Statistics 150: Spring 2007

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1 Introduction

Definition 1.1. A sequence $Y = \{Y_n \colon n \geq 0\}$ of real-valued random variables is a *martingale* with respect to the sequence $X = \{X_n \colon n \geq 0\}$ of random variables if, for all $n \geq 0$,

1. $\mathbb{E}[|Y_n|] < \infty$
2. $\mathbb{E}[Y_{n+1}|X_0, X_1, \ldots, X_n] = Y_n$.

Example 1.2 (Simple random walk)**.** Let X_i be i.i.d. random variables such that $X_i = 1$ with probability p and $X_i = -1$ with probability $q = 1 - p$. Then $S_n = X_1 + X_2 + \cdots + X_n$ satisfies $\mathbb{E}[|S_n|] \leq n$ and

$$\mathbb{E}[S_{n+1}|X_1, X_2, \ldots, X_n] = S_n + (p - q),$$

and $Y_n = S_n - n(p - q)$ defines a martingale with respect to X.

Example 1.3 (De Moivre's martingale)**.** A simple random walk on the set $\{0, 1, 2, \ldots, N\}$ is begun at k and stops when it first hits either of the absorbing barriers at 0 and at N; what is the probability that it stops at the barrier 0?

Write $X_1, X_2, \ldots,$ for the steps of the walk, and S_n for the position after n steps, where $S_0 = k$. Define $Y_n = (q/p)^{S_n}$. We assume that $0 < p < 1$.

The process $\{Y_1, Y_2, \ldots\}$ is a martingale:

$$\mathbb{E}\left[Y_{n+1} | X_1, X_2, \ldots, X_n\right] = Y_n \qquad \text{for all } n.$$

To show that $\mathbb{E}\left[Y_{n+1}|X_1, X_2, \ldots, X_n\right] = Y_n$, first consider the cases in which the process has stopped by time n.

If S_n equals 0 or N, then $S_{n+1} = S_n$; and therefore $Y_{n+1} = Y_n$.

On the other hand, if $0 < S_n < N$, then

$$\begin{aligned}
\mathbb{E}\left[Y_{n+1}|X_1, X_2, \ldots, X_n\right] &= \mathbb{E}\left[(\frac{q}{p})^{S_n + X_{n+1}} \middle| X_1, X_2, \ldots, X_n \right] \\
&= \left(\frac{q}{p}\right)^{S_n} \left[p\left(\frac{q}{p}\right) + q\left(\frac{q}{p}\right)^{-1} \right] \\
&= Y_n,
\end{aligned}$$

Therefore, $(Y_n)_{n \geq 0}$ is a martingale, and in particular, we see that $\mathbb{E}\left[Y_{n+1}\right] = \mathbb{E}\left[Y_n\right]$ for all n, and hence $\mathbb{E}\left[Y_n\right] = \mathbb{E}\left[Y_0\right] = (q/p)^k$ for all n.

Let T be the number of steps before the absorption of the particle at either 0 or N. Since $S_0 = k$, we have that $\mathbb{E}\left[Y_T\right] = \mathbb{E}[Y_0] = (q/p)^k$.

Expanding $\mathbb{E}\left[Y_T\right]$, we have that

$$\mathbb{E}\left[Y_T\right] = (q/p)^0 p_k + (q/p)^N (1 - p_k)$$

where $p_k = \mathbb{P}\{\text{absorbed at } 0 | S_0 = k\}$. Therefore

$$p_k = \frac{\rho^k - \rho^N}{1 - \rho^N} \qquad \text{where } \rho = q/p$$

Example 1.4 (Markov chains). Let X be a discrete-time Markov chain taking values in the countable state space S with transition matrix $\mathbf{P}$. Suppose that $\phi\colon S \to \mathbb{R}$ is bounded and *harmonic*, which is to say that

$$\sum_{j \in S} p_{ij}\phi(j) = \phi(i) \qquad \text{for all } i \in S.$$

It is easily seen that $Y = \{\phi(X_n)\colon n \geq 0\}$ is a martingale with respect to X:

$$\begin{aligned} \mathbb{E}\left[\phi(X_{n+1})|X_1, X_2, \ldots, X_n\right] &= \mathbb{E}\left[\phi(X_{n+1})|X_n\right] \\ &= \sum_{j \in S} p_{X_n,j}\phi(j) = \phi(X_n). \end{aligned}$$

Definition 1.5 (Filtrations)**.** Given a random variable Z we use the shorthand $\mathbb{E}[Z|\mathcal{F}_n]$ for $[Z|X_0, X_1, \ldots X_n]$. We call $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots\}$ a *filtration*.

A sequence of random variables $Y = \{Y_n : n \geq 0\}$ is said to be *adapted* to the filtration $\mathcal{F}$ if Y_n is $\mathcal{F}_n$-measurable for all n, that is, if Y_n is a deterministic function of $X_0, X_1, \ldots, X_n$.

Definition 1.6. Let $\mathcal{F}$ be a filtration and let Y be a sequence of random variables which is adapted to $\mathcal{F}$. We can rewrite our previous definition of a martingale by saying that the pair $(Y, \mathcal{F}) = \{(Y_n, \mathcal{F}_n) \colon n \geq 0\}$ is a *martingale* if for all $n \geq 0$,

1. $\mathbb{E}[|Y_n|] < \infty$
2. $\mathbb{E}\left[Y_{n+1} | \mathcal{F}_n\right] = Y_n$

Note that $\mathbb{E}\left[Y_{n+1} | \mathcal{F}_n\right] = Y_n$ implies that Y_n is $\mathcal{F}_n$-measurable, e.g. that Y is adapted to $\mathcal{F}$.

2 Hoeffding's inequality

Proposition 2.1 (Hoeffding's inequality). *Let* $(Y, \mathcal{F})$ *be a martingale, and suppose that there exists a sequence* $K_1, K_2, \ldots$ *of real numbers such that* $\mathbb{P}\{|Y_n - Y_{n-1}| \leq K_n\} = 1$ *for all* n. *Then*

$$\mathbb{P}\{|Y_n - Y_0| \geq x\} \leq 2\exp\left(-\frac{\frac{1}{2}x^2}{\sum_{i=1}^n K_i^2}\right), \quad x > 0.$$

Proof: Observe that for $\psi > 0$ the function $g(d) = e^{\psi d}$ is convex and

$$e^{\psi d} \le \frac{1}{2}(1-d)e^{-\psi} + \frac{1}{2}(1+d)e^{\psi} \quad \text{if } |d| \le 1.$$

Applying this to a random variable D having mean 0 and satisfying $\mathbb{P}\{|D| \le 1\} = 1$, we obtain

$$\mathbb{E}\left[e^{\psi D}\right] \le \frac{1}{2}(e^{-\psi} + e^{\psi}) < e^{\frac{1}{2}\psi^2},$$

by a comparison of the coefficients of ψ^{2n} for $n \ge 0$.

Using Markov's inequality we have

$$\mathbb{P}\{Y_n - Y_0 \ge x\} \le e^{-\theta x}\mathbb{E}\left[e^{\theta(Y_n - Y_0)},\right] \quad \text{for } \theta > 0.$$

Writing $D_n = Y_n - Y_{n-1}$ and conditioning on $\mathcal{F}_{n-1}$, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{\theta(Y_n - Y_0)} \mid \mathcal{F}_{n-1}\right] &= e^{\theta(Y_{n-1}-Y_0)}\mathbb{E}\left[e^{\theta D_n} \mid \mathcal{F}_{n-1}\right] \\ &\le e^{\theta(Y_{n-1}-Y_0)} \exp(\frac{1}{2}\theta^2 K_n^2).\end{aligned}$$

We take expectations and iterate to find

$$\mathbb{E}\left[e^{\theta(Y_n - Y_0)}\right] \le \mathbb{E}[e^{\theta(Y_{n-1}-Y_0)}] \exp(\frac{1}{2}\theta^2 K_n^2) \le \exp\left(\frac{1}{2}\theta^2 \sum_{i=1}^{n} K_i^2\right).$$

and therefore

$$\mathbb{P}\left\{Y_n - Y_0 \ge x\right\} \le \exp\left(-\theta x + \frac{1}{2}\theta^2 \sum_{i=1}^{n} K_i^2\right), \quad \text{for all } \theta > 0.$$

Suppose that $x > 0$, and set $\theta = \frac{x}{\sum_{i=1}^{n} K_i^2}$ (this is the value which minimizes the exponent); we obtain

$$\mathbb{P}\left\{Y_n - Y_0 \geq x\right\} \leq \exp\left(-\frac{\frac{1}{2}x^2}{\sum_{i=1}^{n} K_i^2}\right).$$

The same argument is valid with $Y_n - Y_0$ replaced by $Y_0 - Y_n$, and the claim of the theorem follows by adding the two (identical) bounds together. $\square$

Example 2.2 (Large deviations)**.** Let $X_1, X_2, \ldots$ be independent random variables, X_i having the Bernoulli distribution with parameter p. We set $S_n = X_1 + \ldots + X_n$ and $Y_n = S_n - np$ to obtain a martingale Y. It is a consequence of Hoeffding's inequality that

$$\mathbb{P}\left\{|S_n - np| \geq x\sqrt{n}\right\} \leq 2\exp(-\frac{1}{2}x^2) \quad \text{for } x > 0.$$

3 Sub- and Supermartingales

Definition 3.1. Let $\mathcal{F}$ be a filtration of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let Y be a sequence of random variables which is adapted to $\mathcal{F}$. We call the pair $(Y, \mathcal{F})$ a *submartingale* if, for all $n \geq 0$,

1. $\mathbb{E}[Y_n^+] < \infty$
2. $\mathbb{E}\left[Y_{n+1} | \mathcal{F}_n\right] \geq Y_n$

   It is a *supermartingale* if, for all $n \geq 0$,

3. $\mathbb{E}[Y_n^-] < \infty$
4. $\mathbb{E}\left[Y_{n+1} | \mathcal{F}_n\right] \leq Y_n$

Definition 3.2. We call the pair $(S, \mathcal{F})$ *predictable* if S_n is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$. We call a predictable process $(S, \mathcal{F})$ increasing if $S_0 = 0$ and $\mathbb{P}\{S_n \leq S_{n+1}\} = 1$ for all n.

Proposition 3.3 (Doob decomposition). *A submartingale* $(Y, \mathcal{F})$ *with finite means may be expressed in the form*

$$Y_n = M_n + S_n$$

where $(M, \mathcal{F})$ *is a martingale, and* $(S, \mathcal{F})$ *is an increasing predictable process. This decomposition is unique.*

The process $(S, \mathcal{F})$ is called the *compensator* of the submartingale $(Y, \mathcal{F})$. Note that compensators have finite mean, since $0 \leq S_n \leq Y_n^+ - M_n$, implying that

$$\mathbb{E}|S_n| \leq \mathbb{E}\left[Y_n^+\right] + \mathbb{E}|M_n|.$$

Proof: We define M and S explicitly as follows:

$$\begin{aligned} M_0 &= Y_0, \\ S_0 &= 0, \\ M_{n+1} - M_n &= Y_{n+1} - \mathbb{E}\left[Y_{n+1}|\mathcal{F}_n\right], \\ S_{n+1} - S_n &= \mathbb{E}\left[Y_{n+1}|\mathcal{F}_n\right] - Y_n. \end{aligned}$$

□

4 Exercises

1) Let $X_1, X_2, \ldots$ be random variables such that the partial sums $S_n = X_1 + X_2 + \cdots + X_n$ determine a martingale. Show that $\mathbb{E}\left[X_i X_j\right] = 0$ if $i \neq j$.

2) Let $X_0, X_1, X_2, \cdots$ be a sequence of random variables with finite means and satisfying $\mathbb{E}[X_{n+1}|X_0, X_1, \cdots, X_n] = aX_n + bX_{n-1}$ for $n \geq 1$, where $0 < a, b < 1$ and $a + b = 1$. Find a value of α for which $S_n = \alpha X_n + X_{n-1}$, $n \geq 1$ defines a martingale with respect to the sequence X.

3) (i) If $(Y, \mathcal{F})$ is a martingale, show that $\mathbb{E}[Y_n] = \mathbb{E}[Y_0]$ for all n.

(ii) If $(Y, \mathcal{F})$ is a submartingale (respectively supermatingale) with finite means, show that $\mathbb{E}[Y_n] \geq \mathbb{E}[Y_0]$ (respectively $\mathbb{E}[Y_n] \leq \mathbb{E}[Y_0]$).

4) Let $(Y, \mathcal{F})$ be a martingale. Show that $\mathbb{E}[Y_{n+m} | \mathcal{F}_n] = Y_n$ for all $n, m \geq 0$.

5) Let $\{S_n : n \geq 0\}$ be a simple symmetric random walk on the integers with $S_0 = k$. Show that $S_n^2 - n$ is a martingale. Arguing as we did for the probability of ruin, find the expected duration of the game for the gambler's ruin problem.

6) Let X be a discrete-time Markov chain with countable state space S and transition matrix $\mathbf{P}$. Suppose that $\psi : S \to \mathbb{R}$ is bounded and satisfies $\sum_{j \in S} p_{ij}\psi(j) \leq \lambda\psi(i)$ for some $\lambda > 0$ and all $i \in S$. Show that $\lambda^{-n}\psi(X_n)$ constitutes a supermartingale.

5 Stopping times

Definition 5.1. A random variable T taking values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ is called a *stopping time* with respect to the filtration $\mathcal{F}$ if the indicator of the event $\{T = n\}$ is $\mathcal{F}_n$-measurable for all $n \geq 0$.

Note that the indicator of the event

$$\{T > n\} = \{T \leq n\}^c$$

is $\mathcal{F}_n$-measurable for all n. We write $[Z|\mathcal{F}_T]$ for $[Z|X_0, X_1, \ldots, X_T]$.

Example 5.2 (First passage times)**.** For each (sufficiently nice) subset B of $\mathbb{R}$ define the *first passage time* of X to B by

$$T_B = \min\{n\colon X_n \in B\}$$

with $T_B = \infty$ if $X_n \notin B$ for all n. It is easily seen that T_B is a stopping time.

Proposition 5.3. *Let $(Y, \mathcal{F})$ be a martingale and let T be a stopping time with respect to $\mathcal{F}$. Then the sequence $(Z, \mathcal{F})$, where $Z_n = Y_{T \wedge n}$, is a martingale.*

Proof: We may write

$$Z_n = \sum_{t=0}^{n-1} Y_t \mathbf{1}_{T=t} + Y_n \mathbf{1}_{T \geq n},$$

whence Z_n is $\mathcal{F}_n$-measurable and

$$\mathbb{E}\left[Z_n\right] \leq \sum_{t=0}^{n} \mathbb{E}\left[Y_t\right] < \infty.$$

Also $Z_{n+1} - Z_n = (Y_{n+1} - Y_n)\mathbf{1}_{T>n}$, whence

$$\mathbb{E}\left[Z_{n+1} - Z_n \mid \mathcal{F}_n\right] = \mathbb{E}\left[Y_{n+1} - Y_n \mid \mathcal{F}_n\right] \mathbf{1}_{T>n} = 0.$$

by the martingale property. □

6 Optional stopping

Proposition 6.1 (Optional sampling theorem, I). *Let* $(Y, \mathcal{F})$ *be a martingale. If* T *is a stopping time for which* $\mathbb{P}\{T \leq N\} = 1$ *for some fixed* $N (< \infty)$, *then* $\mathbb{E}[Y_T] < \infty$ *and* $\mathbb{E}[Y_T \mid \mathcal{F}_0] = Y_0$.

Proof: Suppose $\mathbb{P}\{T \leq N\} = 1$. Let $Z_n = Y_{T \wedge n}$, so that $(Z, \mathcal{F})$ is a martingale. Therefore $\mathbb{E}[Z_N] < \infty$ and

$$\mathbb{E}[Z_N \mid \mathcal{F}_0] = Z_0 = Y_0,$$

and the proof is finished by observing that $Z_N = Y_{T \wedge N} = Y_T$ a.s. $\square$

Example 6.2 (Hitting times of a nearest-neighbor martingale)**.** Let $(S_n)_{n\geq 0}$ be a martingale such that $|S_{n+1} - S_n| \in \{0,1\}$ for all n, and let T be the hitting time of $\{-a, b\}$ for some positive integers a and b, e.g.

$$T = \min\{n \geq 0 : S_n = -a \text{ or } S_n = b\}.$$

Fix an integer $N > 0$, and let $p_{a,b} = \mathbb{P}\{S_{T\wedge N} = -a | T \leq N\}$. Then

$$\begin{aligned} \mathbb{E}[S_{T\wedge N}] &= -a\mathbb{P}\{S_{T\wedge N} = -a\} + b\mathbb{P}\{S_{T\wedge N} = b\} + \mathbb{E}[S_N \mathbf{1}_{T>N}] \\ &= \mathbb{P}\{T \leq N\}(-ap_{a,b} + b(1 - p_{a,b})) + \mathbb{E}[S_N \mathbf{1}_{T>N}], \end{aligned}$$

and by the previous theorem, $\mathbb{E}[S_{T\wedge N}] = \mathbb{E}[S_0] = 0$, so

$$p_{a,b} = \frac{b - \mathbb{E}[S_N \mathbf{1}_{T>N}]/\mathbb{P}\{T \leq N\}}{b+a} \to \frac{b}{a+b} \text{ as } N \to \infty$$

as long as $\mathbb{P}\{T > N\} \to 0$ as $N \to \infty$.

Proposition 6.3 (Optional sampling theorem, II). *Let* $(Y, \mathcal{F})$ *be a martingale and let* T *be a stopping time. If*

1. $\mathbb{P}\{T < \infty\} = 1$,
2. $\mathbb{E}[|Y_T|] < \infty$, *and*
3. $\mathbb{E}[Y_n 1_{T>n}] \to 0$ *as* $n \to \infty$,

then $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$.

Proof: Note that $Y_T = Y_{T\wedge n} + (Y_T - Y_n)1_{T>n}$. Taking expectations and using the fact that $\mathbb{E}\left[Y_{T\wedge n}\right] = \mathbb{E}\left[Y_0\right]$, we find that

$$\begin{aligned}\mathbb{E}\left[Y_T\right] &= \mathbb{E}[Y_T \mathbf{1}_{T\leq n}] + \mathbb{E}[Y_T \mathbf{1}_{T>n}] \\ &= \mathbb{E}\left[Y_{T\wedge n}\right] + \mathbb{E}\left[Y_T \mathbf{1}_{T>n}\right] - \mathbb{E}\left[Y_n \mathbf{1}_{T>n}.\right] \\ &= \mathbb{E}\left[Y_0\right] + \mathbb{E}\left[Y_T \mathbf{1}_{T>n}\right] - \mathbb{E}\left[Y_n \mathbf{1}_{T>n}.\right]\end{aligned}$$

Now $\mathbb{E}[Y_n \mathbf{1}_{T>n}] \to 0$ as $n \to \infty$ by assumption, and

$$\mathbb{E}\left[Y_T \mathbf{1}_{T>n}\right] = \sum_{k=n+1}^{\infty} \mathbb{E}\left[Y_T \mathbf{1}_{T=k}\right]$$

is the tail of the convergent series $\mathbb{E}\left[Y_T\right] = \sum_k \mathbb{E}\left[Y_T \mathbf{1}_{T=k}\right]$; therefore $\mathbb{E}\left[Y_T \mathbf{1}_{T>n}\right] \to 0$ as $n \to \infty$. $\square$

Example 6.4 (Random walk with inertia). Let $\{X_0, X_1, X_2, \ldots\}$ be $\{+1, -1\}$-valued random variables with distribution:

$$\mathbb{P}\{X_0 = +1\} = \mathbb{P}\{X_0 = -1\} = \frac{1}{2}, \quad \text{and}$$

$$X_n = \begin{cases} X_{n-1} & \text{with probability } \frac{1}{2} \\ +1 & \text{with probability } \frac{1}{4} \\ -1 & \text{with probability } \frac{1}{4} \end{cases}.$$

Let $S_n = \sum_{k=1}^{n} X_k$. Notice that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \frac{1}{2}X_n$, which leads us to the martingale

$$Y_n = X_n + S_n.$$

Let $T = \min\{n \geq 0 : S_n \in \{-a, b\}\}$. The last theorem applies to T, so we have that $\mathbb{E}[Y_0] = \mathbb{E}[Y_T]$, and so if we let $p_{a,b} = \mathbb{P}\{S_T = -a\}$, then

$$\begin{aligned} \mathbb{E}[Y_0] &= 0 = \mathbb{E}[Y_T] \\ &= (-1-a)p_{a,b} + (1+b)(1-p_{a,b}), \end{aligned}$$

which implies that

$$p_{a,b} = \frac{1+b}{2+a+b}.$$

Example 6.5 (Markov chains)**.** Let X be an irreducible persistent Markov chain with countable state space S and transition matrix $\mathbf{P}$, and let $\psi\colon S \to \mathbb{R}$ be a bounded function satisfying

$$\sum_{j\in S} p_{ij}\psi(j) = \psi(i) \qquad \text{for all } i \in S.$$

Then $\psi(X_n)$ constitutes a martingale. Let T_i be the first passage time of X to the state i, that is, $T_i = \min\{n\colon X_n = i\}$. The sequence $\{\psi(X_n)\}$ is bounded and we obtain $\mathbb{E}\left[\psi(X_{T_i})\right] = \mathbb{E}\left[\psi(X_0)\right]$, whence $\mathbb{E}\left[\psi(X_0)\right] = \psi(i)$ for all states i and all choices of X_0. Therefore ψ is a constant function.

Proposition 6.6 (Optional Sampling Theorem, III). *Let* $(Y, \mathcal{F})$ *be a martingale, and let* T *be a stopping time. Then* $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$ *if the following hold:*

1. $\mathbb{P}\{T < \infty\} = 1$, $\mathbb{E}[T] < \infty$, *and*
2. *there exists a constant* c *such that* $\mathbb{E}[|Y_{n+1} - Y_n| \mid \mathcal{F}_n] \leq c$ *for all* $n < T$.

We omit the proof.

Example 6.7 (Wald's equation)**.** Let $X_1, X_2, \ldots$ be independent identically distributed random variables with finite mean μ, and let $S_n = \sum_{i=1}^{n} X_i$. Then $Y_n = S_n - n\mu$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$. Now

$$\mathbb{E}\left[|Y_{n+1} - Y_n| \mid \mathcal{F}_n\right] = \mathbb{E}[|X_{n+1} - \mu|] = \mathbb{E}[|X_1 - \mu|] < \infty.$$

Thus $\mathbb{E}\left[Y_T\right] = \mathbb{E}\left[Y_0\right] = 0$ for any stopping time T with finite mean, implying that

$$\mathbb{E}\left[S_T\right] = \mu \mathbb{E}\left[T\right].$$

Example 6.8 (Wald's identity). Let $X_1, X_2, \ldots$ be independent identically distributed random variables with common moment generating function $M(t) = \mathbb{E}\left[e^{tX}\right]$; suppose that there exists at least one value of $t(\neq 0)$ such that $1 \leq M(t) < \infty$, and fix t accordingly. Let $S_n = \sum_{i=1}^n X_i$. Define

$$Y_0 = 1, \quad Y_n = \frac{e^{tS_n}}{M(t)^n} \quad \text{for} \quad n \geq 1.$$

It is clear that $(Y, \mathcal{F})$ is a martingale. Let T be a stopping time with finite mean, and note that

$$\begin{aligned} \mathbb{E}\left[|Y_{n+1} - Y_n| \mid \mathcal{F}_n\right] &= Y_n \mathbb{E}\left[\left|\frac{e^{tX}}{M(t)} - 1\right|\right] \\ &\leq \frac{Y_n}{M(t)} \mathbb{E}\left[e^{tX} + M(t)\right] \\ &= 2Y_n. \end{aligned}$$

Suppose that T is such that

$$|S_n| \le C \quad \text{for} \quad n < T, \tag{6.1}$$

where C is a constant. Now $M(t) \ge 1$, and

$$Y_n = \frac{e^{tS_n}}{M(t)^n} \le \frac{e^{|t|C}}{M(t)^n} \le e^{|t|C} \quad \text{for} \quad n < T.$$

In summary, if T is a stopping time with finite mean such that (6.1) holds, then

$$\mathbb{E}[e^{tS} M(t)^{-T}] = 1 \quad \text{whenever} \quad M(t) \ge 1.$$

Example 6.9 (Simple random walk)**.** Suppose that $\{S_n\}$ is a simple random walk whose steps $\{X_i\}$ take the values 1 and -1 with respective probabilities p and $q(= 1 - p)$. For positive integers a and b, we have from Wald's identity that

$$e^{-at}\mathbb{E}\left[M(t)^{-T}\mathbf{1}_{S_T=-a}\right] + e^{tb}\mathbb{E}\left[M(t)^{-T}\mathbf{1}_{S_T=b}\right] = 1 \quad \text{if} \quad M(t) \geq 1 \tag{6.2}$$

where T is the first exit time of $(-a, b)$ as before, and $M(t) = pe^t + qe^{-t}$.

Setting $M(t) = s^{-1}$ we get $e^t = \lambda_1(s)$ or $e^t = \lambda_2(s)$ where

$$\lambda_1(s) = \frac{1 + \sqrt{1 - 4pqs^2}}{2ps}, \quad \lambda_2(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps}.$$

Substituting these into equation (6.2), we obtain two linear equations in the quantities

$$P_1(s) = \mathbb{E}\left[s^T \mathbf{1}_{S_T=-a,}\right] \quad P_2(s) = \mathbb{E}\left[s^T \mathbf{1}_{S_T=b}\right] \tag{6.3}$$

with solutions

$$P_1(s) = \frac{\lambda_1^a \lambda_2^a (\lambda_1^b - \lambda_2^b)}{\lambda_1^{a+b} - \lambda_2^{a+b}}, \quad P_2(s) = \frac{\lambda_1^a \lambda_2^a}{\lambda_1^{a+b} - \lambda_2^{a+b}}$$

which we add to obtain the probability generating function of T.

$$\mathbb{E}\left[s^T\right] = P_1(s) + P_2(s), \quad 0 < s \leq 1.$$

Suppose we let $a \to \infty$, so that T becomes the time until the first passage to the point b. From (6.3), $P_1(s) \to 0$ as $a \to \infty$ if $0 < s < 1$ and $P_2(s) \to F_b(s)$ where

$$F_b(s) = \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2qs} \right)^b .$$

Notice that $F_b(1) = (\min\{1, p/q\})^b$.

7 Crossing and convergence

Proposition 7.1 (Martingale convergence theorem)**.** *Let* $(Y, \mathcal{F})$ *be a submartingale and suppose that* $\mathbb{E}\left[Y_n^+\right] \leq M$ *for some* M *and all* n*. There exists a random variable* Y_∞ *such that* $Y_n \xrightarrow{a.s.} Y_\infty$ *as* $n \to \infty$*.*

Corollary 7.2. *If* $(Y, \mathcal{F})$ *is either a non-negative supermartingale or a non-positive submartingale, then* $Y_\infty = \lim_{n\to\infty} Y_n$ *exists almost surely.*

Suppose that $y = \{y_n : n \geq 0\}$ is a real sequence and $[a, b]$ a real interval. Let $U_n(a, b; y)$ be the number of up-crossings of $[a, b]$ by the subsequence $y_0, y_1, \ldots, y_n$, and let $U(a, b; y) = \lim_{n\to\infty} U_n(a, b; y)$ be the total number of such up-crossings by y.

Lemma 7.3. *If $U(a, b; y) < \infty$ for all rationals a and b satisfying $a < b$, then $\lim_{n\to\infty} y_n$ exists (but may be infinite).*

Suppose now that $(Y, \mathcal{F})$ is a submartingale, and let $U_n(a, b; Y)$ be the number of up-crossing of $[a, b]$ by Y up to time n.

Proposition 7.4 (Up-crossing inequality)**.** *If* $a < b$ *then*

$$\mathbb{E}[U_n(a, b; Y)] \leq \frac{\mathbb{E}\left[(Y_n - a)^+\right]}{b - a}.$$

Proof. Set $Z_n = (Y_n - a)^+$, so that $U_n(a, b; Y) = U_n(0, b-a; Z)$.

Let $[T_{2k-1}, T_{2k}]$, $k \geq 1$, be the up-crossing by Z of $[0, b-a]$, and define the indicator function

$$I_i = \begin{cases} 1 & \text{if } i \in (T_{2k-1}, T_{2k}] \text{ for some } k, \\ 0 & \text{otherwise} \end{cases}$$

Note that I_i is $\mathcal{F}_{i-1}$-measurable. Now

$$(b-a)U_n(0, b-a; Z) \leq \left(\sum_{i=1}^{n} (Z_i - Z_{i-1}) I_i \right).$$

However

$$\begin{aligned}\mathbb{E}\left[(Z_i - Z_{i-1})I_i\right] &= \mathbb{E}\left[\mathbb{E}[(Z_i - Z_{i-1})I_i \mid \mathcal{F}_{i-1}]\right] \\ &= \mathbb{E}\left[I_i[\mathbb{E}\left[Z_i \mid \mathcal{F}_{i-1}\right] - Z_{i-1}]\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left[Z_i \mid \mathcal{F}_{i-1}\right] - Z_{i-1}\right] \\ &= \mathbb{E}\left[Z_i\right] - \mathbb{E}\left[Z_{i-1}\right]\end{aligned}$$

and so

$$(b-a)U_n(0, b-a; Z) \leq \mathbb{E}\left[Z_n\right] - \mathbb{E}\left[Z_0\right] \leq E(Z_n).$$

□

[Proof of proposition (7.1)] Suppose $(Y, \mathcal{F})$ is a submartingale and $\mathbb{E}\left[Y_n^+\right] \leq M$ for all n. From the up-crossing inequality we have that, if $a < b$,

$$\mathbb{E}U_n(a, b; Y) \leq \frac{\mathbb{E}\left[Y_n^+\right] + |a|}{b - a}$$

so that $U(a, b; Y) = \lim_{n \to \infty} U_n(a, b; Y)$ satisfies

$$\mathbb{E}U(a, b; Y) = \lim_{n \to \infty} \mathbb{E}U_n(a, b; Y) \leq \frac{M + |a|}{b - a}$$

for all $a < b$. Therefore $U(a, b; Y) < \infty$ a.s. for all $a < b$. Since there are only countably many rationals, it follows that, with probability 1, $U(a, b; Y) < \infty$ for all rational a and b. And the sequence Y_n converges almost surely to some limit Y_∞ (which could be infinite).

We omit the proof that Y_∞ is actually finite with probability one. $\square$

Example 7.5 (Random walk)**.** Consider De Moivre's martingale, namely

$$Y_n = (q/p)^{S_n},$$

where S_n is the position after n steps of the usual simple random walk.

The sequence $(Y_n)_{n\geq 0}$ is a non-negative martingale, and hence converges almost surely to some finite limit Y as $n \to \infty$. This is not of much interest if $p = q$, since $Y_n = 1$ for all n in this case. Suppose that $p \neq q$.

The random variable Y_n takes values in the set $\{\rho^k : k = 0, \pm 1, \ldots\}$, where $\rho = q/p$. Certainly Y_n cannot converge to any given (possibly random) member of this set, since this would necessarily entail that S_n converges to a finite limit (which is obviously false).

Therefore Y_n converges to a limit point of the set, not lying within the set. The only such limit point which is finite is 0, and therefore

$$Y_n \to 0 \quad \text{a.s..}$$

Hence,

$$S_n \to -\infty \quad \text{a.s. if } p < q,$$

and

$$S_n \to +\infty \quad \text{a.s. if } p > q.$$

Note that Y_n does not converge in mean, since $\mathbb{E}[Y_n] = \mathbb{E}[Y_0] \neq 0$ for all n.

Lemma 7.6. *Let* $(Y, \mathcal{F})$ *be a martingale. Then* Y_n *converges in mean if and only if there exists a random variable* Z *with finite mean such that* $Y_n = \mathbb{E}[Z \mid \mathcal{F}_n]$. *If* $Y_n \xrightarrow{1} Y_\infty$, *then* $Y_n = \mathbb{E}[Y_\infty \mid \mathcal{F}_n]$.

Remark. That is, Y_∞ is a possible choice of Z, and the unique one that is $\mathcal{F}_\infty$-measurable.

8 Exercises

1) If T_1 and T_2 are stopping times with respect to a filtration $\mathcal{F}$, show that $T_1 + T_2$, $\max\{T_1, T_2\}$, and $\min\{T_1, T_2\}$ are stopping times also.

2) Let $X_1, X_2, \ldots$ be a sequence of non-negative independent random variables and let $N(t) = \max\{n : X_1 + X_2 + \cdots + X_n \leq t\}$. Show that $N(t) + 1$ is a stopping time with respect to a suitable filtration to be specified.

3) Let $(Y, \mathcal{F})$ be a submartingale and let S and T be stopping times satisfying $0 \leq S \leq T \leq N$ for some deterministic N. Show that $\mathbb{E}(Y_0) \leq \mathbb{E}(Y_S) \leq \mathbb{E}(Y_T) \leq \mathbb{E}(Y_N)$.

4) Let $\{S_n\}$ be a simple random walk with $S_0 = 0$ such that $0 < p = \mathbb{P}\{S_1 = 1\} < \frac{1}{2}$. Use de Moivre's martingale to show that $\mathbb{E}(\sup_m S_m) = \frac{p}{1-2p}$.

5) Let $\{S_n : n \geq 0\}$ be a simple symmetric random walk with $S_0 = 0$. Show that

$$Y_n = \frac{\cos\{\lambda[S_n - \frac{1}{2}(b-a)]\}}{(\cos\lambda)^n}.$$

constitutes a martingale if $\cos\lambda \neq 0$.

6) Let $S_n = a + \sum_{r=1}^{n} X_r$ be a simple symmetric random walk. The walk stops at the earliest time T when it reaches either of the two positions 0 or K where $0 < a < K$. Show that $M_n = \sum_{r=0}^{n} S_r - \frac{1}{3}S_n^3$ is a martingale and deduce that $\mathbb{E}(\sum_{r=0}^{T} S_r) = \frac{1}{3}(K^2 - a^2)a + a$.

9 The maximal inequality

Let's denote $Y_n^* = \max\{Y_i : 0 \leq i \leq n\}$.

Proposition 9.1 (Maximal inequality). *1. If $(Y, \mathcal{F})$ is a submartingale, then*

$$\mathbb{P}\{Y_n^* \geq x\} \leq \frac{\mathbb{E}[Y_n^+]}{x} \quad \textit{for} \quad x > 0.$$

2. If $(Y, \mathcal{F})$ is a supermartingale and $\mathbb{E}|Y_0| < \infty$,then

$$\mathbb{P}\{Y_n^* \geq x\} \leq \frac{\mathbb{E}[Y_0] + \mathbb{E}[Y_n^-]}{x} \quad \textit{for} \quad x > 0.$$

Proof. Let $T = \min\{n \colon Y_n \geq x\}$ where $x > 0$. Suppose first that $(Y, \mathcal{F})$ is a submartingale. Then $(Y^+, \mathcal{F})$ is a non-negative submartingale with finite means and $T = \min\{n \colon Y_n^+ \geq x\}$. Applying the optional sampling theorem with stopping times $T_1 = T \wedge n$, $T_2 = n$, we obtain $\mathbb{E}\left[Y_{T\wedge n}^+\right] \leq \mathbb{E}\left[Y_n^+\right]$. However,

$$\begin{aligned} \mathbb{E}\left[Y_{T\wedge n}^+\right] &= \mathbb{E}\left[Y_T^+ \mathbf{1}_{T\leq n}\right] + \mathbb{E}\left[Y_n^+ \mathbf{1}_{T\leq n}\right] \\ &\geq x\mathbb{P}\left\{T \leq n\right\} + \mathbb{E}\left[Y_n^+ \mathbf{1}_{T\leq n}\right] \end{aligned}$$

whence

$$\begin{aligned} x\mathbb{P}\left\{T \leq n\right\} &\leq \mathbb{E}\left[Y_n^+ (1 - \mathbf{1}_{T>n})\right] \\ &= x\mathbb{E}\left[Y_n^+ \mathbf{1}_{T\leq n}\right] \leq \mathbb{E}\left[Y_n^+ .\right] \end{aligned}$$

Suppose next that $(Y, \mathcal{F})$ is a supermartingale. By optional sampling $\mathbb{E}[Y_0] \geq \mathbb{E}[Y_{T\wedge n}]$.

Now

$$\begin{aligned}\mathbb{E}[Y_{T\wedge n}] &= \mathbb{E}[Y_T \mathbf{1}_{T\leq n} + Y_n \mathbf{1}_{T>n}] \\ &\geq x\mathbb{P}\{T \leq n\} - \mathbb{E}\left[Y_n^-,\right]\end{aligned}$$

whence $x\mathbb{P}\{T \leq n\} \leq \mathbb{E}[Y_0] + \mathbb{E}[Y_n^-]$.

Example 9.2 (Doob-Kolmogorov inequality)**.** Let $(Y, \mathcal{F})$ be a martingale such that $\mathbb{E}\left[Y_n^2\right] < \infty$ for all n. Then $(Y_n^2, \mathcal{F}_n)$ is a submartingale, whence

$$\mathbb{P}\left(\max_{0 \le k \le n} |Y_k| \ge x\right) = \mathbb{P}\left(\max_{0 \le k \le n} Y_k^2 \ge x^2\right) \le \frac{\mathbb{E}\left[Y_n^2\right]}{x^2}$$

for $x > 0$.

Example 9.3 (Gambling systems)**.** For a given game, write $Y_0, Y_1, \ldots$ for the sequence of capitals obtained by wagering one unit on each play; that is Y_0 is the initial capital, and Y_n is the capital obtained after n gambles each involving a unit stake.

A general betting strategy would allow the gambler to vary her stake. If she bets S_n on the nth play, her profit is $S_n(Y_n - Y_{n-1})$ since $Y_n - Y_{n-1}$ is the profit resulting from a stake of one unit. Hence the gambler's capital Z_n after n plays satisfies

$$Z_n = Z_{n-1} + S_n(Y_n - Y_{n-1}) = Y_0 + \sum_{i=1}^{n} S_i(Y_i - Y_{i-1}).$$

Notice that $(S, \mathcal{F})$ must be a predictable process. The sequence Z is called the *transform* of Y by S. If Y is a martingale, we call Z a *martingale transform*.

Proposition 9.4. *Let $(S, \mathcal{F})$ be a predictable process, and let Z be the transform of Y by S. Then:*

1. *if $(Y, \mathcal{F})$ is a martingale, then $(Z, \mathcal{F})$ is a martingale as long as $\mathbb{E}|Z_n| < \infty$ for all n,*
2. *if $(Y, \mathcal{F})$ is a submartingale and in addition $S_n \geq 0$ for all n, then $(Z, \mathcal{F})$ is a submartingale as long as $\mathbb{E}Z_n^+ < \infty$ for all n.*

Proof:

$$\begin{aligned}\mathbb{E}\left[Z_{n+1} \mid \mathcal{F}_n\right] - Z_n &= \mathbb{E}[S_{n+1}(Y_{n+1} - Y_n) \mid \mathcal{F}_n] \\ &= S_{n+1}[\mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] - Y_n].\end{aligned}$$

□

A number of special cases are of value.

1. *Optional skipping.* At each play the gambler either wagers a unit stake or skips the round; S equals either 0 or 1.

2. *Optional stopping.* The gambler wagers a unit stake on each play until the (random) time T, when she gambles for the last time. That is,
$$S_n = \begin{cases} 1 & \text{if} \quad n \le T, \\ 0 & \text{if} \quad n > T, \end{cases}$$
and $Z_n = Y_{T \wedge n}$. Now $\{T = n\} = \{S_n = 1, S_{n+1} = 0\} \in \mathcal{F}_n$, so that T is a stopping time. It is a consequence of Proposition (9.4) that $(Y_{T\wedge n}, \mathcal{F}_n)$ is a martingale whenever Y is a martingale, as established earlier.

3. *Optional starting.* The gambler does not play until the $(T+1)$th play , where T a stopping time. In this case $S_n = 0$ for $n \le T$.

Let $(X, \mathcal{F})$ and $(Y, \mathcal{F})$ be two martingales with respect to the filtration $\mathcal{F}$. Let T be a stopping time with respect to $\mathcal{F}$; T is the switching time from X to Y.

Proposition 9.5 (Optional switching)**.** *Suppose that $X_T = Y_T$ on the event $\{T < \infty\}$. Then*

$$Z_n = \begin{cases} X_n & \text{if } n < T, \\ Y_n & \text{if } n \geq T, \end{cases}$$

defines a martingale with respect to $\mathcal{F}$.

Proof: Note that

$$Z_n = X_n \mathbf{1}_{n<T} + Y_n \mathbf{1}_{n \geq T};$$

is $\mathcal{F}_n$-measurable. Also $\mathbb{E}|Z_n| \leq \mathbb{E}|X_n| + \mathbb{E}|Y_n| < \infty$. By the martingale property of X and Y,

$$\begin{aligned} Z_n &= \mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] \mathbf{1}_{n<T} + \mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] \mathbf{1}_{n \geq T} \\ &= \mathbb{E}\left[X_{n+1} \mathbf{1}_{n<T} + Y_{n+1} \mathbf{1}_{n \geq T} \mid \mathcal{F}_n .\right] \end{aligned}$$

Now

$$\begin{aligned} X_{n+1} \mathbf{1}_{n<T} + Y_{n+1} \mathbf{1}_{n \geq T} &= Z_{n+1} + X_{n+1} \mathbf{1}_{n+1=T} - Y_{n+1} \mathbf{1}_{n+1=T} \\ &= Z_{n+1} + (X_T - Y_T) \mathbf{1}_{n+1=T}. \end{aligned}$$

By the assumption that $X_T = Y_T$ on the event $\{T < \infty\}$, we have that $Z_n = \mathbb{E}\left[Z_{n+1} \mid \mathcal{F}_n\right]$, so that $(Z, \mathcal{F})$ is a martingale. □