(1) Consider a Yule process with $X(0) = i$. Given that $X(t) = i + k$, what is the conditional distribution of the birth times of the k individuals born in $(0, t)$?

(2) Consider a Yule process starting with a single individual and suppose that the individuals born are either male or female. Suppose that an individual born at time s will be female with probability $P(s)$, independently of the rest of the process, and male otherwise. Compute the distribution of the number of female individuals born in $(0, t)$.

(3) Suppose that the local weather system can be either in a "hot" pattern or a "cold" pattern, that it switches back and forth between the patterns according to a continuous-time Markov chain, and that rain events occur with different rates depending on the weather pattern. We model this as follows:

Let the "state" of the system be a two-state continuous-time Markov chain with transition rates $\nu_0 = q_{0,1} = \lambda$ and $\nu_1 = q_{1,0} = \mu$. Suppose that when the system is in state 0, "events" occur at the times of a Poisson process with rate α_0, and when the system is in state 1, "events" occur at the times of a Poisson process with rate α_1. Let $N(t)$ denote the number of events in $(0, t)$. You may use the results on two-state Markov chains proved in class.

(a) Find $\lim_{t\to\infty} N(t)/t$.

(b) If the initial state is state 0, find $\mathbb{E}[N(t)]$.

(4) Consider a reversible continuous-time Markov chain on the nonnegative integers with transition parameters ν_i and p_{ij} and with limiting probabilities p_j for $j \geq 0$. Define the *absorbed chain* to be the Markov chain with the same parameters, except $\nu_0 = 0$, so that state 0 is an absorbing state. Suppose now at the time points of a Poisson process with rate λ, independent copies of the absorbed chain are started, and that each begins at state j with probability $p_{0,j}$ for $j > 0$. Let $N_j(t)$ denote the number of chains in state j, at time t, for $j > 0$.

(a) Show that if there are no chains "alive" at time 0, then for any t, the random variables $\{N_j(t) : \ j > 0\}$, are independent and Poisson distributed.

(b) Show that the vector process $((N_1(t), N_2(t), \cdots))_{t\geq 0}$ is time reversible in steady state with stationary probabilities

$$p_{\underline{n}} = \prod_{j=1}^{\infty} e^{-\alpha_j} \frac{\alpha_j^{n_j}}{n_j!}, \text{ for } \underline{n} = (n_1, n_2, \cdots),$$

where $\alpha_j = \lambda p_j / p_0 \nu_0$.