there exists a positive integer $r$ such that $r\eta$ is very ample, and we obtain Corollary B.1 by applying Corollary 5.13 with $\lambda' = \lambda + j\eta$ and $\eta' = r\eta$ for $j = 0, 1, \ldots, r - 1$.

It is still desirable to have a non-asymptotic result; that is, a necessary and sufficient condition for localization to hold for $D$ itself. Let $Z^{(p)}$ be the $\mathbb{Z}$-algebra defined by $kZ_m^{(p)} \cong kpZ_{mp}$ with the obvious product structure. It is clear that $Z^{(p)}$ is isomorphic to the $\mathbb{Z}$-algebra $Z(M, Q, L^p)$.

**Lemma 5.15** For all $p$, the restriction functor $Z\text{-mod}/Z\text{-mod}_{bd} \to Z^{(p)}\text{-mod}/Z^{(p)}\text{-mod}_{bd}$ is an equivalence of categories.

**Proof:** By Theorem 5.8 both the source and the target are equivalent to $D\text{-mod}$, and it is easy to check that these equivalences are compatible with the restriction functor. □

**Proposition 5.16** Localization holds for $D$ if and only if $Z^{(p)}$ is Morita for some $p$.

**Proof:** If $Z^{(p)}$ is Morita, then the functor $\Gamma_S: D\text{-mod} \to A\text{-mod}$ factors as

$$D\text{-mod} \to Z\text{-mod}/Z\text{-mod}_{bd} \to Z^{(p)}\text{-mod}/Z^{(p)}\text{-mod}_{bd} \to A\text{-mod},$$

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where the first functor is the equivalence of Theorem 5.8, the second is the equivalence of Lemma 5.15, and the last is the equivalence of [GS05, §5.5]. Thus localization holds for $D$.

Conversely, assume that localization holds for $D$. By Theorem 5.12 there is an integer $p$ such that $Z[p]$ is Morita, which easily implies that $Z^{(p)}[1]$ is Morita. We need to extend this to show that $Z^{(p)}$ is Morita, which involves showing that the bimodule $pZ_0$ is Morita and the multiplication map $2pZ_p \otimes pZ_0 \to 2pZ_0$ is an isomorphism. The fact that $pZ_0$ is Morita follows from the natural isomorphism of functors

$$pZ_0 \otimes - \cong \Gamma_S(pT'_0 \otimes \text{Loc}(-))$$

along with the fact that localization holds for both $D$ and $D_p$. Similarly, the fact that the multiplication map is an isomorphism follows from the natural isomorphism of functors

$$2pZ_p \otimes - \cong \Gamma_S(2pT'_p \otimes \text{Loc}(-))$$

applied to the module $pZ_0$.

\[ \square \]

**Remark 5.17** The “if” direction of Proposition 5.16 is very close in content to [KR08, 2.10] (though they do not use the language of $\mathbb{Z}$-algebras) and our proof draws heavily on theirs. We note, however, that Proposition 5.12 and Corollary 5.13 have no analogues in [KR08].

### 5.4 Twisted modules and the Kirwan functor

In this section, we consider a quantized moment map $\eta: U(g) \to A$ for an action of a connected reductive algebraic group $G$ on $(X, Q)$, and we let $U$ be the semistable locus for a $G$-equivariant ample line bundle $L$ on $X$. As in Section 3.4, we assume that the $G$ action on $U$ is free and that $S$ acts on $\mathbb{C}[X]$ with non-positive weights, fixing only the constant functions. Fix an element $\xi \in \chi(g)$. Let $X_{\text{red}}$ be the reduced space, $L_{\text{red}}$ its induced ample line bundle, and $D_{\text{red}}$ the associated quantization of $X_{\text{red}}$.

We call a $G$-equivariant object $N$ of $D$-$\text{mod}$ (respectively $D_U$-$\text{mod}$) $\xi$-twisted if, for all $x \in g$, the action of $x$ on $N$ induced by the $G$-structure coincides with left multiplication by the element $\eta(x) - \xi(x) \in A$. Let $D$-$\text{mod}_\xi$ (respectively $D_U$-$\text{mod}_\xi$) denote the full subcategory of $\xi$-twisted objects of $D$-$\text{mod}$ (respectively $D_U$-$\text{mod}$). Kashiwara and Rouquier [KR08, 2.8(ii)] prove that $D_U$-$\text{mod}_\xi$ is equivalent to $D_{\text{red}}$-$\text{mod}$ via the functor that takes $N$ to $\psi_*\text{Hom}(E_\xi, N)$.

Define the functor $\kappa: D$-$\text{mod} \to D_{\text{red}}$-$\text{mod}$ by putting

$$\kappa(N) := \psi_*\text{Hom}(E_\xi, N_U)$$

for all $N$ in $D$-$\text{mod}$. We call $\kappa$ the Kirwan functor in analogy with the Kirwan map in
(equivariant) cohomology. Our main result in this section will be Theorem 5.23, which says
that the Kirwan functor is essentially surjective. To prove this theorem, we introduce all of
the analogous constructions in the context of $\mathbb{Z}$-algebras.

In Section 5, we defined a $\mathbb{Z}$-algebra $Z = Z(\mathcal{X}, \mathcal{D}, \mathcal{L})$ and functor $\Gamma^{Z}_{\mathcal{D}} : \mathcal{D}\text{-mod} \rightarrow \mathcal{Z}\text{-mod}$. We may also define the $\mathbb{Z}$-algebra $Z_{\text{red}} = Z(\mathcal{X}_{\text{red}}, \mathcal{D}_{\text{red}}, \mathcal{L}_{\text{red}})$, with its own sections functor $\Gamma^{Z_{\text{red}}}_{\mathcal{D}_{\text{red}}} : \mathcal{D}_{\text{red}}\text{-mod} \rightarrow Z_{\text{red}}\text{-mod}$. By assumption, we have a ring homomorphism

$$\eta : U(\mathfrak{g}) \rightarrow A = \Gamma_{\mathcal{D}}(\mathcal{D}) \cong \mathbb{Z}_0.$$ 

Moreover, for all $m \geq 0$, there is a unique homomorphism

$$\eta_m : U(\mathfrak{g}) \rightarrow \Gamma_{\mathcal{D}_m}(\mathcal{D}) \cong m\mathbb{Z}_m$$

such that $\eta_0 = \eta$ and for all $x \in \mathfrak{g}$, the action of $x$ on $\mathcal{L}$ induced by the $G$-structure coincides with that induced by the adjoint action, via $\eta_{m+1}$ and $\eta_m$, on the $\mathcal{D}_{m+1} - \mathcal{D}_m$ bimodule $\mathcal{D}_m$ that quantizes $\mathcal{L}$. We say that a $G$-equivariant $Z$-module $N = \oplus_m N$ is $\xi$-twisted if, for all $x \in \mathfrak{g}$, the action of $x$ on $mN$ induced by the $G$-structure coincides with left multiplication by the element $\eta_m(x) - \xi(x) \in m\mathbb{Z}_m$. We denote the category of such modules $Z_{\text{mod}}\xi$.

Consider the $\mathbb{Z} - \mathbb{Z}_{\text{red}}$ bimodule

$$iY_j := iZ_j / iZ_j \cdot \langle \eta_j(x) - \xi(x) \mid x \in \mathfrak{g} \rangle.$$ 

We define the $\mathbb{Z}$-Kirwan functor

$$\kappa^{Z} : \text{Hom}_Z(Y, -) : Z\text{-mod} \rightarrow Z_{\text{red}}\text{-mod}$$

along with its left adjoint

$$\kappa^{Z}_{!} : Y \otimes_{Z_{\text{red}}} - : Z_{\text{red}}\text{-mod} \rightarrow Z\text{-mod}.$$

**Remark 5.18** Every $Z$-module $N$ has a largest submodule $N_{\xi}$ on which $\eta_j(x) - \xi(x)$ acts locally finitely and on which the $\mathfrak{g}$-action integrates to a $G$-action. The $G$-action makes $N_{\xi}$ a $\xi$-twisted equivariant module in a canonical way. Because of the $\xi$-twisted condition, the $G$-invariant part of $N_{\xi}$ is already a module over $Z_{\text{red}}$ in the obvious way, and we have a canonical isomorphism $\kappa^{Z}(N) \cong N^G_{\xi}$.

**Proposition 5.19** The functors $\kappa^{Z}$ and $\kappa^{Z}_{!}$ both preserve boundedness and thus induce functors

$$\kappa^{Z} : Z\text{-mod} / Z\text{-mod}_{\text{bd}} \rightarrow Z_{\text{red}}\text{-mod} / Z_{\text{red}}\text{-mod}_{\text{bd}}.$$
\[ \kappa^Z: \text{Z}_{\text{red}}\text{-mod} / \text{Z}_{\text{red}}\text{-mod}_{\text{bd}} \to \text{Z}\text{-mod} / \text{Z}\text{-mod}_{\text{bd}}. \]

**Proof:** The functor \( \kappa^Z \) obviously sends bounded modules to bounded modules.

To see that \( \kappa^Z \) preserves boundedness, find integers \( N \) and \( M \) such that all of the higher cohomology groups of \( L^N \) and \( L^M_{\text{red}} \) vanish. Then for any non-negative integers \( i, j, k \) with \( i \geq j + N \geq k + N + M \) the associated graded of the multiplication map \( iY_j \otimes j(Z_{\text{red}})_k \to iY_k \) is

\[ \Gamma(X; L^{i-j})^G \otimes \Gamma(X_{\text{red}}; L^{j-k}_{\text{red}}) \to \Gamma(X; L^{i-k})^G. \]

Since \( \bigoplus_n \Gamma(X; L^n)^G \) is a finitely generated module over \( \bigoplus_n \Gamma(X_{\text{red}}; L^n_{\text{red}}) \), there is some \( N' \) such that \( \bigoplus_{N \leq n \leq N'} \Gamma(X; L^n)^G \) generates \( \bigoplus_{N \leq n} \Gamma(X; L^n)^G \). That is, if we fix \( i \) and \( k \) such that \( i - k \geq N \), then \( \Gamma(X; L^{i-k})^G \) is spanned by the images of the maps

\[ \Gamma(X; L^{i-j})^G \otimes \Gamma(X_{\text{red}}; L^{j-k}_{\text{red}}) \to \Gamma(X; L^{i-k})^G \]

for all \( j \) such that \( N \leq i - j \leq N' \).

We may as well assume that \( N' \geq M + N \). Thus, since a map whose associated graded is surjective is itself surjective, we see that the map

\[ \bigoplus_{j \geq i - N'} iY_j \otimes j(Z_{\text{red}})_k \to iY_k \]

is surjective for all \( k \leq i - N' \). If \( M \) is a \( Z_{\text{red}} \)-module, it follows that if \( i \geq N' \) then \( \kappa^Z_i(M)_i \) is spanned by the images of \( iY_j \otimes M_j \) for \( j \geq i - N' \). Then if \( M_p = 0 \) for \( p \geq P \), we have \( \kappa^Z_i(M)_i = 0 \) whenever \( i > P + N' \).

This shows that both \( \kappa^Z \) and \( \kappa^Z_i \) preserve bounded modules and thus induce functors on the quotient categories. \( \square \)

**Proposition 5.20** The following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{D}\text{-mod} & \xrightarrow{\Gamma^Z_S} & \text{Z}\text{-mod} / \text{Z}\text{-mod}_{\text{bd}} \\
\kappa \downarrow & & \downarrow \kappa^Z \\
\mathcal{D}_{\text{red}}\text{-mod} & \xrightarrow{\Gamma^Z_{S_{\text{red}}}} & \text{Z}_{\text{red}}\text{-mod} / \text{Z}_{\text{red}}\text{-mod}_{\text{bd}}
\end{array}
\]

(Note that the horizontal arrows are equivalences by Theorem 5.8)

**Proof:** First note that we can assume that \( \mathcal{N} = \mathcal{N}_\xi \), that is, that \( \mathcal{N} \) has an \( G \)-equivariant structure agreeing with that induced by \( \eta - \xi \). This is because passing to the largest submodule
that has such a structure doesn’t change $\kappa$ or $\kappa^Z$.

With this assumption, we have a restriction map

$$\kappa^Z \left( \Gamma_S^Z(\mathcal{N}) \right) = \left( \Gamma_S^Z(\mathfrak{X}; \mathcal{N}) \right)^G \rightarrow \left( \Gamma_S^Z(\mathfrak{U}; \mathcal{N}) \right)^G \cong \Gamma_S^{Z_{red}}(\kappa(\mathcal{N}))$$

where $\Gamma_S^Z(\mathfrak{X}; -) = \Gamma_S^Z$, and $\Gamma_S^Z(\mathfrak{U}; -)$ denotes the same functor defined using the set $\mathfrak{U}$ of stable points. As in the proof of Theorem 5.8 let $\tilde{N} := N(0)/N(-1)$.

For each $m \in \mathbb{Z}$, the restriction from $\mathfrak{X}$ to $\mathfrak{U}$ gives the following long exact sequence in local cohomology.

$$H^0_{\mathfrak{X}\setminus\mathfrak{U}} (\tilde{N} \otimes \mathcal{L}^m)^G \rightarrow \Gamma (\mathfrak{X}; \tilde{N} \otimes \mathcal{L}^m)^G \rightarrow \Gamma (\mathfrak{U}; \tilde{N} \otimes \mathcal{L}^m)^G \rightarrow H^1_{\mathfrak{X}\setminus\mathfrak{U}} (\tilde{N} \otimes \mathcal{L}^m)^G \rightarrow \cdots$$

The space

$$\bigoplus_{m \geq 0} H^0_{\mathfrak{X}\setminus\mathfrak{U}} (\tilde{N} \otimes \mathcal{L}^m)$$

of sections of twists of $\tilde{N}$ which are supported on $\mathfrak{X} \setminus \mathfrak{U}$ is finitely generated over the ring

$$\bigoplus_{m \geq 0} \Gamma (\mathfrak{X} \setminus \mathfrak{U}; \mathcal{L}^m)$$

of sections of powers of the restriction of $\mathcal{L}$ to $\mathfrak{X} \setminus \mathfrak{U}$. Since $G$ is reductive, the invariant part of (9) is finitely generated over the invariant part of (10). The invariant part of (10) is a single copy of $\mathbb{C}$, since any invariant section of $\mathcal{L}^m$ for $m > 0$ vanishes on all unstable points. Thus $H^0_{\mathfrak{X}\setminus\mathfrak{U}} (\tilde{N} \otimes \mathcal{L}^m)^G$ vanishes for $m > 0$.

The module $H^1_{\mathfrak{X}\setminus\mathfrak{U}} (\tilde{N} \otimes \mathcal{L}^m)^G$ is not in general finitely generated as a module over the invariant section ring. On the other hand, the module $\bigoplus_{m \geq 0} \Gamma (\mathfrak{U}; \tilde{N} \otimes \mathcal{L}^m)^G$ is the sections of the twists of a coherent sheaf on the quotient $\mathfrak{U}/G$, which is projective over an affine variety, and thus finitely generated over the invariant section ring $\bigoplus_{m \geq 0} \Gamma (\mathfrak{U}; \mathcal{L}^m)^G$. In particular, its image in $\bigoplus_{m \geq 0} H^1_{\mathfrak{X}\setminus\mathfrak{U}} (\tilde{N} \otimes \mathcal{L}^m)^G$ under the boundary map is finitely generated over the same ring.

Since any positive degree invariant section of $\mathcal{L}$ vanishes on $\mathfrak{X} \setminus \mathfrak{U}$, its action on local cohomology is locally nilpotent; this implies that there is some integer $k$ such that all invariants of degree $\geq k$ act trivially on the image of $\bigoplus_{m \geq 0} \Gamma (\mathfrak{U}; \tilde{N} \otimes \mathcal{L}^m)^G$ under the boundary map. This in turn implies that the image is trivial for $m$ sufficiently large. Note that we used the fact that the image is finitely generated in both of these steps.

It follows that the restriction map

$$\left( \Gamma(\mathfrak{X}; \tilde{N} \otimes \mathcal{L}^m) \right)^G \rightarrow \left( \Gamma(\mathfrak{U}; \tilde{N} \otimes \mathcal{L}^m) \right)^G$$
is an isomorphism for \( m \gg 0 \). We next observe that

\[
\left( \Gamma(\bar{\mathcal{X}}; \bar{\mathcal{N}} \otimes \mathcal{L}^m) \right)^G \cong \text{gr} \left( \Gamma_{\mathcal{G}}(\bar{\mathcal{X}}; mL_0 \otimes \mathcal{D} \mathcal{N}) \right)^G \cong \text{gr} \left( \Gamma_{\mathcal{G}}(\bar{\mathcal{X}}; \mathcal{N})[m] \right)^G
\]

and similarly

\[
\left( \Gamma(\mathcal{U}; \mathcal{N} \otimes \mathcal{L}^m) \right)^G \cong \text{gr} \left( \Gamma_{\mathcal{G}}(\mathcal{U}; \mathcal{N})[m] \right)^G,
\]

where \([m]\) denotes a shift as in Section \([5]\). Since maps that induce isomorphism on associated graded are isomorphisms, we may conclude that the restriction map

\[
\left( \Gamma_{\mathcal{G}}(\mathcal{X}; \mathcal{N})[m] \right)^G \rightarrow \left( \Gamma_{\mathcal{G}}(\mathcal{U}; \mathcal{N})[m] \right)^G
\]

is an isomorphism for \( m \gg 0 \). This is equivalent to the statement that the kernel and cokernel of the map

\[
\left( \Gamma_{\mathcal{G}}(\mathcal{X}; \mathcal{N}) \right)^G \rightarrow \left( \Gamma_{\mathcal{G}}(\mathcal{U}; \mathcal{N}) \right)^G
\]

are bounded, as desired. \( \square \)

**Lemma 5.21** The Kirwan functor \( \kappa \) has a left adjoint \( \kappa_! \) such that \( \kappa \circ \kappa_! \) is isomorphic to the identity functor on \( \mathcal{D}_{\text{red}} \)-mod.

**Proof:** By Theorem \([5.20]\), we may work instead with the \( \mathcal{Z} \)-Kirwan functor \( \kappa_\mathcal{Z} \) and its left adjoint \( \kappa_! \mathcal{Z} \). Let \( \mathcal{Y}_j' \subset \mathcal{Y}_j \) be the sum of all non-trivial \( G \)-isotypic components. Since \( G \) is reductive, \( \mathcal{Y}_j \) is isomorphic to \( \mathcal{Y}_j' \oplus \mathcal{Y}_j^G \). There is a natural map from \( \mathcal{Y}_j^G \) to \( \mathcal{i}(\mathcal{Z}_{\text{red}}) \) whose associated graded is the map \( \Gamma(\bar{\mathcal{X}}; \mathcal{L}^{i-j})^G \rightarrow \Gamma(\mathcal{X}_{\text{red}}; \mathcal{L}_{\text{red}}^{i-j}) \). This map is an isomorphism when \( i - j \) is sufficiently large, which implies that the same is true of the map \( \mathcal{Y}_j^G \) to \( \mathcal{i}(\mathcal{Z}_{\text{red}}) \). Thus, modulo bounded modules, we have \( \mathcal{Y} \cong \mathcal{Y}' \oplus \mathcal{Z}_{\text{red}} \) as a right module over \( \mathcal{Z}_{\text{red}} \). Then for any \( \mathcal{Z}_{\text{red}} \)-module \( N \), we have

\[
\kappa_{\mathcal{Z}} \circ \kappa_! \mathcal{Z} (N) = \kappa_{\mathcal{Z}} (\mathcal{Y} \otimes \mathcal{Z}_{\text{red}} N) \cong (\mathcal{Y} \otimes \mathcal{Z}_{\text{red}} N)^G \cong \mathcal{Z}_{\text{red}} \otimes \mathcal{Z}_{\text{red}} N \cong N,
\]

modulo bounded modules. \( \square \)

**Remark 5.22** One can use similar principles to construct a right adjoint as well as a left to \( \kappa \). One considers the \( \mathcal{Z}_{\text{red}} \)-\( \mathcal{Z} \) bimodule

\[
i \mathcal{W}_j := i \mathcal{Z}_j / \langle \eta_j(x) - \xi(x) \mid x \in \mathfrak{g} \rangle \cdot i \mathcal{Z}_j.
\]

The obvious guess for the right adjoint based on general nonsense is \( \text{Hom}_{\mathcal{Z}_{\text{red}}} (\mathcal{W}, -) \); however,
we need to exercise care here since $W$ is not finitely generated as a left module. On the other hand, it is (as a left module) the direct sum $W = \bigoplus_{\chi \in \hat{G}} W^\chi$ of its isotypic components $W^\chi$ according to the natural $G$ action, and each isotypic component is finitely generated even after taking the associated graded by a classical theorem of Hilbert. We should emphasize that here $\hat{G}$ is the set of all finite dimensional representations, not just 1-dimensional ones.

A replacement for $\text{Hom}_{\mathbb{Z}}(W, -) = \prod_{\chi \in \hat{G}} \text{Hom}_{\mathbb{Z}}(W^\chi, -)$ with better finiteness properties is the direct sum $\kappa_*(-) = \bigoplus_{\chi \in \hat{G}} \text{Hom}_{\mathbb{Z}}(W^\chi, -)$ which we can consider as the subspace of $\text{Hom}_{\mathbb{Z}}(W, -)$ which kills all but finitely many isotypic components. This is closed under the action of $Z$ acting on the right since the $G$-action on $Z$ is locally finite.

It is still not obvious that $\kappa_*$ takes finitely generated modules to finitely generated modules. When $X$ is the cotangent bundle to a smooth affine $G$-variety, this is proved in a recent preprint by McGerty and Nevins [MNb, 6.1(3)].

The following theorem, which is an immediate consequence of Lemma 5.21, may be regarded as a categorical, quantum version of Kirwan surjectivity.

**Theorem 5.23** The Kirwan functor $\kappa$ is essentially surjective.

**Proof:** For any object of $\mathcal{D}_{\text{red}} \text{-mod}$, we can apply the left adjoint from Lemma 5.21 to obtain a witness to essential surjectivity. \qed

**Remark 5.24** McGerty and Nevins [MNa] always work with symplectic quotients of affine schemes, and the category of quantizations that they consider is by definition the essential image of the Kirwan functor. Thus Theorem 5.23 establishes that their module category is the same as ours. This fact was known to McGerty and Nevins, but the proof does not appear in their paper.

### 5.5 Comparison of the analytic and algebraic categories

Up until this point we have worked exclusively in the algebraic category, quantizing the sheaf of regular functions in the Zariski topology. On the other hand, some other important papers have considered quantizations of the functions on an analytic variety, for example [KR08, KS]. We will need to apply some results from these papers below, so we must prove a comparison theorem relating quantizations and their module categories for a nondegenerate Poisson scheme $\mathcal{X}$ and its analytification $\mathcal{X}^{\text{an}}$.

First, we note that every quantization $\mathcal{Q}$ in the Zariski topology introduces a corresponding quantization $\mathcal{Q}^{\text{an}}$ of the structure sheaf in the analytic category. To see this, we can consider the jet bundle $J_\infty \mathcal{Q}$, which is a pro-vector bundle on $\mathcal{M}$ with flat connection whose sheaf of flat sections is $\mathcal{Q}$, as explained in [BK04, 1.4]. The corresponding sheaf of analytic sections
$(J_\infty Q)^{an}$ again has a flat connection, and we let $Q^{an}$ be its sheaf of flat sections. We have a map $\alpha^{-1} Q \to Q^{an}$ where $\alpha: X^{an} \to X$ is the identity on points. If $Q$ is $S$-equivariant, so is $Q^{an}$.

As in the Zariski topology, we let $D^{an} := Q^{an}[h^{-1/n}]$. Similarly, for any $D$-module $M$, we let $M^{an} := \alpha^{-1} M \otimes_{\alpha^{-1} D} D^{an}$. As in [KR08], we call an $S$-equivariant $D^{an}$-module good if it admits a coherent $S$-equivariant $Q^{an}|_U$-lattice on every relatively compact open subset of $M$. If $M$ is a good $D$-module, $M^{an}$ is a good $D^{an}$-module.

**Theorem 5.25** If $\mathfrak{M}$ is a conical symplectic resolution, then the functor $(-)^{an}: D$-$\text{mod} \to D^{an}$-$\text{mod}$ is an equivalence of categories.

**Proof:** In essence, the proof is simply to observe that a version Theorem 5.8 holds in the analytic topology. More precisely, we define the quantum homogeneous coordinate ring $Z^{an}$ exactly as we defined $Z$. There is a canonical map from $Z$ to $Z^{an}$, and we claim that it is an isomorphism.

In bidegree $(0, 0)$, this map is the map from $\Gamma_S(D)$ to $\Gamma_S(D^{an})$. To see that this is an isomorphism, it is enough to show that the associated graded map $\Gamma(\mathfrak{S}_{\mathfrak{M}}) \to \Gamma(\mathfrak{S}_{\mathfrak{M}}^{an})^{\text{fin}}$ is an isomorphism, where $(-)^{\text{fin}}$ denotes the subalgebra of $S$-locally finite vectors. Since all $S$-weights on $\mathfrak{M}$ are positive, any $S$-weight vector in $\Gamma(\mathfrak{S}_{\mathfrak{M}}^{an})$ can be interpreted as a section of a line bundle on the projectivization of $\mathfrak{M}_0$ for the $S$ action; by the classic GAGA theorem of Serre [Ser56], this is in fact algebraic, and thus arises from an algebraic function on $\mathfrak{M}_0$. The argument in arbitrary bidegree follows from a similar analysis of sections of line bundles.

Now that we know that $Z$ and $Z^{an}$ are isomorphic, we have a functor from $D^{an}$-$\text{mod}$ to $D$-$\text{mod}$ given by the composition

$$D^{an}$-$\text{mod} \xrightarrow{(\Gamma_S^{an})} Z^{an}$-$\text{mod} \cong Z$-$\text{mod} \xrightarrow{\text{Loc}_Z} D$-$\text{mod}.$$  

This functor splits $(-)^{an}$ and is exact (since the cohomology of a sufficiently high twist with $L$ vanishes), so to check that it gives an equivalence, we need only check that it kills no module $K$. Thus, we need only show that for any good $S$-equivariant $D$-module, we must have that $\Gamma_S^{an}$ is not 0. Since $L$ is ample, $K/hK \otimes L^k$ has non-zero sections for $k \gg 0$ unless $K/hK = 0$; then Nakayama’s lemma tells us that $kT_0^{an} \otimes_{D_0} K$ has non-zero sections as well unless $K = 0$. This completes the proof. 

**Remark 5.26** Hou-Yi Chen [Che10] proves a version of Theorem 5.25 in the more general context of $DQ$-algebroids, but subject to the hypothesis that $\mathfrak{M}$ is projective over a point (which is never the case for a conical symplectic resolution of positive dimension). Chen uses
a more direct reduction to Serre’s classic GAGA theorem than we do; it is possible that his techniques could be adapted to our setting, as well.

**Remark 5.27** It might worry the reader that we used some analytic techniques in the proof of Proposition 5.2, used that result in the proof of Theorem 5.8, and then used that in the proof of Proposition 5.25; at first glance, this looks as though it may be circular. In fact, in the proof of Proposition 5.2, we use only the comparison theorem between algebraic and analytic de Rham cohomology; nothing in the vein of GAGA.

### 6 Convolution and twisting

Let $\nu: \mathcal{M} \to \mathcal{M}_0$ be a conical symplectic resolution with Steinberg variety $\mathfrak{Z} := \mathcal{M} \times_{\mathcal{M}_0} \mathcal{M}$. Consider the three different projections $p_{ij}: \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ as well as the two projections $p_i: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. The cohomology $H^2_{\dim \mathcal{M}}(\mathcal{M} \times \mathcal{M}; \mathbb{C})$ with supports in $\mathfrak{Z}$ has a convolution product given by the formula

$$\alpha \star \beta := (p_1)_*(p_{12}^* \alpha \cdot p_{23}^* \beta),$$

making it into a semisimple algebra [CG97, 8.9.8]. For any closed subvariety $\mathcal{L} \subset \mathcal{M}$ with the property that $\mathcal{L} = \nu^{-1}(\nu(\mathcal{L}))$, there is a degree-preserving action of this algebra on the cohomology $H^*_{\mathcal{L}}(\mathcal{M}; \mathbb{C})$ given by the formula

$$\alpha \star \gamma := (p_2)_*(\alpha \cdot p_1^* \gamma).$$

**Example 6.1** When $\mathcal{M}$ is the cotangent bundle of the flag variety, $H^2_{\dim \mathcal{M}}(\mathcal{M} \times \mathcal{M}; \mathbb{C})$ is isomorphic to the group ring of the Weyl group [CG97, 3.4.1], and $H^*(\mathcal{M}; \mathbb{C})$ is isomorphic to the regular representation.

In this section, we explain how to categorify this action. In Section 6.1, we define the category of Harish-Chandra bimodules over a pair of quantizations. There is both an algebraic and a geometric version of this definition, and they are related by the localization and invariant section functors. In Section 6.2, we show that a Harish-Chandra bimodule has a characteristic cycle in $H^2_{\dim \mathcal{M}}(\mathcal{M} \times \mathcal{M}; \mathbb{C})$, and tensor products of bimodules categorify convolution product of cycles. Furthermore, an object $\mathcal{N}$ of $D^b(D\text{-mod})$ has a characteristic cycle in $H^2_{\dim \mathcal{M}}(\mathcal{M}; \mathbb{C})$ for any $\mathcal{L} \subset \mathcal{M}$ containing $\text{Supp} \mathcal{N}$, and we show that the tensor product action of bimodules on modules categorifies the convolution action. In Section 6.3, we define a particularly nice collection of (algebraic) Harish-Chandra bimodules, which we use in Section 6.4 to study a certain collection of auto-equivalences of $D(A\text{-Mod})$ related to twisting functors on BGG category $\mathcal{O}$.  

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6.1 Harish-Chandra bimodules

Recall that, for any $\lambda \in H^2(\mathcal{M}; \mathbb{C})$, we let $A_\lambda := \Gamma_S(D_\lambda)$ be the section ring of the quantization of $\mathcal{M}$ with period $\lambda$. Let $H$ be a finitely generated $A_{\lambda'}-A_\lambda$ bimodule. Recall that $\text{gr} A_\lambda \cong \text{gr} A_{\lambda'} \cong \mathbb{C}[\mathcal{M}]$, thus for any filtration $H_0 \subset H(1) \subset \ldots \subset H$ which is compatible with the filtrations on $A_\lambda$ and $A_{\lambda'}$, the $\mathbb{C}[\mathcal{M}] \otimes \mathbb{C}[\mathcal{M}]$-module $\text{gr} H$ may be interpreted as an $\mathbb{S}$-equivariant sheaf on $\mathcal{M}_0 \times \mathcal{M}_0 \cong \text{Spec} (\mathbb{C}[\mathcal{M}] \otimes \mathbb{C}[\mathcal{M}])$.

When $n$ is greater than 1, we will be interested in a thickened associated graded $\text{gr}_n H := R(H)/hR(H)$. This is a module over $R(A_\lambda \otimes A_{\lambda'}^{\text{op}})/hR(A_\lambda \otimes A_{\lambda'}^{\text{op}}) \cong \mathbb{C}[\mathcal{M}_0] \otimes \mathbb{C}[\mathcal{M}_0] \otimes \mathbb{C}[h^{1/n}]/(h)$, and thus over $\mathbb{C}[\mathcal{M}_0] \otimes \mathbb{C}[\mathcal{M}_0]$. The module $\text{gr}_n H$ is an $n$-fold self-extension of $\text{gr} H$, but this can be a non-split extension, so $\text{gr}_n H$ contains more information.

**Definition 6.2** We say that $H$ is Harish-Chandra if it is finitely generated and it admits a filtration such that $\text{gr}_n H$ is scheme-theoretically supported on the diagonal. Equivalently, we require that if $a_\lambda \in A_\lambda(k)$ and $a_{\lambda'} \in A_{\lambda'}(k)$ are specializations of the same element $a \in \mathcal{A}$, then for all $h \in H(m)$, we have $a_\lambda \cdot h - h \cdot a_{\lambda'} \in H(k + m - n)$.

Let $\mathcal{H}_{\lambda}^{\mathcal{A}}$ be the category of Harish-Chandra bimodules, and let $D^b(\mathcal{H}_{\lambda}^{\mathcal{A}})$ be the full subcategory of $D^b(A_{\lambda'}-\text{mod}-A_\lambda)$ consisting of objects $H$ whose cohomology $H^i(H)$ is Harish-Chandra.

**Proposition 6.3** If $A_{\lambda'}$ has finite global dimension and $H_1 \in D^b(\mathcal{H}_{\lambda}^{\mathcal{A}})$ and $H_2 \in D^b(\mathcal{H}_{\lambda'}^{\mathcal{A}})$, then $H_2 \otimes L H_1 \in D^b(\mathcal{H}_{\lambda'}^{\mathcal{A}})$.

**Proof:** It’s more natural to think about the Rees modules $R(H_1), R(H_2)$ associated to some good filtration. These modules have locally free resolutions over $R(A_{\lambda'} \otimes A_{\lambda'}^{\text{op}})$ and $R(A_{\lambda'} \otimes A_\lambda^{\text{op}})$ such that $f' \otimes 1 - 1 \otimes f$ acts trivially on the cohomology of $R(H_1)$ modulo $h$ if $f' = f'$ (mod $h$) and similarly for $f'' \otimes 1 - 1 \otimes f'$ acting on the cohomology of $R(H_2)$. By a standard result of homological algebra, there is thus a homotopies $p_1$ on the resolution of $R(H_1)$ such that $f' \otimes 1 - 1 \otimes f + \partial p_1 + p_1 \partial$ acts trivially modulo $h$ on the complex itself, and a similar homotopy $p_2$ for $R(H_2)$. Then $p_2 \otimes 1 + 1 \otimes p_1$ is a homotopy that plays the same role for $f'' \otimes 1 - 1 \otimes f$ acting on the tensor product of these complexes. This shows that $f'' \otimes 1 - 1 \otimes f$ acts by 0 modulo $h$ on the cohomology of $R(H_2) \otimes R(H_1)$, so $H_2 \otimes L H_1 \in D^b(\mathcal{H}_{\lambda'}^{\mathcal{A}})$. \hfill \Box

**Definition 6.4** For any $\lambda, \lambda' \in H^2(\mathcal{M}; \mathbb{C})$, let $\mathcal{H}_{\lambda'}^{\mathcal{A}}$ be the category of $\mathcal{D}_{\lambda'} - \mathcal{D}_{\lambda}$ bimodules $\mathcal{H}$ with “thick classical limits” that are scheme-theoretically supported on the Steinberg

\footnote{More precisely, $H$ should be a module over $A_{\lambda'} \otimes A_\lambda^{\text{op}}$, where we consider the completed tensor product over $\mathbb{C}((h^{-1/n}))$. The completion will be implicit in all definitions of bimodules, algebraic or geometric, in the rest of this section. 51
Theorem 6.5 For every \( \lambda, \lambda' \), we have \( \mathbb{R}\Gamma_\Sigma(D^b(\chi, \textbf{HC}^g_\lambda)) \subset D^b(\chi, \textbf{HC}^a_\lambda) \). If \( A_\lambda \) and \( A_{\lambda'} \) have finite global dimension\(^{17} \) \( \mathbb{L}\text{Loc}(D^b(\chi, \textbf{HC}^g_\lambda)) \subset D^b(\chi, \textbf{HC}^g_\lambda) \).

**Proof:** Let \( \mathcal{H} \) be an object in \( \chi, \textbf{HC}^g_\lambda \), and let \( \mathcal{H}(0) \subset \mathcal{H} \) be a lattice satisfying the required condition. For every \( m \), we have a long exact sequence showing that \( H^p(\mathcal{M}; \mathcal{H}(0)/\mathcal{H}(-mn)) \) is an extension of a submodule of \( H^p(\mathcal{M}; \mathcal{H}(0)/\mathcal{H}(-(m - 1)n)) \) and quotient of \( H^p(\mathcal{M}; \mathcal{H}(0)/\mathcal{H}(-n)) \).

Thus, \( H^p(\mathcal{M}; \mathcal{H}(0)/\mathcal{H}(-mn)) \) has an \( m \) step filtration compatible with \( \mathcal{H}(0) \supset \mathcal{H}(-n) \supset \cdots \) such that elements of \( A_{\lambda'} \otimes A_\lambda^{op} \) of the form \( \tilde{f}_{\lambda'} \otimes 1 - 1 \otimes \tilde{f}_\lambda \) act trivially on the associated graded. Since \( H^p(\mathcal{M}; \mathcal{H}(0)) = \lim_{\to} H^p(\mathcal{M}; \mathcal{H}(0)/\mathcal{H}(-mn)) \) by Lemma 4.10, we have an induced filtration on this group such that \( \tilde{f}_{\lambda'} \otimes 1 - 1 \otimes \tilde{f}_\lambda \) acts trivially modulo \( h \). This shows that the cohomology of \( \mathbb{R}\Gamma_\Sigma(\mathcal{H}) \) is Harish-Chandra as well.

Now let \( \mathcal{H} \) be an object of \( \chi, \textbf{HC}^a_\lambda \) and put \( \mathcal{H} := \mathbb{L}\text{Loc}(\mathcal{H}) \). A filtration of \( \mathcal{H} \) induces a lattice in \( H^p(\mathcal{H}) \). For any \( \tilde{f} \in \Gamma(\mathcal{M}; \mathcal{F}(0)) \), we have that

\[
(\tilde{f}_{\lambda'} \otimes 1 - 1 \otimes \tilde{f}_\lambda) \cdot R(\mathcal{H}) \subset h \cdot R(\mathcal{H});
\]

thus, on any projective resolution, the map induced by \( (\tilde{f}_{\lambda'} \otimes 1 - 1 \otimes \tilde{f}_\lambda) \) is null-homotopic mod \( h \); this implies that our lattice in \( H^p(\mathcal{H}) \) has the required property. \( \square \)

Corollary 6.6 If derived localization holds at \( \lambda' \) and \(-\lambda\), then \( \mathbb{L}\text{Loc} \) and \( \mathbb{R}\Gamma_\Sigma \) are inverse equivalences between \( D^b(\chi, \textbf{HC}^a_\lambda) \) and \( D^b(\chi, \textbf{HC}^g_\lambda) \). If localization holds at \( \lambda' \) and \(-\lambda\), then \( \text{Loc} \) and \( \Gamma_\Sigma \) are inverse equivalences between \( \chi, \textbf{HC}^a_\lambda \) and \( \chi, \textbf{HC}^g_\lambda \).

Consider the convolution product

\[
\ast : D^b(\mathcal{D}_{\lambda'}^{op} \mathcal{D}_{\lambda'}^{op} \text{-mod}) \times D^b(\mathcal{D}_{\lambda}^{op} \mathcal{D}_{\lambda}^{op} \text{-mod}) \to D^b(\mathcal{D}_{\lambda'}^{op} \mathcal{D}_{\lambda}^{op} \text{-mod}).
\]

defined by the formula

\[
\mathcal{H}_1 \ast \mathcal{H}_2 := (p_{13})_\ast (p_{12}^* \mathcal{H}_1 \otimes_{\mathcal{D}_{\lambda'}} p_{23}^* \mathcal{H}_2),
\]

\(^{17}\)The finite global dimension hypothesis is truly necessary. By [MNa] 1.1, if \( A_\lambda \) does not have finite global dimension, the derived localization \( \mathbb{L}\text{Loc}(A_\lambda) \) as a bimodule is not bounded and thus not in \( D^b(\chi, \textbf{HC}^g_\lambda) \).
where $p_{ij}$ is one of the three projections from $\mathfrak{M} \times \mathfrak{M} \times \mathfrak{M}$ to $\mathfrak{M} \times \mathfrak{M}$.

**Proposition 6.7** Suppose that derived localization holds for $\lambda, \lambda', -\lambda'$, and $-\lambda''$. The derived sections functor $\mathcal{R} \Gamma_S$ intertwines the convolution of bimodules with derived tensor product. That is, given Harish-Chandra bimodules $\mathcal{H}_1 \in D^b(\mathcal{H}^\mathfrak{S}_\lambda)$ and $\mathcal{H}_2 \in D^b(\mathcal{H}^\mathfrak{S}_\lambda)$, we have an isomorphism

$$\mathcal{R} \Gamma_S(\mathcal{H}_1 * \mathcal{H}_2) \cong \mathcal{R} \Gamma_S(\mathcal{H}_1) \otimes \mathcal{R} \Gamma_S(\mathcal{H}_2).$$

In particular, if $\lambda = \lambda' = \lambda''$ and derived localization holds for $\pm \lambda$, then the derived localization and sections functors are inverse equivalences of tensor categories.

**Proof:** The complex of modules $\mathcal{R} \Gamma_S(\mathcal{H}_1)$ has a free resolution over $A_{\lambda''} \otimes (A_{\lambda'})^\text{op} = \Gamma_S(D_{\lambda''} \boxtimes D_{\lambda'})$

of the form

$$\cdots \to A_{\lambda''} \otimes U_1 \otimes A_{\lambda'} \to A_{\lambda''} \otimes U_0 \otimes A_{\lambda'} \to \cdots , \quad (12)$$

and similarly $\mathcal{R} \Gamma_S(\mathcal{H}_2)$ has a free resolution over $A_{\lambda'} \otimes A_{\lambda'}^\text{op} = \Gamma_S(D_{\lambda'} \boxtimes D_{\lambda'})$

$$\cdots \to A_{\lambda'} \otimes V_1 \otimes A_{\lambda} \to A_{\lambda'} \otimes V_0 \otimes A_{\lambda} \to \cdots . \quad (13)$$

Since derived localization holds, the sheaves $\mathcal{H}_1$ and $\mathcal{H}_2$ have resolutions

$$\cdots \to D_{\lambda''} \otimes U_1 \otimes D_{\lambda'} \to D_{\lambda''} \otimes U_0 \otimes D_{\lambda'} \to \cdots$$

$$\cdots \to D_{\lambda'} \otimes V_1 \otimes D_{\lambda} \to D_{\lambda'} \otimes V_0 \otimes D_{\lambda} \to \cdots .$$

Thus, the convolution $\mathcal{H}_1 * \mathcal{H}_2$ is given by the complex

$$\cdots \to \bigoplus_{i+j=k+1} D_{\lambda''} \otimes U_i \otimes A_{\lambda'} \otimes V_j \otimes D_{\lambda} \to \bigoplus_{i+j=k} D_{\lambda''} \otimes U_i \otimes A_{\lambda'} \otimes V_j \otimes D_{\lambda} \to \cdots . \quad (14)$$

The sections of (14) is the complex

$$\cdots \to \bigoplus_{i+j=k+1} A_{\lambda''} \otimes U_i \otimes A_{\lambda'} \otimes V_j \otimes A_{\lambda} \to \bigoplus_{i+j=k} A_{\lambda''} \otimes U_i \otimes A_{\lambda'} \otimes V_j \otimes A_{\lambda} \to \cdots . \quad (15)$$

This is also the tensor product of the complexes (12) and (13), so this shows that the convolutions and tensor products agree.

Following Căldăraru and Willerton [CW10], we define a 2-category $\text{Qua}^k$ where

- objects are elements of $H^2(\mathfrak{M}; \mathbb{C})$

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• 1-morphisms from $\lambda$ to $\lambda'$ are objects of $D^b(\lambda \cdot HC^e_\lambda)$ with composition given by $\star$, and
• 2-morphisms are the usual morphisms in $D^b(\lambda \cdot HC^e_\lambda)$.

Similarly, we can define a 2-category $Qua^g$ where the 1-morphisms are objects of $D^b(\lambda' \cdot HC^g_\lambda)$ with composition given by tensor product.

Let $Cat$ denote the 2-category of all categories, and consider the functors

$$F^g: Qua^g \to Cat$$

and

$$F^a: Qua^a \to Cat$$

taking $\lambda$ to $D(D_\lambda \cdot Mod)$ and $D(A_\lambda \cdot Mod)$, respectively. On 1-morphisms, $F^g$ takes an object $H$ to the functor given by convolution with $H$, defined exactly as in Equation (11). Similarly, $F^a$ takes an object $H$ to the functor given by tensor product with $H$.

Let $\mathcal{L}_0 \subset \mathcal{M}_0$ be an $\mathbb{S}$-equivariant closed subscheme, and let $\mathcal{L} \subset \mathcal{M}$ be its scheme-theoretic preimage. We would like to use $\mathcal{L}_0$ and $\mathcal{L}$ to define subcategories of $A_\lambda \cdot mod$ and $D_\lambda \cdot mod$ in a way that is analogous to the definitions of algebraic and geometric Harish-Chandra bimodules (Definitions 6.2 and 6.4). In fact, those definitions will specialize to these when $\mathcal{M}$ is replaced by $\mathcal{M} \times \mathcal{M}$ and $\mathcal{L}_0$ is the diagonal subscheme of $\mathcal{M}_0 \times \mathcal{M}_0$.

**Definition 6.8** Let $C^{\mathcal{L}_0}_\lambda$ be the full subcategory of $A_\lambda \cdot mod$ consisting of modules $N$ admitting a filtration with thickened associated graded $gr_n N$ supported on $\mathcal{L}_0$. Equivalently, we require that if the symbol of $a_\lambda \in A_\lambda(k)$ vanishes on $\mathcal{L}_0$, then $a_\lambda \cdot N(m) \subset N(k + m - n)$. Let $D(C^{\mathcal{L}_0}_\lambda)$ be the full subcategory of $D(A_\lambda \cdot Mod)$ consisting of objects with cohomology in $C^{\mathcal{L}_0}_\lambda$.

**Definition 6.9** Let $C^{\mathcal{L}_0}_\lambda$ be the full subcategory of $D_\lambda \cdot mod$ consisting of modules $N$ that have thick classical limits that are scheme-theoretically supported on $\mathcal{L}$. More precisely, we require a lattice $N'(0)$ such that for any section $\tilde{f}$ of $Q$ whose reduction modulo $h$ lies in the ideal sheaf of $\mathcal{L}$, $N'(0)$ is preserved by the action of $h^{-1}\tilde{f}$. Let $D(C^{\mathcal{L}_0}_\lambda)$ be the full subcategory of $D(D_\lambda \cdot Mod)$ consisting of objects with cohomology in $C^{\mathcal{L}_0}_\lambda$.

We then have functors

$$F^g_\mathcal{L}: Qua^g \to Cat$$

and

$$F^a_\mathcal{L}: Qua^a \to Cat$$

taking $\lambda$ to $D(C^{\mathcal{L}_0}_\lambda)$ and $D(C^{\mathcal{L}_0}_\lambda)$, respectively.

**Example 6.10** Suppose that $\mathcal{L}_0 \subset \mathcal{M}$ is the unique $\mathbb{S}$-fixed point; then $\mathcal{L} = \nu^{-1}(0)$ is the core of $\mathcal{M}$ (Remark 2.6), possibly with a non-reduced scheme structure. If the weight $n$ of the symplectic form is equal to 1, then $\mathcal{L}$ is Lagrangian, and $C^{\mathcal{L}_0}_\lambda$ is the category of finite-dimensional $A_\lambda$-modules. When $n$ is greater than 1, the core may be too small, in which case $C^{\mathcal{L}_0}_\lambda$ will be zero. For example, if $\mathcal{M}$ is the Hilbert scheme of points on $\mathbb{C}^2$ and $\mathbb{S}$ acts by scaling
\( \mathbb{C}^2 \) (with \( n = 2 \)), then the core is the punctual Hilbert scheme, which has dimension one less than half the dimension of \( \mathcal{M} \).

**Example 6.11** Suppose that \( \mathcal{M} \) is equipped with a Hamiltonian action of \( \mathbb{T} := \mathbb{C}^\times \) that commutes with the action of \( \mathcal{S} \) and has finite fixed point set \( \mathcal{M}^\mathbb{F} \), and consider the Lagrangian subvariety

\[
\mathcal{L}_0 := \left\{ p \in \mathcal{M}_0 \mid \lim_{t \to 0} t \cdot p \text{ exists} \right\}.
\]

In this case, \( C^\mathcal{M}_\lambda \) is the category of finitely generated \( A_\lambda \)-modules that are locally finite for the action of \( A^+_\lambda \), where \( A^+_\lambda \) is the subring of \( A_\lambda \) consisting of elements with non-negative \( T \)-weight. This is an analogue of a block of BGG category \( \mathcal{O} \), and will be the primary object of study in our forthcoming paper [BLPW] with Licata.

More precisely, let \( \mathcal{M} = T^*(G/B) \) and let \( \rho \in H^2(\mathcal{M}; \mathbb{C}) \) be half of the Euler class of the canonical bundle. Then the ring \( A_{\lambda+\rho} \) is a central quotient of \( U(\mathfrak{g}) \), and \( C^\mathcal{M}_{\lambda+\rho} \) is the category of finitely generated, \( U(\mathfrak{b}) \)-locally finite \( U(\mathfrak{g}) \)-modules with same central character as the Verma module \( V_\lambda \) with highest weight \( \lambda \), where \( H^2(\mathcal{M}; \mathbb{C}) \) is identified with the space of weights \( (h) \) of a Cartan subalgebra via the Chern class map. When \( \lambda \) is a regular integral weight, this category is equivalent in a non-obvious way to the block \( \mathcal{O}_\lambda \) of BGG category \( \mathcal{O} \) [Soe86].

### 6.2 Characteristic cycles

Let \( \mathcal{D} \) be a quantization of \( \mathcal{M} \), and let \( \mathcal{N} \in D^b(\mathcal{D}\text{-mod}) \) be an object of the bounded derived category. We have isomorphisms

\[
\mathcal{H}om^\bullet_{\mathcal{D}}(\mathcal{N}, \mathcal{N}) \cong \mathcal{H}om^\bullet_{\mathcal{D}}(\mathcal{N}, \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{D}_\Delta \otimes_{\mathcal{D}_D^\text{opp}} (\mathcal{N} \otimes \mathcal{H}om^\bullet_{\mathcal{D}}(\mathcal{N}, \mathcal{D})),
\]

and evaluation defines a canonical map to the Hochschild homology

\[
\mathcal{H} \mathcal{H}(\mathcal{D}) := \mathcal{D}_\Delta \otimes_{\mathcal{D}_D^\text{opp}} \mathcal{D}_\Delta.
\]

All this is completely general, and holds in both the Zariski and the classical topology. In the classical topology, we also have an isomorphism \( \mathcal{H} \mathcal{H}(\mathcal{D}^{\text{an}}) \cong C_{\mathfrak{m}_\Delta}[[\dim \mathfrak{m}]]((h)) \) by [KS 6.3.1]. (This is a local calculation, so it suffices to check for the Weyl algebra, where it follows from a Koszul resolution.)

We define the **characteristic cycle**

\[
\text{CC}(\mathcal{N}) \in H^0(\mathcal{H} \mathcal{H}(\mathcal{D}^{\text{an}})) \cong H^{\dim \mathfrak{m}}(\mathcal{M}; \mathbb{C}((h)))
\]

to be the image of \( id \in H^0(\mathcal{H} \mathcal{H}(\mathcal{D}^{\text{an}})) \) along this map. More generally, if \( \mathcal{N} \) is supported on a subvariety \( j: \mathcal{L} \hookrightarrow \mathcal{M} \), then we may consider the identity map of \( \mathcal{N}^{\text{an}} \) to be a
section of \( j^! \text{Hom}_D^\bullet (N^\text{an}, N^\text{an}) \). Applying our map then gives us a class in

\[
\text{CC}(N) \in H^0(j^! \mathcal{H}(D^\text{an})) \cong H^2_\Sigma \mathfrak{M} (\mathfrak{N}; \mathbb{C}(h))
\]

Our abuse of notation is justified by the fact that this class is functorial for inclusions of subvarieties. If we replace the conical symplectic resolution \( \mathfrak{M} \) with the product \( \mathfrak{M} \times \mathfrak{M} \), then this construction associates to a Harish-Chandra bimodule \( \mathcal{H} \in D^b(\chi \mathbf{HC}_k) \) a class \( \text{CC}(\mathcal{H}) \in H^2 \dim \mathfrak{M} (\mathfrak{M} \times \mathfrak{M}; \mathbb{C}(h)) \).

Kashiwara and Schapira [KS 7.3.5] show that the characteristic cycle of a holonomic \( D \)-module (that is, one with Lagrangian support) may be computed in terms of its classical limit.

**Proposition 6.12 (Kashiwara and Schapira)** If \( N \in D\text{-mod} \) is supported on a Lagrangian subvariety \( \mathcal{L} \) with components \( \{ \mathcal{L}_i \} \), then for any \( D(0)\)-lattice \( \mathcal{N}(0) \subset N \),

\[
\text{CC}(N) = \sum_i \text{rk}_{\mathcal{L}_i}(N(0)/N(-1)) \cdot [\mathcal{L}_i] \in H^2 \dim \mathfrak{M} (\mathfrak{M}; \mathbb{C}) \subset H^2 \dim \mathfrak{M} (\mathfrak{M}; \mathbb{C}(h)),
\]

where \( \text{rk}_{\mathcal{L}_i} \) denotes the rank at the generic point of \( \mathcal{L}_i \).

We can also take characteristic cycles in families for modules over quantizations of twistor deformations. For \( \eta \in H^2(\mathfrak{M}; \mathbb{C}) \), let \( \mathcal{M}_\eta \rightarrow \mathbb{A}^1 \) be the twistor deformation defined in Section 2.1 with quantization \( D \) extending \( D \). Let \( N \) be a good \( D \)-module, and consider the image of the identity via the natural morphisms

\[
\text{Hom}_D^\bullet (N, N) \cong \text{Hom}_D^\bullet (N, \mathcal{L}) \otimes_D N \cong \mathcal{L} \otimes_{\mathcal{L} \otimes \mathfrak{M}} \mathfrak{M} \otimes_{\mathfrak{M} \otimes \mathfrak{M}} \text{Hom}_D^\bullet (N, D)
\]

\[
\cong \mathcal{L} \otimes_{\mathcal{L} \otimes \mathfrak{M}} \mathfrak{M} \otimes_{\mathfrak{M} \otimes \mathfrak{M}} \text{Hom}_D^\bullet (N, D) \cong \pi^{-1} \mathfrak{S}_{\mathbb{A}^1} \dim \mathfrak{M} ((h)).
\]

This defines a class in relative cohomology \( \text{CC}(N) \in H^2 \dim \mathfrak{M} (\mathcal{M}_\eta/\mathbb{A}^1; \mathbb{C}(h))) \) for any Lagrangian \( \mathcal{L} \supset \text{Supp}(N) \). If we let \( \mathcal{L} = \mathfrak{M} \cap \mathcal{L} \), then we have a natural restriction map

\[
H^2 \dim \mathfrak{M} (\mathcal{M}_\eta/\mathbb{A}^1; \mathbb{C}(h))) \rightarrow H^2 \dim \mathfrak{M} (\mathfrak{M}; \mathbb{C}(h)))
\]

given by dividing by the coordinate \( t \) on \( \mathbb{A}^1 \). We also have a natural functor of restriction from \( D\text{-mod} \rightarrow D\text{-mod} \) given by \( N|_{\mathfrak{M}} = N \otimes_{\mathfrak{M}[t]} \mathbb{C} \). The following lemma says that these operations are compatible.

**Lemma 6.13** If \( N \) is a good \( D \)-module, then \( \text{CC}(N|_{\mathfrak{M}}) = \text{CC}(N)|_{\mathfrak{M}} \).

**Proof:** Consider the complex [16] of \( \pi^{-1} \mathfrak{S}_{\mathbb{A}^1} \) modules, and take the derived tensor product with \( \mathbb{C} \) over \( \mathbb{C}[t] \). We claim that we obtain corresponding sequence for \( N|_{\mathfrak{M}} \). That is, we
obtain
\[ \text{Hom}^\bullet_{\mathcal{D}}(\mathcal{N}|_{\mathcal{M}}, \mathcal{N}|_{\mathcal{M}}) \cong \text{Hom}^\bullet_{\mathcal{D}}(\mathcal{N}|_{\mathcal{M}}, \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{N}|_{\mathcal{M}} \cong \mathcal{D}_{\Delta} \otimes_{\mathcal{D}_{\Delta} \otimes \mathcal{D}} \text{Hom}^\bullet_{\mathcal{D}}(\mathcal{N}|_{\mathcal{M}}, \mathcal{D}) \rightarrow \mathcal{D}_{\Delta} \otimes_{\mathcal{D}_{\Delta} \otimes \mathcal{D}} \mathcal{D}_{\Delta} \cong \mathbb{C}[\dim \mathcal{M}](h) \]. (17)

It suffices to prove this for \( \mathcal{N} \) locally free. In this case, \( \text{Hom}^\bullet_{\mathcal{D}}(\mathcal{N}, \mathcal{D}) \) is concentrated in degree 0 and is itself locally free, so the statement is clear.

Thus \( \text{CC}(\mathcal{N}|_{\mathcal{M}}) \) can be obtained as the image of the identity under the map (17). By definition \( \text{CC}(\mathcal{N}|_{\mathcal{M}}) \) is the image of the identity under (17), so we are done. \( \square \)

Consider the category \( K(\mathfrak{z}) \) with objects \( H^2(\mathfrak{m}; \mathbb{C}) \) and morphisms \( H^2_{\dim \mathfrak{m}}(\mathfrak{m} \times \mathfrak{m}; \mathbb{C}) \) between any two objects, with composition given by the convolution structure defined at the beginning of this section. We also have a category \( K(\mathbf{HC}^\mathfrak{g}) \) with objects \( H^2(\mathfrak{m}; \mathbb{C}) \) and morphisms \( K(\lambda \mathbf{HC}^\mathfrak{g}) \) from \( \lambda \) to \( \lambda' \), with composition given by convolution; this is simply the decategorification of the 2-category defined in the previous section.

**Proposition 6.14** The characteristic cycle map defines a functor \( K(\mathbf{HC}^\mathfrak{g}) \rightarrow K(\mathfrak{z}) \).

**Proof:** The fact that the characteristic cycle of a morphism in \( K(\mathbf{HC}^\mathfrak{g}) \) is an element of \( H^2_{\dim \mathfrak{m}}(\mathfrak{m} \times \mathfrak{m}; \mathbb{C}) \) rather than \( H^2_{\dim \mathfrak{m}}(\mathfrak{m} \times \mathfrak{m}; \mathbb{C}(h)) \) follows from Proposition 6.12. Since the map \( \mathfrak{z} \times_{\mathfrak{m}} \mathfrak{z} \rightarrow \mathfrak{z} \) is proper, the rest of the proposition follows from [KS, 6.5.4] and the fact that the functor \((-)_{\text{an}}^{\text{an}}\) is monoidal and preserves Hom-spaces. \( \square \)

Now fix a subvariety \( \mathcal{L}_0 \subset \mathcal{M}_0 \), and let \( \mathcal{L} \subset \mathcal{M} \) be its scheme-theoretic preimage as in Section 6.1. We assume for convenience that \( \mathcal{L} \) is Lagrangian. Consider the functor

\[ G_{\mathcal{L}} : K(\mathbf{HC}^\mathfrak{g}) \rightarrow \text{Ab} \]

taking:

- The class \( \lambda \) to \( K(\mathcal{C}^\mathfrak{g}_{\lambda}) \), the Grothendieck group of objects in \( \mathcal{C}^\mathfrak{g}_{\lambda} \) with finitely generated cohomology concentrated in finitely many degrees. Note that by its definition, \( \mathcal{C}^\mathfrak{g}_{\lambda} \) may not be a Serre subcategory, in which case we consider the subgroup of the Grothendieck group of all holonomic \( \mathcal{D} \)-modules generated by the objects in \( \mathcal{C}^\mathfrak{g}_{\lambda} \).

- The class \( [\mathcal{H}] \in K(\lambda \mathbf{HC}^\mathfrak{g}_{\lambda}) \) to the convolution operator

\[ [\mathcal{H}] * - : K(\mathcal{C}^\mathfrak{g}_{\lambda}) \rightarrow K(\mathcal{C}^\mathfrak{g}_{\lambda}) \]

defined by the formula

\[ [\mathcal{N}] * \alpha := (p_{13})_* (p_{12}^* [\mathcal{N}] : p_{23}^* \alpha). \]
We also have a functor

$$H_L: K(HC^g) \to \text{Ab}$$

taking every object $\lambda$ to $H^0_L(H_M; \mathbb{C})$, where the map on morphisms is defined by the convolution action of $H^0_2(H_M \times H_M; \mathbb{C})$ on $H^0_L(H_M; \mathbb{C})$.

Proposition 6.15 The characteristic cycle map

$$CC: K(C^g_L) \to H^0_L(H_M; \mathbb{C})$$

defines a natural transformation from $G_L$ to $H_L$. That is, for all $H \in D^b(H_M; g)$ and $N \in D^b(C^g_L)$,

$$CC(H) \star CC(N) = CC(H \star N).$$

Proof: Since the map $3 \times 3 \rightarrow \mathcal{L}$ is proper, this follows immediately from [KS, 6.5.4].

Thus, these bimodules provide a natural categorification of the convolution algebra of a symplectic singularity, and at least certain of its natural convolution modules. Of course, the characteristic cycle maps need not be isomorphisms, but in many contexts, they are.

Example 6.16 In the case where $\mathcal{M} = T^*(G/B)$, the category $\lambda HC^g_\lambda$ is equivalent to the category of regular twisted D-modules on $G/B \times G/B$ for the twist $(\lambda + \rho, - \lambda + \rho)$ which are smooth on diagonal $G$-orbits; as long as $\lambda + \rho$ is integral, this is the same as the category of perverse sheaves smooth along the same stratification. The fact that these categorify the symmetric group (and thus, implicitly, that CC is an isomorphism in this case) goes back at least as far as [Spr82]. This perspective is Koszul dual to the usual categorification of the symmetric group by projective functors [BG99, 5.16].

Example 6.17 In the case where $\mathcal{M}$ is a hypertoric variety, the map from $K(\lambda HC^g_\lambda)$ to $H^0_2(H_M \times H_M; \mathbb{C})$ is surjective by [BLPW12, 7.11], which allows us to conclude that every irreducible representation of the convolution algebra remains irreducible over $K(HC^g)$. The dimensions of these representations are computed in [PW07] to be $h$-numbers of various matroids.

Example 6.18 In the case of Nakajima quiver varieties, it is more natural to consider all quiver varieties associated to a highest weight $\mu$ jointly, and thus define a 2-subcategory $\text{Qua}(\mu)$ of modules over the exterior products of quantizations of quiver varieties associated to $\lambda$ and possibly different dimension vectors.

However, even with different dimension vectors, we still have a notion of “diagonal” in the product of two quiver varieties with the same highest weight. The affinization of a quiver variety is the moduli space of semi-simple representations of the pre-projective algebra of a
given dimension, and we say a pair of such representations lies in the **stable diagonal** if they become isomorphic after the addition of trivial representations. We can define a 2-categories \( \text{HC}^\otimes(\mu) \) by replacing the diagonal and its vanishing ideal with that of the stable diagonal. The third author relates this construction to works by Cautis and Lauda and Nakajima.

**Proposition 6.19 ([Web, Thm. A])** There is a 2-functor from the version of the 2-quantum group \( U \) defined by Cautis and Lauda [CL] to \( \text{HC}^\otimes(\mu) \) with the property that the induced map of K-groups is exactly the geometric construction of \( U(\mathfrak{g}) \) defined by Nakajima [Nak98].

### 6.3 Twisting bimodules

For the rest of this paper, we will assume that the Picard group of \( \mathcal{M} \) is torsion-free, so that a line bundle is determined by its Euler class in \( H^2(\mathcal{M}; \mathbb{C}) \). This assumption is not strictly necessary, but it greatly simplifies the notation (see Remark 6.21).

Consider the universal Poisson deformation \( \mathcal{M} \) of \( \mathcal{M} \). Let \( \mathcal{L} \) be a line bundle on \( \mathcal{M} \), let \( \mathcal{L} \) be its restriction to \( \mathcal{M} \), and let \( \gamma \in H^2(\mathcal{M}; \mathbb{Z}) \cong H^2(\mathcal{M}; \mathbb{Z}) \) be the Euler class of \( \mathcal{L} \) or \( \mathcal{L} \). Let \( \gamma_\mathcal{L}_0 \) be the quantization of \( \mathcal{L} \) constructed in Proposition 5.2, and let \( \gamma_\mathcal{L}' := \gamma_\mathcal{L}_0[\hbar^{-1/\gamma}] \). This is a right \( \mathcal{D} \)-module and a left module over \( \mathcal{D}_\gamma \), the quantization with period \( I + h\gamma \). Then \( \Gamma_S(\mathcal{M}; \gamma_\mathcal{L}'_0) \) is a family over \( H^2(\mathcal{M}; \mathbb{C}) \) via the right action of \( \mathcal{A} = \Gamma_S(\mathcal{M}; \mathcal{D}) \).

Recall the map \( c: \mathbb{C}[H^2(\mathcal{M}; \mathbb{C})] \to \Gamma(\mathcal{M}; \mathcal{D}) \) from Section 3.3 and the fact that \( h^{-1}c(x) \in \mathcal{A} \) for all \( x \in H^2(\mathcal{M}; \mathbb{C})^\ast \). Also recall that, by Proposition 3.6, the specialization of \( \mathcal{A} \) at \( h^{-1}c(x) = \lambda(x) \) for all \( x \in H^2(\mathcal{M}; \mathbb{C})^\ast \) is isomorphic to \( A_\lambda \).

**Definition 6.20** Let \( \lambda_+ \gamma \mathcal{T}_\lambda \) denote the \( A_{\lambda_+ \gamma} - A_\lambda \) bimodule that we obtain by specializing \( \Gamma_S(\mathcal{M}; \gamma_\mathcal{L}_0') \) at \( h^{-1}c(x) = \lambda(x) \) for all \( x \in H^2(\mathcal{M}; \mathbb{C})^\ast \).

**Remark 6.21** The purpose of the assumption at the beginning of this section was to ensure that the bimodule \( \lambda_+ \gamma \mathcal{T}_\lambda \) is actually determined by \( \lambda \) and \( \gamma \); without the assumption, the bimodule would depend on an additional choice of a line bundle with Euler class \( \gamma \).

**Proposition 6.22** The bimodule \( \lambda_+ \gamma \mathcal{T}_\lambda \) is Harish-Chandra.

**Proof:** By definition, \( \lambda_+ \gamma \mathcal{T}_\lambda \) is a specialization of \( \Gamma_S(\mathcal{M}; \gamma_\mathcal{L}_0') \). It carries a natural filtration, where \( \lambda_+ \gamma \mathcal{T}_\lambda(m) \) is the same specialization of \( \Gamma_S(\mathcal{M}; h^{-m/\gamma} \gamma_\mathcal{L}_0[h^{1/\gamma}]) \). We claim that the associated graded module with respect to this filtration is scheme-theoretically supported on the diagonal.

To see this, consider a function \( f \in \mathbb{C}[\mathcal{M}] \) of \( \mathfrak{S} \)-weight \( \ell \). We can choose a lift \( \tilde{f} \in \Gamma_S(\mathcal{D}(\ell)) \) so that its image in \( \text{gr} \Gamma_S(\mathcal{D}) \cong \mathbb{C}[\mathcal{M}] \) restricts to \( f \) on \( \mathcal{M} \). Let \( \mathcal{D}_\gamma \) be the quantization of \( \mathcal{M} \) with period \( \gamma \); since \( \text{gr} \Gamma_S(\mathcal{D}_\gamma) \cong \text{gr} \Gamma_S(\mathcal{D}) \), we can choose a lift \( \tilde{f}_\gamma \in \Gamma_S(\mathcal{D}_\gamma(\ell)) \) of \( f \) similarly.

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\(^{18}\)We note that all quantizations of \( \mathcal{M} \) are isomorphic as sheaves of algebras, but they are not isomorphic as sheaves of \( \pi^{-1}\mathfrak{S}_{H^2(\mathcal{M}; \mathbb{C})} \)-algebras.
To show that $f \otimes 1 - 1 \otimes f$ annihilates $\text{gr}_m(\lambda_1 T_\lambda)$, it is sufficient to show that $\tilde{f}_\gamma \otimes 1 - 1 \otimes \tilde{f}$ takes $\Gamma_S(\mathcal{M}; h^{-m/n} \mathcal{F}_0(h^{1/n}))$ to $\Gamma_S(\mathcal{M}; h^{1-\ell/m} \gamma \mathcal{F}_0(h^{1/n}))$. This follows from the fact that $\gamma \mathcal{F}_0$ is the quantization of a line bundle on $\mathcal{M}$, so the left action of $\tilde{f}_\gamma$ and the right action of $\tilde{f}$ agree modulo $h$.

The following two propositions are bimodule analogues of Corollary 3.9 and Proposition 3.10. Since their proofs are essentially identical, we omit them.

**Proposition 6.23** Let $\mathcal{M}$ and $\mathcal{M}'$ be two conical symplectic resolutions of the same cone. Fix elements $\lambda, \gamma \in H^2(\mathcal{M}; \mathbb{C}) \cong H^2(\mathcal{M}'; \mathbb{C})$, where $\gamma$ is the Euler class of a line bundle on $\mathcal{M}$ or its strict transform on $\mathcal{M}'$. The isomorphism of rings in Corollary 3.9 induces an isomorphism of bimodules $\lambda_+ T_\lambda \cong \lambda_+ T'_\lambda$.

**Proposition 6.24** For any $\lambda, \gamma \in H^2(\mathcal{M}; \mathbb{C}) \cong H^2(\mathcal{M}'; \mathbb{C})$, where $\gamma$ is the Euler class of a line bundle on $\mathcal{M}$, and any $w \in W$, the isomorphisms of Proposition 3.10 induce isomorphisms of bimodules $\lambda_+ T_\lambda \cong w(\lambda_+ \gamma) T_{w, \lambda}$.

We would like to have an analogue of Proposition 3.6 as well, though an extra hypothesis is needed. The following proposition gives a natural map from $\lambda_+ T_\lambda$ to $\Gamma_S(\mathcal{M}; \lambda_+ T'_\lambda)$, and gives a sufficient (though not necessary) condition for showing that it is an isomorphism. (Note that it is always injective.)

**Proposition 6.25** There is a natural map from the bimodule $\lambda_+ T_\lambda$ to $\Gamma_S(\mathcal{M}; \lambda_+ T'_\lambda)$. If $H^1(\mathcal{M}; \lambda_+ T'_\lambda) = 0$, then this map is an isomorphism.

**Proof:** The pullback of $\gamma \mathcal{F}_0'$ along the map $\Delta \to H^2(\mathcal{M}; \mathbb{C}) \times \Delta$ given by $h \mapsto (h\lambda, h)$ is a quantization of $L$. By the uniqueness of the quantized line bundles constructed in Proposition 5.2, this pullback is isomorphic to $\lambda_+ T'_\lambda$. Since $\lambda_+ T_\lambda$ is obtained from $\gamma \mathcal{F}_0'$ by first taking sections and then specializing, this defines the required map.

Now suppose that $H^1(\mathcal{M}; \lambda_+ T'_\lambda) = 0$. To prove that our map is surjective, we factor the pullback into two steps. Choose $\nu \in H^2(\mathcal{M}; \mathbb{C})$ with $\mathcal{M}_\nu(\infty)$ affine. Let $\lambda_+ T'_\lambda \mathcal{F}(\nu)$ be the bimodule on $\mathcal{M}_\nu$ obtained by pulling $\gamma \mathcal{F}_0'$ back along the map $A^1 \times \Delta \to H^2(\mathcal{M}; \mathbb{C}) \times \Delta$ taking $(t, h)$ to $(t\nu + h\lambda, h)$. Thus $\lambda_+ T'_\lambda \mathcal{F}(\nu)$ is obtained from this sheaf by pulling back further by the map $\Delta \to A^1 \times \Delta$ given by $h \mapsto (0, h)$. Let $\lambda_+ O_\lambda := \Gamma_S(\mathcal{M}_\nu; \lambda_+ T'_\lambda \mathcal{F}(\nu))$. To show that our map is surjective, it will suffice to show that

1. the map from $\Gamma_S(\mathcal{M}_\nu; \gamma \mathcal{F}_0')$ to $\lambda_+ O_\lambda$ is surjective, and
2. the map from $\lambda_+ O_\lambda$ to $\Gamma_S(\mathcal{M}; \lambda_+ T'_\lambda)$ is surjective.
Applying Lemma 4.14, we see that $h$ is impossible by assumption, so $0$ cannot be a root. Thus, $h$ has finite length. In particular, if $0$ is a root of this minimal polynomial, the map

\[ \Gamma_S(\mathcal{M}; \gamma_{\mathcal{R}_0'}) \cong \Gamma_S(\mathcal{N}; \gamma_{\mathcal{R}_0}). \]

Similarly, we define a sheaf $\lambda+\gamma_{\mathcal{R}_0'}^\nu$ on $\mathcal{N}$ obtained from $\lambda+\gamma_{\mathcal{R}_0}^\nu$ by first restricting it to $\mathcal{N}$ and then pushing it forward to $\mathcal{N}$, and we have

\[ \lambda+\gamma O_\lambda \cong \Gamma_S(\mathcal{N}_\nu; \lambda+\gamma_{\mathcal{R}_0'}^\nu). \]

To see that the map from $\Gamma_S(\mathcal{M}; \gamma_{\mathcal{R}_0'})$ to $\lambda+\gamma O_\lambda$ is surjective, it suffices to check that the associated graded is surjective. When we pass to the associated graded, we obtain a map between spaces of sections of two reflexive sheaves on $\mathcal{N}$, namely the classical limits of $\gamma_{\mathcal{R}_0'}$ and $\lambda+\gamma_{\mathcal{R}_0'}^\nu$. The fact that this map is surjective follows from the fact that the second classical limit is the quotient of the first by the ideal sheaf of $\mathcal{N}_\nu \subset \mathcal{N}$, and $\mathcal{N}$ is affine.

We now turn to the second surjectivity statement. Consider the exact sequence

\[ 0 \rightarrow \lambda+\gamma_{\mathcal{R}_0^\nu} \xrightarrow{h^{-1}t} \lambda+\gamma_{\mathcal{R}_0^\nu} \rightarrow \lambda+\gamma T_{\lambda} \rightarrow 0 \]

of sheaves on $\mathcal{M}_\nu$ and its associated long exact sequence

\[ 0 \rightarrow \lambda+\gamma O_\lambda \xrightarrow{h^{-1}t} \lambda+\gamma O_\lambda \rightarrow \Gamma_S(\mathcal{M}_\nu; \lambda+\gamma T_{\lambda}^\nu) \rightarrow H^1(\mathcal{M}_\nu; \lambda+\gamma_{\mathcal{R}_0^\nu})^S \xrightarrow{h^{-1}t} H^1(\mathcal{M}_\nu; \lambda+\gamma_{\mathcal{R}_0^\nu})^S \rightarrow H^1(\mathcal{M}_\nu; \lambda+\gamma T_{\lambda}^\nu)^S \rightarrow \cdots. \]

The surjectivity statement that we need is equivalent (by exactness) to injectivity of the action of $h^{-1}t$ on $H^1(\mathcal{M}_\nu; \lambda+\gamma_{\mathcal{R}_0^\nu})^S$.

Since the generic fiber of $\mathcal{M}_\nu$ is affine, $H^1(\mathcal{M}_\nu; \lambda+\gamma_{\mathcal{R}_0^\nu})^S$ is supported on the fiber over $0$. This bimodule is Harish-Chandra, so its localization has Lagrangian support in $\mathcal{M} \times \mathcal{M}$. Applying Lemma 4.14, we see that $h^{-1}t$ satisfies a polynomial equation on $H^1(\mathcal{M}_\nu; \lambda+\gamma_{\mathcal{R}_0^\nu})^S$, so the bimodule is the sum of finitely many generalized eigenspaces for $h^{-1}t$ and $h^{-1}t$ acts with finite length. In particular, if $0$ is a root of this minimal polynomial, the map $h^{-1}t$ is not surjective (since its stable image is a proper summand), and thus $H^1(\mathcal{M}_\nu; \lambda+\gamma T_{\lambda})$ is not $0$. This is impossible by assumption, so $0$ cannot be a root. Thus, $h^{-1}t$ does act invertibly, so the desired map is surjective. 

\[ \square \]
The following proposition says that derived tensor product with a twisting bimodule does not change the characteristic cycle of the localization. Let $N$ be an object of $D^b(A_\lambda\text{-mod})$, so that $\mathbb{L}\text{Loc}(\lambda+\gamma T^L \otimes N)$ is an object of $D^b(D_{\lambda+\gamma}\text{-mod})$.

**Proposition 6.26** Assume derived localization holds at $\lambda$ and $\lambda + \gamma$. Then we have that

$$\text{CC}(\mathbb{L}\text{Loc}(N)) = \text{CC}(\mathbb{L}\text{Loc}(\lambda+\gamma T^L \otimes N)).$$

**Proof:** As in the proof of Proposition 6.25, choose $\nu \in H^2(\mathfrak{M}; \mathbb{C})$ such that $\mathcal{M}_\nu(\infty)$ is affine, and consider the sheaf $\lambda+\gamma \mathcal{F}^{(\nu)}_\lambda$. At any point $p$ of $\mathbb{A}^1$, the derived functor of base change to the fiber $\pi^{-1}(p\nu)$ over $p$ sends $\lambda+\gamma \mathcal{F}^{(\nu)}_\lambda$ to the derived localization $\mathbb{L}\text{Loc}(\lambda+\gamma O_\lambda/(t-p))$ as a module over a quantization of $\mathcal{M}_\nu \times \mathcal{M}_\nu$, since the module $\lambda+\gamma O_\lambda$ is flat over $\mathbb{A}^1$.

If $p$ is not 0, then the fiber is affine, and $\mathbb{L}\text{Loc}(\lambda+\gamma O_\lambda/(t-p))$ is a line bundle on the diagonal in $\pi^{-1}(p\nu) \times \pi^{-1}(p\nu)$. In particular, the class $\text{CC}(\lambda+\gamma \mathcal{F}^{(\nu)}_\lambda)$ thus must be the class of the diagonal over every non-zero point in $\mathbb{A}^1$. By Lemma 6.13, we thus have that

$$\text{CC}(\mathbb{L}\text{Loc}(\lambda+\gamma T_\lambda)) = \text{CC}(\lambda+\gamma \mathcal{F}^{(\nu)}_\lambda|_{\pi^{-1}(0)}) = [\mathcal{M}_\Delta].$$

By Proposition 6.14, the characteristic cycle map intertwines derived tensor product with convolution. Since convolution with the diagonal is trivial, this implies the desired equality. $\square$

We conclude this section by computing these bimodules explicitly in the case where $\mathfrak{M}$ is a symplectic quotient of a vector space, as in Example 2.2. Let $G$ be a connected reductive algebraic group acting on a vector space $V$ with flat moment map $\mu : T^*V \to \mathfrak{g}^*$; let $\mathfrak{M}$ be the symplectic quotient of $T^*V$ at a generic character $\theta$ of $G$, and suppose that the Kirwan map $K : \chi(\mathfrak{g}) \to H^2(\mathfrak{M}; \mathbb{C})$ is an isomorphism. Let $A_{T^*V}$ be the section ring of the unique quantization of $T^*V$; this is isomorphic to the ring of differential operators on $V$. Fix a quantized moment map $\eta : U(\mathfrak{g}) \to A_{T^*V}$ and an element $\xi \in \chi(\mathfrak{g})$, and let $D_\xi$ be the associated quantization of $\mathfrak{M}$ (Section 3.4) with section ring $A = \Gamma_G(D)$. By Proposition 3.13, we have $A \cong \text{End}_{A_{T^*V}}(Y_\xi)$.

Fix a second character $\xi'$ such that $\xi' - \xi$ integrates to a character of $G$, and consider the $A' - A$ bimodule

$$\text{Hom}(Y_{\xi'}, Y_\xi) \cong \left( A_{T^*V}/A_{T^*V} \cdot (\eta(x) - \xi(x) \mid x \in \mathfrak{g}) \right)^{\xi' - \xi}. \quad (18)$$

By Proposition 5.4, we have a natural map from $\text{Hom}(Y_{\xi'}, Y_\xi)$ to $K_{(\xi')} T_{K(\xi)}$. This map is always injective but it need not be an isomorphism; the restriction to the semistable locus can cause new sections to appear.
Lemma 6.27  If \( \xi' = \xi + m\theta \) for \( m \gg 0 \), then the map from \( \text{Hom}(Y_{\xi'}, Y_{\xi}) \) to \( \kappa_{(\xi')} T_{\kappa(\xi)} \) is an isomorphism.

**Proof:** The associated graded of this map is the natural map from \( \mathbb{C}[\mu^{-1}(0)]_{m\theta} \) to \( \Gamma(M, L_{m\theta}) \), where the subscript in the source indicated the \( S \)-weight space. This map is an isomorphism for sufficiently large \( m \), thus so is our original map. \( \square \)

Remark 6.28 We note that, by Corollary 3.9 and Proposition 6.23, the source and target of the map in Lemma 6.27 (along with the map itself) are independent of the choice of conical symplectic resolution. Thus Lemma 6.27 simply says that our map is an isomorphism when \( \xi \) and \( \xi' \) are sufficiently far apart in any generic direction.

6.4 Twisting functors

By Theorem 2.20, the set \( I \) of isomorphism classes of conical symplectic resolutions of \( M_0 \) is finite. For each \( i \in I \), let \( M_i \) be a representative resolution. By Remark 2.21, the chambers of the hyperplane arrangement \( H \) are in canonical bijection with \( I \times W \), where \( W \) is the Weyl group from Section 2.2. For each pair \((i, w)\), let \( \Pi_{i,w} \subset P_\mathbb{R} \) be the set of parameters \( \lambda \) in the corresponding chamber of \( H \) with the additional property that localization holds at \( w\lambda \) on \( M_i \). Let

\[
\Pi := \bigcup_{i \times W} \Pi_{i,w} \subset P_\mathbb{R}.
\]

By Theorem 4.17(ii), derived localization holds at every element of \( \Pi \) on all of the different conical symplectic resolutions. By Corollary 4.18 derived localization also holds at every point in \( -\Pi \).

Lemma 6.29 If \( w\eta \) is an ample class on \( M_i \), then for any \( \lambda \), the class \( \lambda + k\eta \) lies in \( \Pi_{i,w} \) for all but finitely many \( k \in \mathbb{Z}_{\geq 0} \).

**Proof:** Recall from Remark 2.21 that the chamber of \( H \) indexed by \((i, w)\) is equal to the \( w \) translate of the ample cone of \( M_i \). Since \( w\eta \) is ample on \( M_i \), so is \( w(\lambda + k\eta) \) when \( k \) is sufficiently large. The fact that localization holds at \( w(\lambda + k\eta) \) for large \( k \) follows from Corollary 5.13. \( \square \)

Let \( A_\lambda \) be the invariant section ring of the quantization with period \( \lambda \). (Note that, by Corollary 3.9, the ring \( A_\lambda \) does not depend on the choice of resolution of \( M_0 \).) For any pair of elements \( \lambda, \lambda' \in \Pi \) that differ by an integral class, let

\[
\Phi^{\lambda,\lambda'} : D(A_\lambda \text{-Mod}) \to D(A_{\lambda'} \text{-Mod}) \tag{19}
\]

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be the functor obtained by derived tensor product with the bimodule $\chi T_\lambda$. For any $\lambda \in \Pi$ and $w \in W$, let
\[
\Phi^\lambda_w: D(Aw_\lambda \text{-Mod}) \to D(A_\lambda \text{-Mod})
\] (20)
be the equivalence obtained from the isomorphism of Proposition 3.10. Note that the compatibility in the statement of Proposition 3.10 implies that the composition $\Phi^w_\lambda \circ \Phi^\lambda_w$ is naturally isomorphic to the identity functor.

**Proposition 6.30** Suppose that $\lambda' \in \Pi_{i,w}$. Then the functor $\Phi^{\lambda',\lambda}$ is naturally isomorphic to the composition
\[
D(A_\lambda \text{-Mod}) \xrightarrow{\Phi^\lambda_w} D(Aw_\lambda \text{-Mod}) \xrightarrow{\mathrm{LLoc}_i} D(Dw_\lambda \text{-Mod}) \xrightarrow{w\lambda' T_{w\lambda} \otimes -} D(Dw_{\lambda'} \text{-Mod}) \xrightarrow{\mathrm{R}\Gamma_{\mathcal{S},i}} D(Aw_{\lambda'} \text{-Mod}) \xrightarrow{\Phi^{w_1}_{w'}} D(A_{\lambda'} \text{-Mod}),
\]
where the subscript $i$ on $\mathrm{R}\Gamma_{\mathcal{S}}$ and $\mathrm{LLoc}$ refers to the fact that we are using the resolution $\mathcal{M}_i$.

**Proof:** Since $\lambda' \in \Pi_{i,w}$, localization holds at $w\lambda'$, which implies that the higher cohomology of $w\lambda' T_{w\lambda}$ is trivial. Then Proposition 6.25 tells us that $w\lambda' T_{w\lambda} \cong \mathrm{R}\Gamma_{\mathcal{S},i}(w\lambda' T'_{w\lambda})$, and therefore that $w\lambda' T_{w\lambda} \cong \mathrm{LLoc}_i(w\lambda' T_{w\lambda})$. The proposition follows immediately using Proposition 6.24. \( \square \)

**Corollary 6.31** The functor $\Phi^{\lambda',\lambda}: D(A_{\lambda} \text{-Mod}) \to D(A_{\lambda'} \text{-Mod})$ is an equivalence of triangulated categories inducing an equivalence $D^b(A_{\lambda} \text{-mod}) \to D^b(A_{\lambda'} \text{-mod})$.

**Proof:** The functors $\mathrm{LLoc}_i$ and $\mathrm{R}\Gamma_{\mathcal{S},i}$ are equivalences because both $\lambda$ and $\lambda'$ are in $\Pi$, while $w\lambda' T_{w\lambda} \otimes -$ is biadjoint to $w\lambda' T'_{w\lambda} \otimes -$ by the uniqueness part of Proposition 5.2. Since $\lambda, \lambda' \in \Pi$, $A_\lambda$ has finite global dimension and derived tensor product with a module sends bounded complexes to bounded complexes; similarly, its adjoint is Hom with a bimodule, which also sends bounded complexes to bounded complexes. \( \square \)

**Corollary 6.32** If $\lambda$ and $\lambda'$ lie in the same chamber of $\mathcal{H}$, then $\Phi^{\lambda',\lambda} \circ \Phi^{\lambda',\lambda}$ is naturally isomorphic to the identity functor.

**Proof:** This follows similarly from Propositions 5.2 and 6.30. \( \square \)

Fixing a particular $\lambda \in \Pi$, we define **twisting functors** to be the group of endofunctors of $D(A_{\lambda} \text{-Mod})$ (or of the full subcategory $D^b(A_{\lambda} \text{-mod})$) obtained by composing functors of the form (19) and (20) and their inverses, and we define **pure twisting functors** to be the
subgroup obtained using only functors of the form (19) and their inverses. Note that Corollary 6.32 implies that any such composition that never leaves the chamber in which $\lambda$ lives is trivial. However, when one crosses a wall and then crosses back, one can and does obtain something nontrivial (see Proposition 6.37 for the case of the Springer resolution).

For Lemma 6.33 we adopt the notational convention, introduced in Section 4.3, whereby we fix $\eta \in H^2(\mathcal{M}; \mathbb{Z})$ and $\lambda \in H^2(\mathcal{M}; \mathbb{C})$ and use $k$ in a subscript or superscript in place of $\lambda + k\eta$.

**Lemma 6.33** Suppose that $w\eta$ is very ample on $\mathcal{M}_i$. Then for any natural numbers $k_\ell > k_{\ell-1} > \cdots > k_1 \geq k_0$, there is a natural isomorphism of functors

$$\Phi^{k_\ell,k_0}_{\ell} \simeq \Phi^{k_\ell,k_{\ell-1}}_{\ell-1} \circ \cdots \circ \Phi^{k_1,k_0}_{1}.$$ 

**Proof:** Let $\mathcal{L}$ be the line bundle on $\mathcal{M}_i$ with Euler class $\eta$. For any $k' > k$, the higher cohomology of $\mathcal{L}^{k'-k}$ vanishes. Therefore the higher cohomology of $k' T^k_i$ vanishes as well. By the same argument that we used in the proof of Proposition 6.30, Proposition 6.25 tells us that $k T^k_i \cong R \Gamma_S(k T^k_i)$ as desired. Since $\Phi^{k_\ell,k_0} = k T^k_i \otimes_{A_{i,k_0}} -$, the isomorphism follows. 

Let

$$E := H^2(\mathcal{M}; \mathbb{C}) \setminus \bigcup_{H \in \mathcal{H}} H_{\mathbb{C}}$$

be the complement of the complexification of $\mathcal{H}$. The main theorem of this section says that the fundamental group of $E/W$ acts on our category by twisting functors.

**Theorem 6.34** For any $\lambda \in \Pi$, there is a natural homomorphism from $\pi_1(E/W, [\lambda])$ to the group of twisting functors on $D(A_{\lambda,-}\text{-Mod})$. The subgroup $\pi_1(E, \lambda)$ maps to the group of pure twisting functors.

**Proof:** For each element $(i, w) \in I \times W$, choose an integral class $\eta_{i,w}$ such that $w\eta$ is ample on $\mathcal{M}_i$. By Lemma 6.29 we may choose a natural number $k_{i,w}$ such that $\lambda_{i,w} := \lambda + k_{i,w}\eta_{i,w}$ lies in $\Pi_{i,w}$. The Deligne groupoid of $\mathcal{H}$ is the full sub-groupoid of the fundamental groupoid of $E$ with objects $\{\lambda_{i,w} \mid (i, w) \in I \times W\}$. Note that different choices would lead to a canonically isomorphic groupoid; the only important thing is that we have chosen one representative of each chamber.
The Deligne quiver of a real hyperplane arrangement is the quiver with nodes indexed by chambers and arrows in both directions between any two adjacent chambers. Paris [Par93] proves that the Deligne groupoid is isomorphic to the quotient of the fundamental groupoid of the Deligne quiver obtained by identifying any pair of positive paths of minimal length between the same two nodes. Thus, to construct an action of the Deligne groupoid, it is sufficient to first define an action of the Deligne quiver and then check Paris’s relations.

Recall that the chambers of $H$ are in bijection with $I \times W$. We begin by associating the category $D(A_{\lambda_i,w^{-}\text{-Mod}})$ to the node indexed by $(i,w)$. If the chambers indexed by $(i,w)$ and $(j,v)$ are adjacent, then we assign the functor $\Phi_{\lambda_j,v,\lambda_i,w}$ to the corresponding arrow in the Deligne quiver. We now need to check the relations. By Randell’s presentation of the fundamental group of $E$ [Ran82, Ran85], it is sufficient to compare minimal positive paths between chambers that are opposite among those adjacent to some face $F$ of codimension 2. For any two such chambers, there are exactly two minimal positive paths between them. Suppose we are given two such chambers, labeled by $(i,w)$ and $(i',w')$. Let $H$ be a generic cooriented hyperplane that contains $F$ and bisects both chambers. Figure 1 illustrates a 2-dimensional slice transverse to $F$, so that $F$ appears as a point and $H$ appears as a line; we draw $H$ as a dotted line.

Choose elements $\mu$ and $\nu$ of $\Pi_{i,w}$ that differ from $\lambda_{i,w}$ by an integral class, with $\mu$ on the positive side and $\nu$ on the negative side of $H$. Choose $\mu'$ and $\nu'$ in $\Pi_{i',w'}$ similarly. Let $\mu = \mu_1, \mu_2, \ldots, \mu_n = \mu'$ be colinear integral representatives of all the chambers on the positive side of $H$, and let $\nu = \nu_1, \ldots, \nu_\ell = \nu'$ be colinear representatives of all the chambers on the negative side of $H$. We may arrange these classes such that for all $k$, $\mu_k - \mu_{k+1}$ and $\nu_k - \nu_{k+1}$ both lie in the chamber indexed by $(i,w)$. Put differently, we may assume that $w\mu_k - w\mu_{k+1}$ and $w\nu_k - w\nu_{k+1}$ are both ample on $M_i$. All of this is illustrated in Figure 1.

By Corollary [6.32], we may reduce the theorem to checking that the functors

$$\Phi_{\lambda_{i,w}^{\mu'},\mu'} \circ \Phi_{\mu',\mu-1} \circ \cdots \Phi_{\mu_2,\mu} \circ \Phi_{\mu,\lambda_{i,w}}$$

and

$$\Phi_{\lambda_{i,w}^{\nu'},\nu'} \circ \Phi_{\nu',\nu-1} \circ \cdots \Phi_{\nu_2,\nu} \circ \Phi_{\nu,\lambda_{i,w}}$$

from $D(A_{\lambda_i,w^{-}\text{-Mod}})$ to $D(A_{\lambda_{i,w}^{\mu'}^{-}\text{-Mod}})$ are naturally isomorphic. By Corollary [6.32] and Lemma [6.33], both are equivalent to $\Phi_{\lambda_{i,w}^{\mu'},\lambda_{i,w}}$.

We have now established that $\pi_1(E,\lambda)$ acts on $D(A_\lambda^{-}\text{-Mod})$ via pure twisting functors. To complete the proof of the theorem, we let an element $w \in W$ act on $D(A_\lambda^{-}\text{-Mod})$ by the impure twisting functor $\Phi_w^\lambda \circ \Phi_{\lambda,w^\lambda}$. Using Proposition [6.24], we may conclude that this defines an action of the semidirect product $\pi_1(E,\lambda) \rtimes W \cong \pi_1(E/W, [\lambda])$.\[\]19 A path can travel forward or backward along arrows; a positive path is one that always travels forward.
Remark 6.35 We have already remarked that $D$-$\text{mod}$ (and therefore $A$-$\text{mod}$, when localization holds) may be thought of as a twisted algebraic version of the Fukaya category of $\mathfrak{M}$ (Remark 4.2). In this interpretation, we expect the action in Conjecture 6.34 to be given by parallel transport in the universal deformation, along the lines of the construction in [SS06] for Slodowy slices of type A.

Remark 6.36 As in Section 6.1, we may replace $D(A_\lambda$-$\text{Mod}$) in the statement of Theorem 6.34 with $D(C_L^0)$ for any $S$-equivariant $L_0 \subset M_0$.

If $M$ is a hypertoric variety and $L$ is as in Example 6.11, we obtain the twisting functors studied in [BLPW10, §6] and [BLPW12, 8.4]. To see this, we need to apply Lemma 6.27 and Remark 6.28 because the functors in [BLPW12, 8.4] are defined using the bimodules in Equation (18).

If $M = T^*(G/B)$ and $L$ is as in Example 6.11 then for any regular integral weight $\lambda$, the category $C_{\lambda+\rho}^2$ is is equivalent to $O_\lambda$ by Soergel’s functor. This equivalence identifies the functors we call twisting functors with Arkhipov’s twisting functors [Ark04, AS03]. More precisely, Arkhipov defines a collection of auto-equivalences $\{T_w \mid w \in W\}$ of the category $O_\lambda$ satisfying the relation $T_w \circ T_{w'} \cong T_{ww'}$ whenever the length of $ww'$ is equal to the sum of the lengths of $w$ and $w'$, which means that these functors generate an action of the generalized braid group. In this case the discriminantal arrangement is equal to the Coxeter arrangement for $W$, so the fundamental group $\pi_1(E/W, [\lambda])$ is also isomorphic to the generalized braid group.

Proposition 6.37 Suppose that $\mathfrak{M} = T^*(G/B)$ and let $L$ be as in Example 6.11. If $\lambda \in H^2(\mathfrak{M}; \mathbb{C})$ is regular, integral, and dominant, then Soergel’s equivalence from the block $O_\lambda$ of
BGG category $\mathcal{O}$ to the category $C_{\lambda+\rho}^\mathcal{L}$ intertwines Arkhipov’s twisting action with the twisting action from Theorem 6.34.

**Proof:** We begin by showing that Arkhipov’s twisting functors are uniquely characterized by the following two properties:

- $T_w$ strongly commutes with projective functors [AS03, Lemma 2.1]. That is, for any projective functor $F$, there is an isomorphism $T_w \circ F \cong F \circ T_w$, and these isomorphisms are compatible with natural transformations of projective functors.

- For all $w \in W$, $T_w V_{\lambda} \cong V_{w \cdot \lambda}$, where $V_{\lambda}$ is the Verma module with highest weight $\lambda$.

Indeed, let $\{T'_w \mid w \in W\}$ be any other collection of functors satisfying these conditions. By [BG80, 3.3(iib)], for any irreducible projective object of $\mathcal{O}_{\lambda}$, there is a projective functor taking $V_{\lambda}$ to that object. Since $\mathcal{O}_{\lambda}$ has enough projectives, for any object $N$ of $\mathcal{O}_{\lambda}$, there is a complex $F_N$ of projective functors taking $V_{\lambda}$ to $N$. Furthermore, projective functors may be regarded as modules over $g \times g$ [Bac01], and we have $\text{Hom}_{g \times g}(N, N') \cong \text{Hom}_{g \times g}(F_N, F_{N'})$. We therefore have

$$T_w' N \cong T_w' F_N V_{\lambda} \cong F_N T'_w V_{\lambda} \cong F_N V_{w \cdot \lambda} \cong T_w F_N V_{\lambda} \cong T_w N,$$

and the strong commutativity condition ensures that this induces an isomorphism of functors.

Since $\lambda$ is dominant, Soergel’s equivalence between $\mathcal{O}_{\lambda}$ and $C_{\lambda+\rho}^\mathcal{L}$ is given by composing the functors

$$\begin{array}{ccc}
\mathcal{O}_{\lambda} & \xleftarrow{(-)^{\circ} \otimes V_{\lambda}} & 1_{\mathcal{H}_{\lambda}^\infty} \xrightarrow{\lim_{\leftarrow i} (- \otimes V_{\lambda}^i)} C_{\lambda+\rho}^\mathcal{L} \\
\text{Hom}_{C_{\mathcal{L}}}^{\text{fin}}(V_{\lambda}, -)^{\circ} & & \lim_{\leftarrow i} \text{Hom}_{C_{\mathcal{L}}}^{\text{fin}}(V_{\lambda}^i, -)
\end{array}$$

(21)

where

- $1_{\mathcal{H}_{\lambda}^\infty}$ denotes the category of Harish-Chandra bimodules (in the usual sense) for $U(g)$ with generalized central character $\lambda$ for both the left and right actions, with the center acting on the left semi-simply,

- $\text{Hom}_{C_{\mathcal{L}}}^{\text{fin}}(V_{\lambda}, N)$ is the Harish-Chandra bimodule of $U(g)_\Delta$-locally finite $\mathbb{C}$-linear maps $V_{\lambda} \to N$,

- $(-)^{\circ}$ denotes the functor on $U(g)$-$U(g)$ bimodules which switches the left and right actions, twisting by the antipode of $U(g)$,

- $V_{\lambda}^i$ denotes the length $i$ thickened Verma module $V_{\lambda}^i := U(g) \otimes_{U(b)} (U(h)/m_{\lambda})$, where $m_{\lambda}$ is the kernel of the action of $U(h)$ on the $\lambda$-weight space.
Thus, we need only show that our twisting functors on $D(C^L_{\lambda+\rho})$, transported to $D(O_\lambda)$ via Soergel’s equivalence, satisfy these two conditions.

For any element $w \in W$, let $R^w_\lambda := \Phi^w(\lambda+\rho)T_{\lambda+\rho}$, where $w(\lambda+\rho)T_{\lambda+\rho}$ is regarded as a left $A_{w(\lambda+\rho)}$-module. Consider the twisting functor

$$S_w := \Psi^\lambda \circ \Phi^w(\lambda+\rho) \cong R^w_\lambda \otimes -$$. 

Under the bi-adjoint equivalences of $C^L_{\lambda+\rho}$ with $\lambda \mathcal{H}_\lambda^{\infty}$ described in Equation (21) of Example 6.11, this functor is intertwined with $R^w_\lambda \otimes -$; now regarded as a functor on Harish-Chandra bimodules, since tensor product on the left commutes with $\lim\limits_{\leftarrow}(- \otimes V_i^j)$. On the other hand, the equivalence to $O_\lambda$, described in the same equation, involves exchanging the left and right actions. Thus, any projective functor $F \cong F(U(g)) \otimes_{U(g)} -$ is intertwined with $- \otimes_{U(g)} F(U(g))^0 : \lambda \mathcal{H}_\lambda^{\infty} \rightarrow \lambda \mathcal{H}_\lambda^{\infty}$, which obviously commutes with $R^w_\lambda \otimes -$.

Checking the second condition is an easy geometric calculation. Since $\lambda$ is dominant and regular, localization holds at $\lambda$ $[\text{BBS1}]$. The localization of $V_\lambda$ is an object of $D_{\lambda+\rho}$-mod, which we may regard as a twisted $D$-module by Proposition 4.4. Concretely, it is the restriction of the line bundle $L_\lambda$ to the open Bruhat cell, where only the action of $g$ depends on $\lambda$. Tensoring with $w(\lambda+\rho)T'_{\lambda+\rho}$ takes us to the restriction of $L_w \cdot \lambda$ to that cell. The sections of that restriction are exactly the Verma module $V_w \cdot \lambda$, since it is generated by a unique $U$-invariant section of weight $w \cdot \lambda$ (here $U$ is the nilpotent radical of $B$), and the dimension of weight spaces matches the character of the Verma module.

We end by analyzing the twisting action of Theorem 6.34 on the level of the Grothendieck group. Assume $\lambda \in \Pi$. Every twisting functor $\Phi : D^b(A_\lambda\text{-mod}) \rightarrow D^b(A_\lambda\text{-mod})$ is induced by derived tensor product with an algebraic Harish-Chandra bimodule $K_\Phi$; by Proposition 6.7, this implies that the corresponding functor $\mathbb{L}\text{Loc} \circ \Phi \circ \mathbb{R}\Gamma_S : D^b(D_\lambda\text{-mod}) \rightarrow D^b(D_\lambda\text{-mod})$ is induced by convolution with a geometric Harish-Chandra bimodule $F_\Phi \in \lambda \mathcal{HC}_\lambda^g$. By Proposition 6.14, the effect of $\Phi$ on characteristic cycles is given by convolution with the characteristic cycle $CC(F_\Phi)$. Thus we obtain an algebra homomorphism

$$\alpha : C[\pi_1(E/W, [\lambda])] \rightarrow H^2_{\mathfrak{M}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C})$$. 

Proposition 6.38 The subalgebra $C[\pi_1(E, \lambda)] \subset C[\pi_1(E/W, [\lambda])]$ is contained in the kernel of $\alpha$, thus we obtain an induced homomorphism

$$\bar{\alpha} : C[W] \rightarrow H^2_{\mathfrak{M}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C})$$. 

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Proof: By Proposition 6.26, pure twisting functors preserve characteristic cycles. Since the subalgebra $\mathbb{C}[\pi_1(E, \lambda)] \subset \mathbb{C}[\pi_1(E/W, [\lambda])]$ acts by pure twisting functors, the result follows. □

Remark 6.39 The map $\bar{\alpha}$ also has a direct geometric construction, which precisely matches that given by Chriss and Ginzburg [CG97, 3.4.1] for $\mathfrak{m} = T^*G/B$. Applying the argument of the proof of Proposition 6.26 to an impure twisting functor shows that the class corresponding to $w$ is a specialization of the graph of the map $w: \pi^{-1}(\nu) \to \pi^{-1}(w \cdot \nu)$.

References


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