(1) Let $(Y_n, \mathcal{F}_n)_{n\geq 0}$ be a martingale.

(i) Show that $\mathbb{E}[Y_n] = \mathbb{E}[Y_0]$ for all n.

(ii) Show that $\mathbb{E}[Y_{n+m}|\mathcal{F}_n] = Y_n$ for all $n, m \geq 0$.

(2) A *supermartingale* is a sequence $(X_k, \mathcal{F}_k)_{k\geq 0}$ for which $\mathbb{E}[X_k^-] < \infty$ and $\mathbb{E}[X_{k+1}|\mathcal{F}_k] \leq X_k$. Let X be a discrete-time Markov chain with countable state space S and transition matrix $\mathbf{P}$. Suppose that $\psi : S \to \mathbb{R}$ is bounded and satisfies $\sum_{j\in S} p_{ij}\psi(j) \leq \lambda\psi(i)$ for some $\lambda > 0$ and all $i \in S$. Show that $\lambda^{-n}\psi(X_n)$ constitutes a supermartingale.

(3) Let S_n be a symmetric, simple random walk with $S_0 = 0$. Use Hoeffding's inequality to show that

$$\mathbb{P}\{|S_n^2 - n| \geq xn^{3/2}\} \leq 2e^{-x^2/6}.$$

(4) If T_1 and T_2 are stopping times with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$, show that $T_1 + T_2$, $\max\{T_1, T_2\}$, and $\min\{T_1, T_2\}$ are stopping times also.

(5) Let $\{S_n\}$ be a simple random walk with $S_0 = 0$ such that $0 < p = \mathbb{P}\{S_1 = 1\} < \frac{1}{2}$. Note that the walk escapes to $-\infty$ a.s., so there exists some (random) point above which the walk never reaches. Use de Moivre's martingale to show that $\mathbb{E}[\sup_{m\geq 0} S_m] = \frac{p}{1-2p}$.