(1) Consider an irreducible, reversible, discrete-time Markov chain on the state space $\{0, 1, 2, \ldots\}$ with transition probabilities P_{ij} and stationary probabilities π_i. Suppose we restrict the chain to the states $\{0, 1, \ldots, M\}$ by defining the transition probabilities

$$\overline{P}_{ij} = \begin{cases} P_{ij} + \sum_{k>M} P_{ik}, & 0 \leq i \leq M, j = i \\ P_{ij}, & 0 \leq i \neq j \leq M \\ 0, & \text{otherwise.} \end{cases}$$

Show that the restricted chain is also time reversible and that the distribution

$$\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i}$$

is a stationary distribution for the restricted chain, in the sense that it solves the equation $\overline{\pi}\,\overline{P} = \overline{\pi}$. Note that the restricted chain need not be irreducible, and explain this result in the case when it is not.

(2) Suppose M balls are initially distributed among m urns. At each stage one of the balls is selected uniformly at random, taken from whichever urn it is in, and placed into a uniformly chosen one of the other $m-1$ urns. Consider the discrete-time Markov chain whose state at any time is the vector $(n_1, \ldots, n_m)$, where n_i denotes the number of balls in urn i. Guess at the stationary probabilities for this Markov chain and verify your guess, showing at the same time that the Markov chain is time reversible.

(3) A small colony of fruit flies consists only of males and females, and at time t is composed of $N_m(t)$ males and $N_f(t)$ females. Suppose that in any small time interval of length h, each individual has a probability $\mu h + o(h)$ of dying, independently of the other individuals. Also suppose that in any small time interval of length h, each female has a probability $\lambda h + o(h)$ of mating with a male (if there are any present) and producing a single offspring, which is equally likely to be male or female.

Derive the transition parameters of the continuous-time Markov chain $((N_m(t), N_f(t))_{t\geq 0}$, appealing to results we know about Poisson processes.

(4) Suppose that a one-celled organism can be in one of two states-either A or B. An individual in state A will change to state B at an exponential rate α; an individual in state B divides into two new individuals of type A at an exponential rate β. Define an appropriate continuous-time Markov chain for a population of such organisms and determine the appropriate transition parameters for this model.