CHARACTER SHEAVES

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Goal: study a representation using some category of sheaves.

Let X be a reasonable space. Local systems on X are the same as $\pi_1(X)$. In particular, if X is simply-connected, then the only irreducible local system is the constant sheaf.

We will be dealing with $D^b_c(X)$, $D^b(D_X\text{-mod})$ and $D^b_c(X/\overline{F_q})$.

0.1. **Motivation for character sheaves.** We have many finite groups of Lie type.

Lusztig writes down their character using the following steps:

(1) Find representatives $SS(G^\vee)$ of semisimple conjugacy classes in $G^\vee$.
$$\mathrm{Irr}(G) = \sqcup_{s \in SS(G^\vee)} \mathcal{E}(G, s).$$
(2) Fix $s \in SS(G^\vee)$. There is a 1-1 correspondence between $\mathcal{E}(G, S)$ and $\mathcal{E}(C_{G^\vee}(s), 1)$.

For example, for $G = SL_n$ we want to look at $\mathcal{E}(SL_n, 1)$, which is in bijection with irreducible character of the Weyl group S_n, i.e. partitions of n. The representations appearing on the right correspond to principal series. But we may have other unipotent characters, which correspond to cuspidal representations.

1. Character sheaves

Lusztig's definition: character sheaves are certain G-equivariant perverse sheaves on G.

Irreducible B-equivariant local systems on BwB are in bijection with irreducible local systems $\mathcal{L}$ on T, such that $w^*\mathcal{L} \cong \mathcal{L}$.

Let $j_w : BwB \hookrightarrow G$.

Given a (tame) local system on T, think of it as living over BwB (with $w^*\mathcal{L} \cong \mathcal{L}$).

Define $K_w^{\mathcal{L}} := \Gamma_B^G(j_{w!})\mathcal{L}[\dim G/B]$. This is not a perverse sheaf. Can take irreducible perverse constituents are the character sheaves.

To get all character sheaves let $\mathcal{L}$ and w vary.

Data: $\mathcal{L}$ on T is the same as a central character.Unipotent means $\mathcal{L}$ is constant.

1.1. **Mirkovic-Vilonen characterization.** Consider a group G over $\mathbf{C}$, so that we can consider $\mathcal{D}$-modules and their characteristic varieties. In particular, there isn't a notion of characteristic variety for ℓ-adic sheaves $(G/\overline{\mathbf{F}}_q)$.

Let $\mathcal{D}_G(G)$ be the category of G-equivariant $\mathcal{D}$-modules on G. When is an irreducible perverse sheave $\mathcal{F} \in \mathcal{D}_G(G)$ a character sheaf?

Look at $\mathrm{Ch}(\mathcal{F}) \subseteq T^*G \cong G \times \mathfrak{g}^* \cong G \times \mathfrak{g}$. Then $\mathcal{F}$ is a character sheaf iff $\mathrm{Ch}(\mathcal{F}) \subseteq G \times \mathcal{N}$, where $\mathcal{N}$ is the nilpotent cone.

In particular, one can look at the Harish-Chandra system on $\mathfrak{g}$;

$$N : \begin{cases} \langle [A, x], \partial_x \rangle \widetilde{u} = 0, & A \in \mathfrak{g} \\ (P(x) - P(t))\widetilde{u} = 0, & P \in \mathbf{C}[\mathfrak{g}]^G \\ (Q(\partial_x) - Q(\partial_t))\widetilde{u} = 0, & Q \in S(\mathfrak{g})^G. \end{cases}$$

Consider the Grothendieck-Springer resolution

$$\widetilde{\mathfrak{g}} = \{(g, B) \in \mathfrak{g} \in \mathcal{B} | g \in \mathrm{Lie}\, B\}$$

given by $\mu : \widetilde{\mathfrak{g}} \to \mathfrak{g}$. The Grothendieck-Springer sheaf is $\mu^* \mathbf{C}_{\widetilde{\mathfrak{g}}}$. One can similarly define it for $\widetilde{G} \to G$. This is a character sheaf.

Another version is given by the horocycle correspondence.

$$\begin{array}{ccc} G \times G/B & & \\ \downarrow \text{proj} & \searrow q & \\ G & & G/B \times G/B \end{array}$$

Here q is the action map of G on the flag variety.

Define

$$p_* q^! : D_G(G/B \times G/B) \to D_G(G).$$

Define the unipotent character sheaves to be exactly the irreducible perverse constituents of $p_* q^!(\mathcal{F})$ for any $\mathcal{F}$.

Observe, that $D_G(G/B \times G/B) \cong D_B(G/B)$.

More general horocycle correspondence (twisted Hecke category):

$$\begin{array}{ccc} & G \times G/B & \\ {}^{p}\swarrow & & \downarrow{}^{q} \\ G & & (G/N \times G/N)/T = Y \end{array}$$

Here Y is known as the horocycle space.

We get the map

$$p_* q^! : D_G((G/N \times G/N)/T) \to D_G(G).$$

To get all character sheaves, take irreducible perverse constituents of $p_* q^!(\mathcal{F})$.

For $\mathcal{L}$ a local system on the torus, then $D_G^{\mathcal{L}}((G/N \times G/N)/T)$ produces character sheaves with $\mathcal{L}$-central character.

Let us restrict our attention to the unipotent case. If we start with $D_B(G/B)$, consider its Grothendieck group and take its center. This doesn't work. (One only gets principal series characters.)

Key idea: taking the center and Grothendieck group is not interchangeable. Exercise: come up with the right procedure.