(1) We have n items labeled with $1, 2, \ldots, n$. Each day an item is requested; it is the one with the ith label with probability $P_i, i \geq 1, \sum_{i=1}^{n} P_i = 1$. These item are at all times arranged in an ordered list that is revised as follows: the item selected is moved to the front of the list, and the relative positions of all other item remain unchanged. Define the state at any time to be the ordering of the list at that time.

(a) Show that the above is Markov chain.

(b) For any state $(i_1, \ldots, i_n)$ (which is a permutation of $(1, 2, \ldots, n)$) let $\pi(i_1, \ldots, i_n)$ denote the stationary distribution. Show that

$$\pi(i_1, \ldots, i_n) = P_{i_1} \frac{P_{i_2}}{1 - P_{i_1}} \cdots \frac{P_{i_{n-1}}}{1 - P_{i_1} - \cdots - P_{i_{n-2}}}.$$

(2) Consider the Markov chain $(X_n)_{n \geq 0}$ on $\{0, 1, 2, \ldots\}$ that with transition probabilities: $p_{0,1} = 1$ and for $k \geq 1$,

$$p_{k,k-1} = \frac{1}{2}, \qquad p_{k,k+1} = \frac{1}{2(k+1)}, \qquad p_{k,k} = 1 - p_{k,k-1} - p_{k,k+1} = \frac{k}{2(k+1)}$$

(a) Find the stationary distribution of the Markov chain.

(b) Suppose that $(X_n)_{n \geq 0}$ is the number of balls in a bag, and come up with a "nice" way of simulating the chain. (you may color balls; use additional coins, etcetera; but the simpler, the better)

(3) Let $\{X_n, n \geq 0\}$ be a Markov chain with stationary probabilities π_j, $j \geq 0$. Suppose that $X_0 = 0$ and define

$$T = \min\{n > 0 : X_n = 0\}.$$

Let $Y_j = X_{T-j}, j = 0, 1, \ldots, T$. Show that $\{Y_j, j = 0, \ldots, T\}$ is distributed as the states visited by a Markov chain (the "reversed" Markov chain) with transition probabilities $P^*_{ij} = \pi_j P_{ji} / \pi_i$ started in state 0 and watched until it returns to 0.

(4) Consider a finite Markov chain on the state space $\{0, 1, 2, \ldots, N\}$ with transition probability matrix $\mathbf{P} = (P_{ij})_{i,j=0}^{N}$, and divide the state space into the three classes $\{0\}$, $\{1, 2, \ldots, N-1\}$ and $\{N\}$. Let 0 and N be absorbing states, both accessible from all states in $1, \ldots, N-1$, and let $\{1, 2, \ldots, N-1\}$ be a transient class.

Let k be a transient state. Define an auxiliary process (the "return process") with transition matrix $\widetilde{P}$ by altering the first and last row of $\mathbf{P}$ so that $\widetilde{P}_{0k} = \widetilde{P}_{Nk} = 1$ and leave the other rows unchanged.

The return process is clearly irreducible. Prove that the expected time until absorption μ_k with initial state k in the original process equals $1/(\pi_0 + \pi_N) - 1$ where $\pi_0 + \pi_N$ is the stationary probability of being in state 0 or N for the return process. Hint: use the relation between stationary probabilities and expected recurrence times to states.