## CHARACTER SHEAVES

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Goal: study a representation using some category of sheaves.

Let X be a reasonable space. Local systems on X are the same as  $\pi_1(X)$ . In particular, if X is simply-connected, then the only irreducible local system is the constant sheaf.

We will be dealing with  $D_c^b(X)$ ,  $D^b(D_X$ -mod) and  $D_c^b(X/\overline{F_q})$ .

0.1. **Motivation for character sheaves.** We have many finite groups of Lie type.

Lusztig writes down their character using the following steps:

(1) Find representatives  $SS(G^{\vee})$  of semisimple conjugacy classes in  $G^{\vee}$ .

$$Irr(G) = \sqcup_{s \in SS(G^{\vee})} \mathcal{E}(G, s).$$

(2) Fix  $s \in SS(G^{\vee})$ . There is a 1-1 correspondence between  $\mathcal{E}(G, S)$  and  $\mathcal{E}(C_{G^{\vee}}(s), 1)$ .

For example, for  $G = SL_n$  we want to look at  $\mathcal{E}(SL_n, 1)$ , which is in bijection with irreducible character of the Weyl group  $S_n$ , i.e. partitions of n. The representations appearing on the right correspond to principal series. But we may have other unipotent characters, which correspond to cuspidal representations.

## 1. Character sheaves

Lusztig's definition: character sheaves are certain G-equivariant perverse sheaves on G.

Irreducible B-equivariant local systems on BwB are in bijection with irreducible local systems  $\mathcal{L}$  on T, such that  $w^*\mathcal{L} \cong \mathcal{L}$ .

Let  $j_w: BwB \hookrightarrow G$ .

Given a (tame) local system on T, think of it as living over BwB (with  $w^*\mathcal{L} \cong \mathcal{L}$ ).

Define  $K_w^{\mathcal{L}} := \Gamma_B^G(j_{w!}) \mathcal{L}[\dim G/B]$ . This is not a perverse sheaf. Can take irreducible perverse constituents are the character sheaves.

To get all character sheaves let  $\mathcal{L}$  and w vary.

Data:  $\mathcal{L}$  on T is the same as a central character. Unipotent means  $\mathcal{L}$  is constant.

1.1. Mirkovic-Vilonen characterization. Consider a group G over  $\mathbb{C}$ , so that we can consider  $\mathcal{D}$ -modules and their characteristic varieties. In particular, there isn't a notion of characteristic variety for  $\ell$ -adic sheaves  $(G/\overline{\mathbb{F}}_q)$ .

Let  $\mathcal{D}_G(G)$  be the category of G-equivariant  $\mathcal{D}$ -modules on G. When is an irreducible perverse sheave  $\mathcal{F} \in \mathcal{D}_G(G)$  a character sheaf?

Look at  $Ch(\mathcal{F}) \subseteq T^*G \cong G \times \mathfrak{g}^* \cong G \times \mathfrak{g}$ . Then  $\mathcal{F}$  is a character sheaf iff  $Ch(\mathcal{F}) \subseteq G \times \mathcal{N}$ , where  $\mathcal{N}$  is the nilpotent cone.

In particular, one can look at the Harish-Chandra system on g;

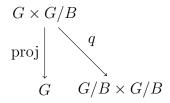
$$N: \begin{cases} \langle [A,x], \partial_x \rangle \widetilde{u} = 0, & A \in \mathfrak{g} \\ (P(x) - P(t)) \widetilde{u} = 0, & P \in \mathbf{C}[\mathfrak{g}]^G \\ (Q(\partial_x) - Q(\partial_t)) \widetilde{u} = 0, & Q \in S(\mathfrak{g})^G. \end{cases}$$

Consider the Grothendieck-Springer resolution

$$\widetilde{\mathfrak{g}}=\{(g,B)\in\mathfrak{g}\in\mathcal{B}|g\in\operatorname{Lie}B\}$$

given by  $\mu: \widetilde{\mathfrak{g}} \to \mathfrak{g}$ . The Grothendieck-Springer sheaf is  $\mu^* \mathbf{C}_{\widetilde{\mathfrak{g}}}$ . One can similarly define it for  $\widetilde{G} \to G$ . This is a character sheaf.

Another version is given by the horocycle correspondence.



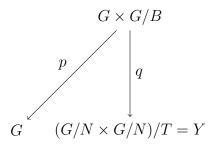
Here q is the action map of G on the flag variety. Define

$$p_*q^!: D_G(G/B \times G/B) \to D_G(G).$$

Define the unipotent character sheaves to be exactly the irreducible perverse constituents of  $p_*q^!(\mathcal{F})$  for any  $\mathcal{F}$ .

Observe, that  $D_G(G/B \times G/B) \cong D_B(G/B)$ .

More general horocycle correspondence (twisted Hecke category):



Here Y is known as the horocycle space.

We get the map

$$p_*q^!: D_G((G/N \times G/N)/T) \to D_G(G).$$

To get all character sheaves, take irreducible perverse constituents of  $p_*q^!(\mathcal{F})$ .

For  $\mathcal{L}$  a local system on the torus, then  $D_G^{\mathcal{L}}((G/N \times G/N)/T)$  produces character sheaves with  $\mathcal{L}$ -central character.

Let us restrict our attention to the unipotent case. If we start with  $D_B(G/B)$ , consider its Grothendieck group and take its center. This doesn't work. (One only gets principal series characters.)

Key idea: taking the center and Grothendieck group is not interchangeable. Exercise: come up with the right procedure.