# Statistics 150: Spring 2007

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# 1 Limiting Probabilities

If the discrete-time Markov chain with transition probabilities $p_{ij}$ is irreducible and positive recurrent; then the limiting probabilities $p_j = \lim_{t\to\infty} P_{ij}(t)$ are given by

$$p_j = \frac{\pi_j/\nu_j}{\sum_i \pi_i/\nu_i}$$

where the $\pi_j$ are the unique nonnegative solution of

$$\pi_j = \sum_i \pi_i p_{ij}, \quad \sum_i \pi_i = 1.$$

We see that the $p_j$ are unique nonnegative solution of

$$\nu_j p_j = \sum_i \nu_i p_i p_{ij}, \quad \sum_j p_j = 1,$$

or, equivalently, using $q_{ij} = \nu_i p_{ij}$,

$$\nu_j p_j = \sum_i p_i q_{ij}, \quad \sum_j p_j = 1.$$

Another way of obtaining the equations for the $p_i$, is by way of the forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t).$$

If we assume that the limiting probabilities $p_j = \lim_{t \to \infty} P_{ij}(t)$ exists, then $P'_{ij}(t)$ would necessarily converge to 0 as $t \to \infty$. Hence, assuming that we can interchange limit and summation in the above, we obtain upon letting $t \to \infty$,

$$0 = \sum_{k \neq j} p_k q_{kj} - \nu_j p_j.$$

Let us now determine the limiting probabilities for a birth and death process. The relevant equations are

$$\lambda_0 p_0 = \mu_1 p_1,$$

$$\lambda_n p_n = \mu_{n+1} p_{n+1} + (\lambda_{n-1} p_{n-1} - \mu_n p_n), \quad n \geq 1.$$

or, equivalently,

$$\lambda_0 p_0 = \mu_1 p_1,$$

$$\lambda_1 p_1 = \mu_2 p_2 + (\lambda_0 p_0 - \mu_1 p_1) = \mu_2 p_2,$$

$$\lambda_2 p_2 = \mu_3 p_3 + (\lambda_1 p_1 - \mu_2 p_2) = \mu_3 p_3,$$

$$\lambda_n p_n = \mu_{n+1} p_{n+1} + (\lambda_{n-1} p_{n-1} - \mu_n p_n) = \mu_{n+1} p_{n+1}.$$

Solving in terms of $p_0$ yields

$$p_1 = \frac{\lambda_0}{\mu_1} p_0,$$

$$p_2 = \frac{\lambda_1}{\mu_2} p_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} p_0,$$

$$p_3 = \frac{\lambda_2}{\mu_3} p_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} p_0,$$

$$p_n = \frac{\lambda_{n-1}}{\mu_n} p_{n-1} = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} p_0.$$

Using $\sum_{n=0}^{\infty} p_n = 1$ we obtain

$$1 = p_0 + p_0 \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_1 \lambda_0}{\mu_n \cdots \mu_2 \mu_1}$$

or

$$p_0 = [1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}]^{-1},$$

and hence

$$p_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n (1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n})}, \quad n \geq 1.$$

The above equations also show us what condition is needed for the limiting probabilities to exist. Namely,

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty.$$

**Example 1.1. The $M/M/1$ Queue**

In the $M/M/1$ queue $\lambda_n = \lambda$, $\mu_n = \mu$, and thus

$$p_n = \frac{(\lambda/\mu)^n}{1 + \sum_{n=1}^{\infty} (\frac{\lambda}{\mu})^n} = (\frac{\lambda}{\mu})^n (1 - \frac{\lambda}{\mu}), \quad n \geq 0$$

provided that $\lambda/\mu < 1$. Customers arrive at rate $\lambda$ and are served at rate $\mu$, and thus if $\lambda > \mu$, they will arrive at a faster rate than they can be served and the queue size will go to infinity. The case $\lambda = \mu$ is null recurrent and thus has no limiting probabilities.

## 2 Time Reversibility

Consider an ergodic continuous-time Markov chain and suppose that it has been in operation an infinitely long time; that is, suppose that it started at time $-\infty$. Such a process will be stationary, and we say that it is in *steady state.*

Let us consider this process going backwards in time. Now, since the forward process is a continuous-time Markov chain it follows that given the present state, call it $X(t)$, the past state $X(t-s)$ and the future states $X(y), y > t$ are independent.

Therefore,

$$\mathbb{P}\{X(t-s)=j|X(t)=i,X(y),y>t\}=\mathbb{P}\{X(t-s)=j|X(t)=i\}$$

and so we can conclude that the reverse process is also a continuous-time Markov chain. Also, since the amount of time spent in a state is the same whether one is going forward or backward in time it follows that the amount of time the reverse chain spends in state $i$ on a visit is exponential with the same rate $\nu_i$ as in the forward process.

The sequence of states visited by the reverse process constitutes a discrete-time Markov chain with transition probabilities $p^*_{ij}$ given by

$$p^*_{ij} = \frac{\pi_j p_{ji}}{\pi_i}$$

where $\{\pi_j, j \geq 0\}$ are the stationary probabilities of the embedded discrete-time Markov chain with transition probabilities $p_{ij}$. Let

$$q^*_{ij} = \nu_i p^*_{ij}$$

denote the infinitesimal rates of the reverse chain. We see that

$$q^*_{ij} = \frac{\nu_i \pi_j p_{ji}}{\pi_i}.$$

Recalling that

$$p_k = \frac{\pi_k/\nu_k}{C}, \text{ where } C = \sum_i \pi_i/\nu_i,$$

we see that

$$\frac{\pi_j}{\pi_i} = \frac{\nu_j p_j}{\nu_i p_i}$$

and so,

$$q^*_{ij} = \frac{\nu_j p_j p_{ji}}{p_i} = \frac{p_j q_{ji}}{p_i}.$$

That is,

$$p_i q^*_{ij} = p_j q_{ji}.$$

**Definition 2.1** (Reversibility)**.** A stationary continuous-time Markov chain is said to be time reversible if the reverse process follows the same probabilistic law as the original process. That is, it is *time reversible* if for all $i$ and $j$

$$q^*_{ij} = q_{ij}$$

which is equivalent to

$$p_i q_{ij} = p_j q_{ji}, \text{ for all } i, j.$$

**Proposition 2.2.** *Any ergodic birth and death process in steady state is time reversible.*

**Corollary 2.3.** *Consider an $M/M/s$ queue in which customers arrive in accordance with a Poisson process having rate $\lambda$ and are served by any one of $s$ servers–each having exponentially distributed service time with rate $\mu$. If $\lambda < s\mu$, then the output process of customers departing is, in steady state, a Poisson process with rate $\lambda$.*

**Proof:** *Let* $X(t)$ *denote the number of customers in the system at time* $t$*. Since the* $M/M/s$ *process is a birth and death process, it follows that* $\{X(t), t \geq 0\}$ *is time reversible. Now going forward in time, the time points at which* $X(t)$ *increases by 1 constitute a Poisson process since these are just the arrival times of customers. hence, by time reversibility, the time points at which the* $X(t)$ *increases by 1 when we go backwards in time also constitute a Poisson process. But these latter points are exactly the points of time when customers depart. Hence, the departure times constitute a Poisson process with rate* $\lambda$*.* $\square$

# 3 Exercises

1) Suppose that the "state" of the system can be modeled as a two-state continuous-time Markov chain with transition rates $\nu_0 = \lambda, \nu_1 = \mu$. When the state of the system is $i$, "events" occur in accordance with a Poisson process with rate $\alpha_i, i = 0, 1$. Let $N(t)$ denote the number of events in $(0, t)$.

(a) Find $\lim_{t\to\infty} N(t)/t$.

(b) If the initial state is state 0, find $\mathbb{E}[N(t)]$.

2) Consider a continuous-time Markov chain with stationary probabilities $\{p_i, i \geq 0\}$, and let $T$ denote the first time the chain has been in state 0 for $t$ consecutive time unites. Find $\mathbb{E}[T|X(0) = 0]$.

3) Find the limiting probabilities for the $M/M/s$ system and determine the condition needed for these to exist.

4) Consider a time-reversible continuous-time Markov chain having parameters $\nu_i, p_{ij}$ and having limiting probabilities $p_j, j \geq 0$. Choose some state – say state 0 – and consider the new Markov chain, which makes state 0 an absorbing state. That is, reset $\nu_0$ to equal 0. Suppose now at time points chosen according to a Poisson process with rate $\lambda$, Markov chains – all of the above type (having 0 as an absorbing state) – are started with the initial states chosen to be $j$ with probabilities $p_{0j}$. All the existing chains are assumed to be independent. Let $N_j(t)$ denote the number of chains in state $j$, $j > 0$, at time $t$.

(a) Argue that if there are no chains "alive" at time $0$, then $N_j(t)$, $j > 0$, are independent Poisson random variables.

(b) Argue that the vector process $\{(N_1(t), N_2(t), \cdots)\}$ is time reversible in steady state with stationary probabilities

$$p_{\underline{n}} = \prod_{j=1}^{\infty} e^{-\alpha_j} \frac{\alpha_j^{n_j}}{n_j!}, \text{ for } \underline{n} = (n_1, n_2, \cdots),$$

where $\alpha_j = \lambda p_j / p_0 \nu_0$.