**Question:** Find the mean of a branching process with immigration (described in Question 5.4.5), in terms of the mean and variance of the offspring and immigrant distributions.

**Solution:** Note first that the general solution to the recursion $a_{n+1} = ra_n + c$ is given by $a_n = (a_0 - a^*)r^n + a^*$, where $a^*$, the steady state solution, is defined by $a^* = \frac{c}{1-r}$ (see e.g. Wikipedia). A simple way to see this is to divide through by $r^{-(n+1)}$, and solve the resulting recursion for $b_n = r^{-n}a_n$.

Let the mean of the offspring distribution be $\mu_o \neq 1$ and let the mean of the immigration distribution be $\mu_i$. Then

$$E[Z_{n+1}] = E[E[Z_{n+1}|Z_n]] = E[\mu_o Z_n + \mu_i] = \mu_o E[Z_n] + \mu_i.$$ 

You may also arrive at the recursion by differentiating the recursion for the generating function of $Z_n$. Solving the recursion,

$$E[Z_n] = E[Z_0] \mu_o^n + \mu_i \frac{\mu_o^n - 1}{\mu_o - 1}.$$ 

If $\mu_o = 1$, on the other hand, then we have that $E[Z_n] = Z_0 + n\mu_i$.

**Question:**

(a) Show, for any two linear fractional transformations

$$f(s) = \frac{as + b}{cs + d} \quad \text{and} \quad g(s) = \frac{es + f}{gs + h}$$

that

$$f(g(s)) = \frac{As + B}{Cs + D},$$

where $A, B, C$ and $D$ are defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

(b) Show that if $\mathbb{P}(Y = 0) = d$ and $\mathbb{P}(Y = k) = (1 - d)p(1 - p)^{k-1}$ for $k > 0$, then the generating function of $Y$ has this form.

(c) Explain how to find the generating function of the branching process whose offspring distribution is $Y$.

**Solution:**

(a)

$$f(g(s)) = f\left(\frac{es + f}{gs + h}\right) = \frac{a_{cs + f} + b}{gs + h} = \frac{(ae + bg)s + (af + bh)}{(ce + dg)s + (cf + dh)}$$
(b)

\[
E[s^Y] = 1 \cdot \mathbb{P}(Y = 0) + \sum_{k=1}^{\infty} s^k \mathbb{P}(Y = k)
\]

\[
= d + ps(1 - d) \sum_{k=1}^{\infty} s^{k-1} (1 - p)^{k-1}
\]

\[
= d + \frac{ps(1-d)}{1-s(1-p)} = \frac{s(p-d) + d}{s(p-1) + 1}
\]

(c) \(G_{Z_n}(s) = \frac{A_n + B_n}{C_n + D_n}\) where

\[
\begin{pmatrix}
A_n & B_n \\
C_n & D_n
\end{pmatrix} = \begin{pmatrix}
p - d & d \\
p - 1 & 1
\end{pmatrix}^n
\]

From this expression it is possible (although not necessary for the homework) to calculate \(A_n, B_n, C_n\) and \(D_n\) fairly easily. Although the matrix is not symmetric, it is (like all two-by-two matrices) similar to a diagonal matrix. Let \(\mu = i\sqrt{d/(1-p)}\) (we may suppose that \(p < 1\)). Then, diagonalizing the matrix, if

\[
V = \begin{pmatrix}
\mu & 1 \\
-1/\mu & 1
\end{pmatrix},
\]

and

\[
\Lambda = \begin{pmatrix}
1 - d & 0 \\
0 & p
\end{pmatrix},
\]

we can check that

\[
\begin{pmatrix}
p - d & d \\
p - 1 & 1
\end{pmatrix}^n = V \Lambda^n V^{-1}
\]

Since \(\Lambda\) is diagonal, this is easy to compute.