(1) Let \((Y_n, \mathcal{F}_n)_{n \geq 0}\) be a martingale.

(i) Show that \(\mathbb{E}[Y_n] = \mathbb{E}[Y_0]\) for all \(n\).

(ii) Show that \(\mathbb{E}[Y_{n+m}|\mathcal{F}_n] = Y_n\) for all \(n, m \geq 0\).

(2) A supermartingale is a sequence \((X_k, \mathcal{F}_k)_{k \geq 0}\) for which \(\mathbb{E}[X^{-}_k] < \infty\) and \(\mathbb{E}[X_{k+1}|\mathcal{F}_k] \leq X_k\). Let \(X\) be a discrete-time Markov chain with countable state space \(S\) and transition matrix \(P\). Suppose that \(\psi: S \to \mathbb{R}\) is bounded and satisfies \(\sum_{j \in S} P_{ij}\psi(j) \leq \lambda \psi(i)\) for some \(\lambda > 0\) and all \(i \in S\). Show that \(\lambda^{-n} \psi(X_n)\) constitutes a supermartingale.

(3) Let \(S_n\) be a symmetric, simple random walk with \(S_0 = 0\). Use Hoeffding’s inequality to show that
\[
\mathbb{P}\{|S_n^2 - n| \geq xn^{3/2}\} \leq 2e^{-x^2/6}.
\]

(4) If \(T_1\) and \(T_2\) are stopping times with respect to a filtration \((\mathcal{F}_n)_{n \geq 0}\), show that \(T_1 + T_2\), \(\max\{T_1, T_2\}\), and \(\min\{T_1, T_2\}\) are stopping times also.

(5) Let \(\{S_n\}\) be a simple random walk with \(S_0 = 0\) such that \(0 < p = \mathbb{P}\{S_1 = 1\} < \frac{1}{2}\). Note that the walk escapes to \(-\infty\) a.s., so there exists some (random) point above which the walk never reaches. Use de Moivre’s martingale to show that \(\mathbb{E}[\sup_{m \geq 0} S_m] = \frac{p}{1-2p}\).