Analysis: Assignment 1
Due on Friday, October 12, 2012
Lin 11:00am

A digital copy of this document can be found at http://pages.uoregon.edu/raies

Dan Raies
Last edited October 22, 2012
Contents

Exercise 1 ................................................................. 3
Exercise 2 ................................................................. 4
Exercise 3 ................................................................. 5
Exercise 4 ................................................................. 6
  Part (a) ................................................................. 6
  Part (b) ................................................................. 6
Exercise 5 ................................................................. 7
  Part (a) ................................................................. 7
  Part (b) ................................................................. 7
  Part (c) ................................................................. 7
Exercise 6 ................................................................. 9
Exercise 7 ................................................................. 10
  Part (a) ................................................................. 10
  Part (b) ................................................................. 11
Exercise 8 ................................................................. 12
  Part (a) ................................................................. 12
  Part (b) ................................................................. 13
  Part (c) ................................................................. 13
Exercise 9 ................................................................. 14
Exercise 10 ............................................................... 15
Exercise 11 ............................................................... 16
  Part (a) ................................................................. 17
  Part (b) ................................................................. 17
Exercise 12 ............................................................... 18
Exercise 13 ............................................................... 19
Exercise 14 ............................................................... 21
Exercise 15 ............................................................... 22
Exercise 16 ............................................................... 23
Exercise 17 ............................................................... 25
Exercise 1

Find the $\sigma$-algebra on $\mathbb{R}$ that is generated by the collection of all one-point subsets of $\mathbb{R}$.

Solution to Exercise 1:

Let $\Sigma$ be the set of all subsets $S \subseteq \mathbb{R}$ such that either $S$ is countable or $S^c$ is countable. Note that clearly $\Sigma$ contains all the singletons of $\mathbb{R}$ as a set of the form $\{x\}$ is countable.

(i) Since $\emptyset$ is countable, $\emptyset \in \Sigma$.

(ii) If a set $S \in \Sigma$ then either $S = (S^c)^c$ is countable or $S^c$ is countable so $S^c \in \Sigma$.

(iii) Let $\{S_n\}_{n \in \mathbb{N}} \subseteq \Sigma$.

Case 1: If $S_n$ is countable for each $n \in \mathbb{N}$ then $\bigcup_{n \in \mathbb{N}} S_n$ is the countable union of countable sets so $\bigcup_{n \in \mathbb{N}} S_n \in \Sigma$.

Case 2: If there exists a $k \in \mathbb{N}$ such that $S_k$ is uncountable then $S_k^c$ must be countable which forces

$$
\left( \bigcup_{n \in \mathbb{N}} S_n \right)^c = \bigcap_{n \in \mathbb{N}} S_n^c = S_k \cap \left( \bigcap_{n \in \mathbb{N} \setminus \{k\}} S_n \right)
$$

(1)

to be countable since the intersection of anything with a countable set is still countable. Hence $\bigcup_{n \in \mathbb{N}} S_n \in \Sigma$.

In both cases we have that $\bigcup_{n \in \mathbb{N}} S_n \in \Sigma$, as desired.

It follows that $\Sigma$ is a $\sigma$-algebra.

Let $\Sigma'$ be any $\sigma$-algebra of $\mathcal{B}$ which contains the singletons in $\mathbb{R}$. Let $S$ be any countable subset of $\mathbb{R}$. Then $S \in \Sigma'$ as $S = \bigcup_{x \in S} \{x\}$ and $\Sigma'$ is a $\sigma$-algebra. Additionally, since $S \in \Sigma'$ we must have that $S^c \in \Sigma'$. It follows that $\Sigma \subseteq \Sigma'$ so that $\Sigma$ is the smallest $\sigma$-algebra of $\mathbb{R}$ generated by the singletons, as desired.
Exercise 2

Show that $\mathcal{B}(\mathbb{R})$ is generated by the collection of all compact subsets of $\mathbb{R}$.

Solution to Exercise 2:

Let $\mathcal{B}_1$ be the $\sigma$-algebra generated by the compact subsets of $\mathbb{R}$. Since $\mathcal{B}(\mathbb{R})$ contains all of the open sets and is closed under complementation, $\mathcal{B}(\mathbb{R})$ must contain all of the closed sets and hence must contain all of the compact sets so that $\mathcal{B}_1 \subseteq \mathcal{B}(\mathbb{R})$. Choose an open set $B \subseteq \mathbb{R}$. Then $\mathbb{R} \setminus B$ is closed. If $\mathbb{R} \setminus B$ is also bounded then it is compact so that $B \in \mathcal{B}_1$. Even if $\mathbb{R} \setminus B$ is not bounded then it is equal to the the countable union of closed intervals which are compact. Since $\mathcal{B}_1$ is closed under countable union $\mathbb{R} \setminus B$ is in $\mathcal{B}_1$ and since $\mathcal{B}_1$ is closed under complementation, $B \in \mathcal{B}_1$ from whence it follows that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}_1$. Finally, $\mathcal{B}_1 = \mathcal{B}(\mathbb{R})$, as desired.
Exercise 3

Let $X$ be a set and $E$ be a collection of subsets of $X$. Prove that there exists a unique monotone class $M(E)$ such that $E \subseteq M(E)$ and if $M_1$ is another monotone class containing $E$ then $M_1 \supseteq M(E)$.

Lemma 1. If $M_1$ and $M_2$ are both monotone classes $M_1 \cap M_2$ is also a monotone class.

Proof. Let $\{A_n\}_{n \in \mathbb{N}} \subseteq M_1 \cap M_2$ be a sequence of sets such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$. Since $\{A_n\}_{n \in \mathbb{N}} \subseteq M_1$ and $M_1$ is a monotone class then $\bigcup_{n \in \mathbb{N}} A_n \in M_1$. Similarly, since $\{A_n\}_{n \in \mathbb{N}} \subseteq M_2$ and $M_2$ is a monotone class then $\bigcup_{n \in \mathbb{N}} A_n \in M_2$. It follows that $\bigcup_{n \in \mathbb{N}} A_n \in M_1 \cap M_2$.

Let $\{A_n\}_{n \in \mathbb{N}} \subseteq M_1 \cap M_2$ be a sequence of sets such that $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$. Since $\{A_n\}_{n \in \mathbb{N}} \subseteq M_1$ and $M_1$ is a monotone class then $\bigcap_{n \in \mathbb{N}} A_n \in M_1$. Similarly, since $\{A_n\}_{n \in \mathbb{N}} \supseteq M_2$ and $M_2$ is a monotone class then $\bigcap_{n \in \mathbb{N}} A_n \in M_2$. It follows that $\bigcap_{n \in \mathbb{N}} A_n \in M_1 \cap M_2$.

Based on the above, $M_1 \cap M_2$ is a monotone class.

Solution to Exercise 3:

Let $X$ be a set and $E$ be a collection of subsets of $X$. Let $\mathcal{M}(E)$ be the collection of all monotone classes which contain $E$. The collection $\mathcal{M}(E)$ is nonempty since $\mathcal{P}(X)$, the power set of $X$, is clearly a monotone class which contains $E$. Now define

$$M(E) = \bigcap_{M \in \mathcal{M}(E)} M,$$  \hspace{1cm} (2)

that is $M(E)$ is the intersection of all monotone classes which contain $E$. By Lemma 1, $M(E)$ must be a monotone class. It is clearly unique and must be contained in all other monotone classes which contain $E$. The result then follows.
Exercise 4

Prove that a \( \sigma \)-ring is monotone and a monotone class which is a ring is always a \( \sigma \)-ring.

Part (a)

Prove that a \( \sigma \)-ring is a monotone class.

_Solution to Exercise 4(a):_

Let \( R \) be a \( \sigma \)-ring.

(i) Let \( \{ E_n \}_{n \in \mathbb{N}} \subseteq R \) such that \( E_n \subseteq E_{n+1} \). Since \( R \) is closed under all countable unions it follows that \( \bigcup_{n \in \mathbb{N}} E_n \in R \).

(ii) Let \( \{ E_n \}_{n \in \mathbb{N}} \subseteq R \) such that \( E_n \supseteq E_{n+1} \). Note that \( E_1 \subseteq E_n \) for all \( n \in \mathbb{N} \) and \( R \) is closed under set subtraction so \( E_1 \setminus E_n \) for all \( n \in \mathbb{N} \). Since \( R \) is also closed under countable union we have that

\[
E_1 \setminus \bigcap_{n \in \mathbb{N}} E_n = \bigcup_{n \in \mathbb{N}} (E_1 \setminus E_n) \in R. \tag{3}
\]

Since \( E_1 \in R \) and \( E_1 \setminus \bigcap_{n \in \mathbb{N}} E_n \in R \) it follows that

\[
\bigcap_{n \in \mathbb{N}} E_n = E_1 \setminus \left( E_1 \setminus \bigcap_{n \in \mathbb{N}} E_n \right) \in R \tag{4}
\]

as desired.

Finally it follows that \( R \) is a monotone class.

Part (b)

Prove that a monotone class which is a ring is always a \( \sigma \)-ring.

_Solution to Exercise 4(b):_

Let \( M \) be a monotone class and a ring and choose \( \{ A_n \}_{n \in \mathbb{N}} \subseteq M \). Now let \( B_n = \bigcup_{k=1}^{n} A_n \) for each \( n \in \mathbb{N} \). Since \( M \) is a ring and is closed under finite unions it follows that \( \{ B_n \}_{n \in \mathbb{N}} \subseteq M \). Since \( B_{n+1} = B_n \cup A_{n+1} \) for all \( n \in \mathbb{N} \) we have \( B_n \subseteq B_{n+1} \) for all \( n \in \mathbb{N} \). Then since \( M \) is a monotone class, \( \bigcup_{n \in \mathbb{N}} B_n \in M \). It is clear, though, that \( \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n \) so that \( \bigcup_{n \in \mathbb{N}} A_n \in M \) and \( M \) is a \( \sigma \)-ring.
Exercise 5

Suppose that $\mu$ is a finite measure on $\{X, \mathcal{A}\}$.

**Part (a)**

Show that if $A$ and $B$ belong to $\mathcal{A}$ then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$  \hfill (5)

**Solution to Exercise 5(a):**

Choose $A, B \in \mathcal{A}$ and write $A \cup B = (A \setminus B) \sqcup (A \cap B) \sqcup (B \setminus A)$, which is a disjoint union. Then

$$\mu(A) + \mu(B) = \mu((A \setminus B) \sqcup (A \cap B)) + \mu((B \setminus A) \sqcup (A \cap B))$$
$$= \mu(A \setminus B) + \mu(A \cup B) + \mu(B \setminus A) + \mu(A \cap B)$$
$$= \mu((A \setminus B) \sqcup (A \cap B) \sqcup (B \setminus A)) + \mu(A \cap B)$$
$$= \mu(A \cup B) + \mu(A \cap B).$$  \hfill (6)

After rearranging Equation 6 we have that $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$, as desired.

**Part (b)**

Show that if $A, B$, and $C$ belong to $\mathcal{A}$ then

$$\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C) - \mu(A \cap B)$$
$$- \mu(A \cap C) - \mu(B \cap C) + \mu(A \cap B \cap C).$$ \hfill (7)

**Solution to Exercise 5(b):**

Choose $A, B, C \in \mathcal{B}(\mathbb{R})$. Using part (a) we have

$$\mu(A \cup B \cup C) = \mu(A \cup (B \cup C))$$
$$= \mu(A) + \mu(B \cup C) - \mu(A \cap (B \cup C))$$
$$= \mu(A) + \mu(B) + \mu(C) - \mu(B \cap C) - \mu((A \cap B) \cup (A \cap C))$$
$$= \mu(A) + \mu(B) + \mu(C) - \mu(B \cap C)$$
$$- (\mu(A \cap B) + \mu(A \cap C) - \mu((A \cap B) \cap (A \cap C)))$$
$$= \mu(A) + \mu(B) + \mu(C) - \mu(A \cap B)$$
$$- \mu(A \cap C) - \mu(B \cap C) + \mu(A \cap B \cap C)$$ \hfill (8)

as desired.

**Part (c)**

Find and prove a corresponding formula for the measure of the union of $n$ sets.

**Solution to Exercise 5(c):**

For $k, n \in \mathbb{N}$ with $k \leq n$ define

$$\Omega^n_k = \{S \subseteq \{1, 2, \ldots, n\} \mid |S| = k\}. \hfill (9)$$
As an example, if \( n = 10 \) and \( k = 3 \), one element of \( \Omega^n_k \) is \( \{3, 4, 9\} \). I claim that for any collection of sets \( \{A_m\}_{m=1}^n \subseteq \mathcal{A} \),
\[
\mu \left( \bigcup_{m=1}^n A_m \right) = \sum_{m=1}^n (-1)^{m+1} \sum_{S \in \Omega_m^n} \mu \left( \bigcap_{k \in S} A_k \right). \tag{10}
\]
I wish to prove the above claim by induction. Part (a) proved the base case so assume that Equation 10 holds for any collection of \( n \) sets in \( \mathcal{A} \) and let \( \{A_m\}_{m=1}^{n+1} \subseteq \mathcal{A} \). Then
\[
\mu \left( \bigcup_{m=1}^{n+1} A_m \right) = \mu \left( \left( \bigcup_{m=1}^n A_m \right) \cup A_{n+1} \right) \\
= \mu \left( \bigcup_{m=1}^n A_m \right) + \mu(A_{n+1}) - \mu \left( \bigcup_{m=1}^n A_m \cap A_{n+1} \right) \\
= \sum_{m=1}^n (-1)^{m+1} \sum_{S \in \Omega_m^n} \mu \left( \bigcap_{k \in S} A_k \right) + \mu(A_{n+1}) - \mu \left( \bigcup_{m=1}^n (A_m \cap A_{n+1}) \right) \\
= \sum_{m=1}^n (-1)^{m+1} \sum_{S \in \Omega_m^n} \mu \left( \bigcap_{k \in S} A_k \right) + \mu(A_{n+1}) \\
- \sum_{m=1}^n (-1)^{m+1} \sum_{S \in \Omega_m^{n+1}} \mu \left( \bigcap_{k \in S} (A_k \cap A_{n+1}) \right) \\
= \sum_{m=1}^{n+1} (-1)^{m+1} \sum_{S \in \Omega_m^{n+1}} \mu \left( \bigcap_{k \in S} A_k \right) \tag{11}
\]
where the last equality in Equation 11 holds because
\[
\bigcup_{m=1}^{n+1} \Omega_m^{n+1} = \left( \bigcup_{m=1}^n \Omega_m^n \right) \cup \{n + 1\} \cup \left( \bigcup_{m=1}^n \{C \cup \{n + 1\} \mid C \in \Omega_m^n \} \right). \tag{12}
\]
By Equation 11 we have that
\[
\mu \left( \bigcup_{m=1}^{n+1} A_m \right) = \sum_{m=1}^{n+1} (-1)^{m+1} \sum_{S \in \Omega_m^{n+1}} \mu \left( \bigcap_{k \in S} A_k \right) \tag{13}
\]
and hence by induction, Equation 10 holds for every \( n \in \mathbb{N} \).
Exercise 6

Let $\mathcal{A}$ be the $\sigma$-algebra of all subsets of $\mathbb{N}$ and let $\mu$ be the counting measure on $(\mathbb{N}, \mathcal{A})$. Give a decreasing sequence $\{A_k\}_{k \in \mathbb{N}}$ of sets in $\mathcal{A}$ such that $\mu(\bigcap_{k \in \mathbb{N}} A_k) \neq \lim_{k \to \infty} \mu(A_k)$.

Solution to Exercise 6:

Let $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and let $\mu$ be the counting measure on $(\mathbb{N}, \mathcal{A})$. Define the sequence $\{N_k\}_{k \in \mathbb{N}}$ such that

$$N_k = \{n \in \mathbb{N} \mid n \geq k\} \text{ for all } k \in \mathbb{N}. \tag{14}$$

Note that $\{N_k\}_{k \in \mathbb{N}}$ is a decreasing sequence and $\mu(N_k) = \infty$ for all $k \in \mathbb{N}$ so that $\lim_{k \to \infty} \mu(N_k) = \infty$. Additionally, $\bigcap_{k \in \mathbb{N}} N_k = \emptyset$ so that $\mu(\bigcap_{k \in \mathbb{N}} N_k) = 0$. It follows that $\{N_k\}_{k \in \mathbb{N}}$ is a decreasing sequence such that

$$\mu\left(\bigcap_{k \in \mathbb{N}} N_k\right) \neq \lim_{k \to \infty} \mu(N_k) \tag{15}$$

as desired.
Exercise 7

Let $\mu$ be a measure on measurable space $(X, \mathcal{A})$ and let $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$.

**Lemma 2.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{E_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$.

(i) if $E_i \subseteq E_{i+1}$ for all $i \in \mathbb{N}$ then

$$\lim_{i \to \infty} \mu(E_i) = \mu \left( \bigcup_{i \in \mathbb{N}} E_i \right)$$

and

(ii) if $E_i \supseteq E_{i+1}$ for all $i \in \mathbb{N}$ and $\mu(E_i) < \infty$ for all $i \in \mathbb{N}$ then

$$\lim_{i \to \infty} \mu(E_i) = \mu \left( \bigcap_{i \in \mathbb{N}} E_i \right).$$

I need Lemma 2 for the solution below. I’m omitting the proof because it’s rather trivial but I wanted to state the result explicitly since it was not mentioned in class.

**Part (a)**

Prove that if there is a $k \in \mathbb{N}$ such that $\mu(\bigcup_{n=k}^{\infty} E_n) < \infty$ then

$$\mu \left( \bigcap_{j \in \mathbb{N}} \bigcup_{n=j}^{\infty} E_n \right) \geq \limsup_{n \to \infty} \mu(E_n).$$

**Solution to Exercise 7(a):**

It is clear that, without loss of generality, we can assume $k = 1$ as both sides of Equation 18 are limits which ignore the first finite number of terms. For convenience I will define

$$\limsup_{n \to \infty} E_n = \bigcap_{j \in \mathbb{N}} \left( \bigcup_{n=j}^{\infty} E_n \right).$$

Define $F_j = \bigcup_{n=j}^{\infty} E_n$ for all $j \in \mathbb{N}$. It then follows that $\{F_j\}_{j \in \mathbb{N}}$ is a monotone sequence which decreases to $\limsup E_n$. Since $\mu(\bigcup_{n \in \mathbb{N}} E_n) < \infty$ it follows that $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$ so that

$$\lim_{j \to \infty} \mu(F_j) = \mu(\limsup E_n).$$

Now choose $j \in \mathbb{N}$. Since $E_n \subseteq F_j$ for each $n \geq j$ we have that $\mu(E_n) \leq \mu(F_j)$ from which we have that $\sup_{n \geq j} \mu(E_n) \leq \mu(F_j)$. It then follows that

$$\limsup_{n \to \infty} \mu(E_n) = \lim_{j \to \infty} \left( \inf_{n \geq j} \mu(E_n) \right) \leq \lim_{j \to \infty} \mu(F_j).$$

Finally

$$\limsup_{n \to \infty} \mu(E_n) \leq \mu(\limsup E_n)$$

as desired.
Part (b)

Prove that

\[ \mu \left( \bigcup_{j \in \mathbb{N}} \bigcap_{n=j}^{\infty} E_j \right) \leq \liminf_{n \to \infty} \mu(E_n). \]  

(Solution to Exercise 7(b):

For convenience I will define

\[ \liminf E_n = \bigcup_{j \in \mathbb{N}} \left( \bigcap_{n=j}^{\infty} E_n \right). \]  

(24)

Now define \( F_j = \bigcap_{n=j}^{\infty} E_n \) for all \( j \in \mathbb{N} \). We then have that \( \{F_j\}_{j \in \mathbb{N}} \) is a monotone sequence which increases to \( \liminf E_n \). It then follows that

\[ \lim_{j \to \infty} \mu(F_j) = \mu(\liminf E_n). \]  

(25)

Now choose \( j \in \mathbb{N} \). Since \( F_j \subseteq E_n \) for all \( n \geq j \) we have that \( \mu(F_j) \leq \mu(E_n) \) from which we have that \( \inf_{n \geq j} \mu(E_n) \geq \mu(F_j) \). It then follows that

\[ \liminf_{n \to \infty} \mu(E_n) = \lim_{j \to \infty} \left( \sup_{n \geq j} \mu(E_n) \right) \geq \lim_{j \to \infty} \mu(F_j). \]  

(26)

Finally

\[ \liminf_{n \to \infty} \mu(E_n) \geq \mu(\liminf E_n) \]  

(27)

as desired.
Exercise 8

Let \((X, \mathcal{A}, \mu)\) be a measure space and define \(\mu^* : \mathcal{A} \to [0, \infty)\) by

\[
\mu^*(A) = \sup \{ \mu(B) \mid B \subseteq A, B \in \mathcal{A}, \mu(B) < \infty \}.
\]  

(28)

Lemma 3. Let \(A \in \mathcal{A}\) such that \(\mu(A) < \infty\). Then \(\mu(A) = \mu^*(A)\).

Proof. Suppose \(A \in \mathcal{A}\) is any set such that \(\mu(A) < \infty\). Since \(\mu\) is a measure \(\mu(A') \leq \mu(A)\) whenever \(A' \subseteq A\). It is clear that \(A \subseteq A\), \(A \in \mathcal{A}\), and \(\mu(A) < \infty\) so that

\[
\mu^*(A) = \sup \{ \mu(B) \mid B \subseteq A, B \in \mathcal{A}, \mu(B) < \infty \} = \mu(A)
\]  

(29)

as desired. \(\square\)

Part (a)

Show that \(\mu^*\) is a measure on \((X, \mathcal{A})\).

Solution to Exercise 8(a):

First observe that \(\mu^*(\emptyset) = 0\) as the only set in \(\mathcal{A}\) contained in \(\emptyset\) is the empty set itself and \(\mu(\emptyset) = 0\).

Now suppose that \(\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}\) such that \(A_i \cap A_j\) whenever \(i \neq j\) and write \(A = \bigcup_{i \in \mathbb{N}} A_i\).

Case 1: Suppose that \(\mu^*(A_k) = \infty\) for at least one \(k \in \mathbb{N}\). Since \(A_k \subseteq A\) we have that \(\infty = \mu^*(A_k) \leq \mu^*(A)\) by definition and it is clear that \(\infty = \mu^*(A_k) \leq \sum_{i \in \mathbb{N}} \mu^*(A_i)\) so that \(\mu^*(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu^*(A_i)\).

Case 2: Suppose that \(\mu^*(A_i) < \infty\) for all \(i \in \mathbb{N}\) and that \(\mu^*(A) < \infty\). Then, by Lemma 3,

\[
\mu^*\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i) = \sum_{i \in \mathbb{N}} \mu^*(A_i)
\]  

(30)

so that \(\mu^*\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu^*(A_i)\).

Case 3: Suppose that \(\mu^*(A_i) < \infty\) for all \(i \in \mathbb{N}\) and that \(\mu^*(A) = \infty\). Let \(B_i = \bigcup_{k=1}^{i} A_k\). Since \(A_i\) is \(\mu^*\)-finite for every \(i \in \mathbb{N}\) it follows that \(B_i\) must also be \(\mu^*\)-finite. Since \(B_i\) is a monotonic sequence which increases to \(\bigcup_{i \in \mathbb{N}} A_i\) we have that

\[
\mu^*(A) = \sup \{ \mu(B) \mid B \subseteq A, B \in \mathcal{A}, \mu(B) < \infty \}
\]  

\[
= \lim_{i \to \infty} \mu(B_i)
\]  

\[
= \lim_{i \to \infty} \sum_{k=1}^{i} \mu(A_k)
\]  

\[
= \lim_{i \to \infty} \sum_{k=1}^{i} \mu^*(A_k)
\]  

\[
= \sum_{k \in \mathbb{N}} \mu^*(A_k)
\]  

(31)

so that \(\mu^*\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu^*(A_i)\).

In any case, \(\mu^*\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu^*(A_i)\) so that \(\mu^*\) is a measure, as desired.
Part (b)

Show that if $\mu$ is $\sigma$-finite then $\mu^* = \mu$.

Solution to Exercise 8(b):

If $\mu$ is $\sigma$-finite then there exists a countable set $\{X_i\}_{i \in \mathbb{N}}$ such that $X_i$ is $\mu$-finite for each $i \in \mathbb{N}$ and $X = \bigcup_{i \in \mathbb{N}} X_i$. Then for any $A \in \mathcal{A}$ we have that $A = \bigcup_{i \in \mathbb{N}} (X_i \cap A)$ and $\mu\left(\bigcup_{k=1}^i (X_i \cap A)\right)$ is finite for all $i \in \mathbb{N}$ so that

$$
\mu^*(A) = \sup\{\mu(B) \mid B \subseteq A, B \in \mathcal{A}, \mu(B) < \infty\}
$$

$$
= \lim_{i \to \infty} \mu\left(\bigcup_{k=1}^i (X_i \cap A)\right)
$$

$$
= \mu\left(\bigcup_{k \in \mathbb{N}} (X_k \cap A)\right)
$$

$$
= \mu(A).
$$

Hence $\mu^*(A) = \mu(A)$ as desired.

Part (c)

Find $\mu^*$ if $X$ is nonempty and $\mu$ is the measure defined such that $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ if $A \in \mathcal{A} \setminus \emptyset$.

Solution to Exercise 8(c):

If $A \in \mathcal{A}$ note that clearly $\emptyset \subseteq A$. Also, since the only set in $\mathcal{A}$ with finite $\mu$-measure is $\emptyset$ so the only set $B \subseteq A$ such that $B \in \mathcal{A}$ and $\mu(B) < \infty$ is $B = \emptyset$. Thus $\mu^*(A) = \mu(\emptyset) = 0$ for all sets $A \in \mathcal{A}$. 


Exercise 9

Let $C$ be a countable subset of $\mathbb{R}$. Using only the definition of $\lambda^*$, show that $\lambda^*(C) = 0$.

Solution to Exercise 9:

Let $C$ be a countable subset of $\mathbb{R}$ and write $C = \{c_n\}_{n \in \mathbb{N}}$. Fix $\varepsilon > 0$ and for $n \in \mathbb{N}$ let

$$I_n = \left( c_n - \frac{\varepsilon}{2^{n+1}}, c_n + \frac{\varepsilon}{2^{n+1}} \right).$$

(33)

Now, observe that

$$\lambda(C) = \inf \left\{ \sum_{n \in \mathbb{N}} (b_n - a_n) \left| C \subseteq \bigcup_{n \in \mathbb{N}} (a_n, b_n) \right\} \leq \sum_{n \in \mathbb{N}} \lambda(I_n) = \sum_{n \in \mathbb{N}} \frac{\varepsilon}{2^n} = \varepsilon.$$  

(34)

By letting $\varepsilon \to 0$ we have that $\lambda(C) = 0$, as desired.
Exercise 10

Show that for each subset \( A \) of \( \mathbb{R} \) there is a Borel subset \( B \in \mathcal{B}(\mathbb{R}) \) that includes \( A \) and satisfies \( \lambda(B) = \lambda^*(A) \).

Solution to Exercise 10:

Let \( A \) be a subset of \( \mathbb{R} \). If \( \lambda(A) = \infty \) then \( \mathbb{R} \in \mathcal{B}(\mathbb{R}) \) and \( \lambda(\mathbb{R}) = \lambda^*(A) \) so we can assume that \( \lambda(A) < \infty \). By definition,

\[
\lambda^*(A) = \inf \left\{ \sum_{k \in \mathbb{N}} (b_k - a_k) \mid A \subseteq \bigcup_{k \in \mathbb{N}} (a_k, b_k) \right\}. \tag{35}
\]

Notice that a set of the form \( \bigcup_{k \in \mathbb{N}} (a_k, b_k) \) must be in \( \mathcal{B}(\mathbb{R}) \) and hence for each \( n \in \mathbb{N} \) we can find a set \( B_n \in \mathcal{B}(\mathbb{R}) \) such that \( A \subseteq B_n \) and \( \lambda^*(A) + \frac{1}{n} \geq \lambda(B_n) \). Now let \( B = \bigcap_{n \in \mathbb{N}} B_n \) and notice that \( B \in \mathcal{B}(\mathbb{R}) \).

Since \( A \subseteq B \) and \( B \in \mathcal{B}(\mathbb{R}) \) we have

\[
\lambda^*(A) \leq \lambda^*(B) = \lambda(B). \tag{36}
\]

Additionally, by construction,

\[
\lambda^*(A) + \frac{1}{n} \geq \lambda(B_n) \geq \lambda(B) \text{ for all } n \in \mathbb{N}. \tag{37}
\]

Since Equation 37 holds for all \( n \in \mathbb{N} \) it follows, by letting \( n \to \infty \), that \( \lambda^*(A) \geq \lambda(B) \).

Finally, \( \lambda^*(A) = \lambda(B) \) as desired.
Exercise 11

Lemma 4. If $E \subseteq \mathbb{R}^d$ is a Lebesgue measurable set and $T : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear transformation then $T(E)$ is Lebesgue measurable and $\lambda(T(E)) = |\det T| \lambda(E)$.

I am not always sure what I am allowed to assume. However, I think it is fair to assume that an invertible linear map $T$ takes intervals of volume $A$ in $\mathbb{R}^d$ to parallelepipeds of area $|\det T| A$ in $\mathbb{R}^d$. This is a fairly easy result to prove in an undergraduate calculus class. I will use this fact in the proof below of Lemma 4.

Proof. Let $E \subseteq \mathbb{R}^d$ be a Lebesgue measurable set and let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transformation. Since $T$ is a homeomorphism (with the usual topology on $\mathbb{R}^d$) it must map Borel sets to Borel sets and hence must map Lebesgue measurable sets to Lebesgue measurable sets so that $T(E)$ is measurable.

For $\varepsilon > 0$ choose intervals $\{I_k\}_{k \in \mathbb{N}}$ in $\mathbb{R}^d$ such that $E \subseteq \bigcup_{k \in \mathbb{N}} I_k$ and $\sum_{k \in \mathbb{N}} \lambda(I_k) < \lambda(E) + \varepsilon$. Since $T$ is a bijection it follows that $T(E) \subseteq \bigcup_{k \in \mathbb{N}} T(I_k)$. Then since $\lambda(T(I_k)) = |\det T| \lambda(I_k)$ for every $k \in \mathbb{N}$, we have that

$$\lambda(T(E)) \leq \sum_{k \in \mathbb{N}} \lambda(T(I_k)) = |\det T| \sum_{k \in \mathbb{N}} \lambda(I_k) < |\det T| (\lambda(E) + \varepsilon) \quad (38)$$

and letting $\varepsilon \to 0$ in Equation 38 yields

$$\lambda(T(E)) \leq |\det T| \lambda(E). \quad (39)$$

If $\delta > 0$ then since $E$ is measurable, choose an open set $G$ such that $E \subseteq G$ and $\lambda(G \setminus E) < \delta$. Since $G$ is an open set $G$ must also be an $F_\sigma$ set (by Proposition 1.1.5) which means that we can write $G$ as the countable union of non-overlapping closed intervals\(^1\) in $\mathbb{R}^d$ so write $G = \bigcup_{k \in \mathbb{N}} J_k$ where $J_k$ is an interval for each $k \in \mathbb{N}$ and $\lambda(J_p \cap J_q) = 0$. Since $T$ is a bijection we have that $T(G) = \bigcup_{k \in \mathbb{N}} T(J_k)$ and since the intervals $J_k$ are non-overlapping it follows that

$$\lambda(T(G)) = \sum_{k \in \mathbb{N}} \lambda(T(J_k)) = |\det T| \sum_{k \in \mathbb{N}} \lambda(J_k) = |\det T| \lambda(G). \quad (40)$$

Since $E \subseteq G$ it is clear that $|\det T| \lambda(E) \leq |\det T| \lambda(G) = \lambda(L(G))$. Additionally,

$$\lambda(T(G)) \leq \lambda(T(E)) + \lambda(T(G \setminus E)) \leq \lambda(T(E)) + |\det T| \delta \quad (41)$$

so that after substitution, $|\det T| \lambda(E) \leq \lambda(T(E)) + |\det T| \delta$. Letting $\delta \to 0$ yields

$$|\det T| \lambda(E) \leq \lambda(T(E)). \quad (42)$$

Combining Equation 39 and Equation 42 yields $|\det T| \lambda(E) = \lambda(T(E))$, as desired. \(\square\)

---

\(^1\)Note that non-overlapping is different than disjoint. Disjoint intervals intersect trivially whereas non-overlapping intervals intersect to a set of measure zero.
Part (a)

Prove that every straight line in \( \mathbb{R}^2 \) has Lebesgue measure zero.

**Solution to Exercise 11(a):**

For \( n \in \mathbb{Z} \) consider the set \( L_n = \{(n + x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\} \). Let \( \varepsilon > 0 \) and write \( I_\varepsilon = (n - \varepsilon, n + 1 + \varepsilon) \times (-\varepsilon, \varepsilon) \). We have that \( I_\varepsilon \) is an interval and \( L_n \subseteq I_\varepsilon \) so that

\[
\lambda(L_n) \leq \inf \{\lambda(I_\varepsilon) \mid \varepsilon > 0\} = \lim_{\varepsilon \to 0} (2\varepsilon + 4\varepsilon^2) = 0. \tag{43}
\]

Now consider the line \( L = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} \) and observe that \( L = \bigcup_{n \in \mathbb{Z}} L_n \). Then

\[
\lambda(L) = \lambda \left( \bigcup_{n \in \mathbb{Z}} L_n \right) \leq \sum_{n \in \mathbb{Z}} \lambda(L_n) = 0 \tag{44}
\]

so that \( L \) has measure zero.

Now, by Lemma 4, the image of \( L \) under any invertible linear transformation must also have measure zero from which it follows that every line through the origin must have measure zero. We showed in class that translation preserves measure so the translation of any line through the origin must have measure zero. Finally, every line in \( \mathbb{R}^2 \) must have measure zero.

Part (b)

Prove that every circle in \( \mathbb{R}^2 \) has Lebesgue measure zero.

**Lemma 5.** Let \( E \subseteq \mathbb{R} \) and let \( \hat{E} = \{(x_1 - x_2, y_1 - y_2) \in \mathbb{R}^2 \mid (x_1, y_1), (x_2, y_2) \in E\} \). Then \( \lambda(E) = 0 \) if and only if \( \hat{E} \) contains an interval of \( \mathbb{R}^2 \) around \((0, 0)\).

In class we proved the analog of Lemma 5 in \( \mathbb{R} \) and the proof of Lemma 5 mimics that proof exactly. The proof of part (b) can be done similarly to that of part (a) but I think the proof which employs Lemma 5 is more powerful and interesting.

**Solution to Exercise 11(b):**

Let \( C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \) be the unit circle in \( \mathbb{R}^2 \). Now let \( C_I = \{(x, y) \in C \mid x, y \geq 0\} \) and write \( \hat{C}_I = \{(x_1 - x_2, y_1 - y_2) \in \mathbb{R}^2 \mid (x_1, y_1), (x_2, y_2) \in C_I\} \). It is clear that \( \hat{C}_I \) cannot contain an interval around the origin because otherwise there would be a point in \( C_I \) which is a distance different from 1 away from the origin. It follows from Lemma 5 that \( \lambda(C_I) = 0 \). Now let \( C_{II} \) be the points in \( C \) which are in the second quadrant, and define \( C_{III} \) and \( C_{IV} \) similarly. It is clear that \( C_{II}, C_{III}, \) and \( C_{IV} \) are the images of \( C_I \) under linear transformations so that each set has measure zero by Lemma 4. Then, since \( C = C_I \cup C_{II} \cup C_{III} \cup C_{IV} \), we have

\[
\lambda(C) = \lambda(C_I \cup C_{II} \cup C_{III} \cup C_{IV}) \leq \lambda(C_I) + \lambda(C_{II}) + \lambda(C_{III}) + \lambda(C_{IV}) = 0. \tag{45}
\]

Every circle in \( \mathbb{R}^2 \) centered at the origin is the image of \( C \) under a linear transformation and so must have measure zero by Lemma 4. Since every circle in \( \mathbb{R}^2 \) is the translation of a circle centered at the origin it follows that every circle in \( \mathbb{R}^2 \) has measure zero, as desired.
Exercise 12

Let $A$ be a subset of $\mathbb{R}^d$. Show that the following are equivalent:

(i) $A$ is Lebesgue measurable.

(ii) $A$ is the union of an $F_\sigma$ set and a set of Lebesgue measure zero.

(iii) There is an $F_\sigma$ set $B$ that satisfies $\lambda^*(A \Delta B) = 0$.

Solution to Exercise 12:
I proceed to show that (i) $\iff$ (ii) and (ii) $\iff$ (iii) which proves the desired result.

(i) $\Rightarrow$ (ii): Assume that $A$ is Lebesgue measurable. For every $n \in \mathbb{N}$ there exists an open set $B_n$ such that $A \subseteq B_n$ and $\lambda^*(A) + 1/n \geq \lambda^*(B_n)$. It then follows that $\lambda^*(A \setminus B_n) \leq 1/n$ for all $n \in \mathbb{N}$. Define $B = \bigcap_{n \in \mathbb{N}} B_n$ which is clearly an $F_\sigma$ set. Now, for all $n \in \mathbb{N}$,

$$0 \leq \lambda^*(A \setminus B_n) = \lambda^* \left( A \setminus \bigcup_{n \in \mathbb{N}} B_n \right) \leq \lambda^*(A \setminus B_n) \leq 1/n$$

so that, by letting $n \to \infty$, we have $\lambda^*(A \setminus B) = 0$. Finally we have that $A = B \cup (A \setminus B)$ where $B$ is an $F_\sigma$ set and $A \setminus B$ has Lebesgue measure zero.

(ii) $\Rightarrow$ (i): Assume that $A = F \cup Z$ where $F$ is an $F_\sigma$ set and $\lambda^*(Z) = 0$. Then $F$ and $Z$ must both be Lebesgue measurable so that $A$ is Lebesgue measurable.

(ii) $\Rightarrow$ (iii): Assume that $A = F \cup Z$ where $F$ is an $F_\sigma$ set and $\lambda^*(Z) = 0$. Let $B = A \setminus Z$ and note that $B$ is an $F_\sigma$ set. Then

$$\lambda^*(A \Delta B) = \lambda^*((A \setminus B) \cup (B \setminus A)) = \lambda^*(Z \cup \emptyset) = 0.$$  \hspace{1cm} (47)

We now have that $B$ is an $F_\sigma$ set such that $\lambda^*(A \Delta B) = 0$, as desired.

(iii) $\Rightarrow$ (ii): Assume that there is an $F_\sigma$ set $B$ such that $\lambda^*(A \Delta B) = 0$. Then

$$0 = \lambda^*(A \Delta B) = \lambda^*((A \setminus B) \cup (B \setminus A)) = \lambda^*(A \setminus B) + \lambda^*(B \setminus A).$$  \hspace{1cm} (48)

It follows that $\lambda^*(A \setminus B) = 0 = \lambda^*(B \setminus A)$. Finally $A = B \cup (B \setminus A)$ where $B$ is an $F_\sigma$ set and $B \setminus A$ has Lebesgue measure zero.
Exercise 13

Show that for each number \( \alpha \) that satisfies \( 0 < \alpha < 1 \) there is a closed subset \( D \) of \([0,1]\) such that \( \lambda(D) = \alpha \) and includes no non-empty open set.

**Solution to Exercise 13:**

Choose a fixed but arbitrary real number \( \alpha \in (0,1) \) and let \( \delta = 1 - \alpha \). Define a sequence \( \{D_k\}_{k=0}^{\infty} \) such that \( D_k \subseteq \mathcal{P}(\mathbb{R}) \). Let \( D_0 = \{[0,1]\} \) and for \( k > 0 \) define

\[
D_k = \left\{ \left[ a, \frac{a+b}{2} - \frac{\delta}{2 \cdot 3^k} \right] \cup \left[ \frac{a+b}{2} + \frac{\delta}{2 \cdot 3^k}, b \right] \middle| [a,b] \in D_{k-1} \right\}. \tag{49}
\]

Note that \( D_k \) is a collection of disjoint closed subintervals of \([0,1]\) for each \( k \). Now define a sequence \( \{D_k\}_{k=0}^{\infty} \) such that \( D_k = \bigcup D_k \) is the union of all the intervals in \( D_k \). Then let \( D = \bigcap_{k=0}^{\infty} D_k \).

We start by showing that \( D \) has measure \( \alpha \). Since \( D_k \) is the union of disjoint closed intervals each \( D_k \) is closed and since \( D \) is the countable intersection of closed sets, \( D \) is closed and hence must be measurable. Since \( D_k \subseteq D_{k-1} \) for each \( k \geq 0 \) we have that \( \lambda(D) = \lim_{k \to \infty} \lambda(D_k) \), hence it suffices to find \( \lambda(D_k) \) for each \( k \geq 0 \). Observe that \( D_1 \) is the union of two subintervals of \([0,1]\) and the number of subintervals doubles each time \( k \) increases by 1 so that \( D_k \) is the union of \( 2^k \) such subintervals. Choose \( k \geq 0 \) and consider \( E = D_{k-1} \setminus D_k \). Now \( D_k \) is attained by removing one interval from each of the intervals in \( D_{k-1} \) so \( E \) is the collection of \( 2^{k-1} \) closed intervals. By the construction we have that each of the closed intervals which make up \( E \) have length \( \delta 3^{-k} \). It follows that \( \lambda(E) = \delta 2^{k-1} 3^{-k} \) and \( D_k \) is a disjoint union of \( D_{k-1} \) and \( E \). Combining all of things we have

\[
\lambda(D_0) = 1 \quad \text{and} \quad \lambda(D_k) = \lambda(D_k) - \delta 2^{k-1} 3^{-k} \quad \text{whenever} \quad k > 0. \tag{50}
\]

After expanding \( \lambda(D_k) \) to a sum and simplifying we have

\[
\lambda(D_k) = 1 - \frac{\delta}{3} \sum_{n=0}^{k} \left( \frac{2}{3} \right)^n
= 1 - \frac{\delta}{3} \left( 1 - \left( \frac{2}{3} \right)^{k+1} \right) \left( 1 - \left( \frac{2}{3} \right) \right)
= 1 - \delta + \delta \left( \frac{2}{3} \right)^{k+1}. \tag{51}
\]

Finally,

\[
\lambda(D) = \lim_{k \to \infty} \lambda(D_k) = \lim_{k \to \infty} \left( 1 - \delta + \delta \left( \frac{2}{3} \right)^{k+1} \right) = 1 - \delta = \alpha \tag{52}
\]

and the measure of \( D \) is \( \alpha \).

We now wish to show that \( D \) contains no open intervals. Choose \( x \in D \). Since \( D = \bigcap_{k=0}^{\infty} D_k \) it follows that \( x \in D_k \) for each \( k \geq 0 \). Since \( D_k = \bigcup D_k \) there must exist an interval in \( D_k \) which contains \( x \); denote this interval \( X_k \). That is, \( X_k \subseteq D_k \) and \( x \in X_k \). Note also that since \( D_k \) is the union of disjoint intervals, \( X_k \) is the largest connected subset of \( D_k \) which contains \( x \). It follows that \( X = \bigcap_{k=0}^{\infty} X_k \) is the largest connected subset of \( D \) which contains \( x \). We have already shown that

\[
\lambda(X_k) = \frac{1 - \delta + \delta \left( \frac{2}{3} \right)^{k+1}}{2^k}. \tag{53}
\]
for every $k \geq 0$. Since $X_k$ is a decreasing sequence of closed intervals and $\lim_{k \to \infty} \lambda(X_k) = 0$ it follows by the Cantor Intersection Theorem that $X$ contains a single point, namely $x$. Hence, for any given $x \in D$, the largest connected component of $D$ which contains $x$ is the singleton $\{x\}$. Now suppose that there exists an open subset $B \subseteq D$ and choose any $x \in B$. Since $B$ is open there must exist an open interval $I$ such that $x \in I \subseteq B \subseteq D$ and hence the largest connected component of $D$ which contains $x$ must contain $I$, which is a contradiction. Hence $D$ cannot contain any open subsets.

Finally, $D \subseteq [0, 1]$ with $\lambda(D) = \alpha$ and $D$ includes no open sets. Since $\alpha \in (0, 1)$ was chosen arbitrarily such a set exists for any $\alpha \in (0, 1)$.
Exercise 14

Show that if $B$ is a subset of $\mathbb{R}$ that satisfies $\lambda^*(B) > 0$ then $B$ includes a set that is not Lebesgue measurable.

Solution to Exercise 14:

Choose any set $B \subseteq \mathbb{R}$ such that $\lambda^*(B) > 0$. By proposition 1.4.9 of Cohn’s text there exists a set $A \subseteq \mathbb{R}$ such that every Lebesgue measurable set contained in either $A$ or $A^c$ has measure zero. It is clear that $B = (B \cap A) \cup (B \cap A^c)$ and that $B \cap A \subseteq A$ and $B \cap A^c \subseteq A^c$. Suppose that both $B \cap A$ and $B \cap A^c$ are Lebesgue measurable. Then, by the assumption on $A$, both $B \cap A$ and $B \cap A^c$ must have measure zero which then implies that $B$ has measure zero which is a contradiction. Hence at least one of either $B \cap A \subseteq B$ or $B \cap A^c \subseteq B$ must not be Lebesgue measurable and the assertion is proved.
Exercise 15

Show that there is a Lebesgue measurable subset of $\mathbb{R}^2$ whose projection on $\mathbb{R}$ under the map $(x, y) \mapsto x$ is not Lebesgue measurable.

Solution to Exercise 15:

Let $F \subseteq \mathbb{R}$ be any set that is not Lebesgue measurable; we showed in class that such a set must exist. Now let $F' = \{(x, x) \mid x \in F\}$ so that clearly the projection of $F'$ to $\mathbb{R}$ in the first coordinate is $F$. In exercise 11(a) we showed that the set $L = \{(x, x) \mid x \in \mathbb{R}\}$ is a Lebesgue measurable set with measure zero. Since $F'$ is a subset of $L$ and $L$ has measure zero by Exercise 11 we have that $F'$ also has measure zero. It follows that $F'$ is a Lebesgue measurable subset of $\mathbb{R}^2$ whose projection onto $\mathbb{R}$ is not Lebesgue measurable.
Exercise 16

Let \( X \) be a set and \( R \) be a ring on \( X \). Denote by \( S(R) \) the \( \sigma \)-ring generated by \( R \) and denote by \( M(R) \) the monotone class generated by \( R \). Prove that \( S(R) = M(R) \).

Lemma 6. Suppose that \( X \) is a set and \( R \) is a collection of subsets of \( R \). If both

(i) \( X \setminus A \in R \) whenever \( A \in R \) and

(ii) \( A \cap B \in R \) whenever \( A, B \in R \)

then \( R \) is a ring.

Proof. Choose \( E_1, E_2 \in R \). Then \( E_1 \cup E_2 = X \setminus [(X \setminus E_1) \cap (X \setminus E_2)] \in R \) and \( E_1 \setminus E_2 = E_1 \cap (X \setminus E_2) \in R \).

It follows that \( R \) is a ring. \( \square \)

Solution to Exercise 16:

We have already shown in Referencesex4 that \( S(R) \) is a monotone class and since \( M(R) \) is the smallest monotone class containing \( R \) it follows that \( M(R) \subseteq S(R) \). In order to show that \( S(R) \subseteq R(R) \) it suffices to show that \( M(R) \) is a \( \sigma \)-ring since \( S(R) \) is the smallest \( \sigma \)-ring containing \( R \).

I start by showing that \( M(R) \) is closed under complements. Write

\[
N_0 = \{ A \in M(R) \mid X \setminus A \in M(R) \}.
\]  

Since \( R \) is a ring and \( R \subseteq M(R) \) it follows that \( R \subseteq N_0 \). Let \( \{ A_n \}_{n \in \mathbb{N}} \subseteq N_0 \) be an increasing sequence and write \( A = \bigcup_{n \in \mathbb{N}} A_n \). Since \( N_0 \subseteq M(R) \) and \( M(R) \) is a monotone class it follows that \( A \in M(R) \).

Since each \( A_n \in N_0 \) we have \( X \setminus A_n \in M(R) \) by the definition of \( N_0 \). Then

\[
X \setminus A = \bigcap_{n \in \mathbb{N}} (X \setminus A_n)
\]

and \( \{ X \setminus A_n \}_{n \in \mathbb{N}} \) is a decreasing sequence in the monotone class \( M(R) \) so \( X \setminus A \in M(R) \). Then, since \( A \in M(R) \) and \( X \setminus A \in M(R) \) it follows that \( A \in N_0 \) by the definition of \( N_0 \) so that \( N_0 \) is closed under countable increasing unions. Let \( \{ A_n \}_{n \in \mathbb{N}} \subseteq N_0 \) be a decreasing sequence and write \( A = \bigcap_{n \in \mathbb{N}} A_n \). Since \( N_0 \subseteq M(R) \) and \( M(R) \) is a monotone class it follows that \( A \in M(R) \). Since each \( A_n \in N_0 \) we have \( X \setminus A_n \in M(R) \) by the definition of \( N_0 \). Then

\[
X \setminus A = \bigcup_{n \in \mathbb{N}} (X \setminus A_n)
\]

and \( \{ X \setminus A_n \}_{n \in \mathbb{N}} \) is an increasing sequence in the monotone class \( M(R) \) so \( X \setminus A \in M(R) \). Then, since \( A \in M(R) \) and \( X \setminus A \in M(R) \) it follows that \( A \in N_0 \) by the definition of \( N_0 \) so that \( N_0 \) is closed under countable decreasing intersections. Thus \( N_0 \) is a monotone class containing \( R \). Since \( M(R) \) is the smallest monotone class containing \( R \) we must have that \( M(R) \subseteq N_0 \) but \( N_0 \) was defined to be a subset of \( M(R) \) so that \( M(R) = N_0 \). Hence \( M(R) \) is closed under complementation.

Next I wish to show that \( M(R) \) is closed under finite intersections. I first show that \( A \cap B \in M(R) \) whenever \( A \in R \) and \( B \in M(R) \). Fix \( A \in R \) and define

\[
N_1 = \{ B \in M(R) \mid A \cap B \in M(R) \}.
\]  

Since \( R \) is a ring, \( R \subseteq N_1 \). Let \( \{ B_n \}_{n \in \mathbb{N}} \subseteq N_1 \) be an increasing sequence and write \( B = \bigcup_{n \in \mathbb{N}} B_n \). Since \( N_1 \subseteq M(R) \) and \( M(R) \) is a monotone ring, \( B \in M(R) \). Since \( B_n \in N_1 \) we have that \( A \cap B_n \in M(R) \) by definition for all \( n \in \mathbb{N} \) and

\[
A \cap B = \bigcup_{n \in \mathbb{N}} (A \cap B_n)
\]
where \( \{A \cap B_n\}_{n \in \mathbb{N}} \subseteq N_1 \) is an increasing sequence in the monotone class \( M(R) \) so that \( A \cap B \in M(R) \). Because \( B \in M(R) \) and \( A \cap B \in M(R) \) we have that \( B \in N_1 \) by definition and hence \( N_1 \) is closed under countable increasing unions. Let \( \{B_n\}_{n \in \mathbb{N}} \subseteq N_1 \) be a decreasing sequence and write \( B = \bigcap_{n \in \mathbb{N}} B_n \). Since \( N_1 \subseteq M(R) \) and \( M(R) \) is a monotone ring, \( B \in M(R) \). Since \( B_n \in N_1 \) we have that \( A \cap B_n \in M(R) \) by definition for all \( n \in \mathbb{N} \) and

\[
A \cap B = \bigcap_{n \in \mathbb{N}} (A \cup B_n)
\]  

(59)

where \( \{A \cap B_n\}_{n \in \mathbb{N}} \subseteq N_1 \) is an increasing sequence in the monotone class \( M(R) \) so that \( A \cap B \in M(R) \). Because \( B \in M(R) \) and \( A \cap B \in M(R) \) we have that \( B \in N_1 \) by definition and hence \( N_1 \) is closed under countable decreasing intersections. Thus \( N_1 \) is a monotone class which contains \( M(R) \) from which it follows that \( N_1 = M(R) \). Since the choice of \( A \) was arbitrary we have that \( A \cap B \in M(R) \) whenever \( A \in R \) and \( B \in M(R) \).

Now choose \( C \in M(R) \) and let \( N_2 = \{D \in M(R) \mid C \cap D \in M(R)\} \). We know that \( D \cap C \in M(R) \) whenever \( D \in R \) so that \( R \subseteq N_2 \). By an argument identical to the one used in the previous paragraph we have that \( N_2 \) is a monotone class so that \( N_2 = M(R) \) and hence \( M(R) \) is closed under finite intersection.

Since \( M(R) \) is closed under finite intersection and complementation, \( M(R) \) is a ring by Lemma 6. We have already shown that a monotone class which is also a ring is a \( \sigma \)-ring so that \( M(R) \) is a \( \sigma \)-ring. As previously discussed, the result follows.
Exercise 17

Let \((X, \mathcal{A}, \mu)\) be a measurable space. Suppose that there is a sequence \(\{E_n\}_{n \in \mathbb{N}}\) such that \(X = \bigcup_{n \in \mathbb{N}} E_n\). Define

\[
\mu^*(E) = \inf \{\mu(F) \mid E \subseteq F \in \mathcal{A}\}
\]

for all \(E \subseteq X\). Show that \(\mu^*\) is an outer measure.

Solution to Exercise 17:

Let \((X, \mathcal{A}, \mu)\) be a measurable space such that there exists a sequence \(\{E_n\}_{n \in \mathbb{N}}\) such that \(X = \bigcup_{n \in \mathbb{N}} E_n\) and define \(\mu^* : \mathcal{P}(\mathbb{R}) \to [0, \infty)\) as in Equation 60. There are three conditions which must be satisfied in order to verify that \(\mu^*\) is an outer measure.

(i) I first wish to ensure that \(\mu^*(\emptyset) = 0\). We have that

\[
\mu^*(\emptyset) = \inf \{\mu(F) \mid \emptyset \subseteq F \in \mathcal{A}\}
\]  

but since \(\emptyset \in \mathcal{A}\), \(\mu(\emptyset) = 0\), and \(\mu(F) \geq 0\) for all \(F \in \mathcal{A}\) it follows that \(\mu^*(\emptyset) = 0\) as desired.

(ii) Now I wish to show that \(\mu^*(A) \leq \mu^*(B)\) whenever \(A \subseteq B \subseteq X\). Let \(\mathcal{S}_A\) be the set of all sets in \(\mathcal{A}\) which contain \(A\) and let \(\mathcal{S}_B\) be the set of all sets in \(\mathcal{A}\) which contain \(B\). It then follows that

\[
\mu^*(A) = \inf_{F \in \mathcal{S}_A} \{\mu(F)\} \quad \text{and} \quad \mu^*(B) = \inf_{F \in \mathcal{S}_B} \{\mu(F)\}.
\]  

Since \(\mathcal{S}_A \subseteq \mathcal{S}_B\) as \(A \subseteq B\) we then have that

\[
\mu^*(A) = \inf_{F \in \mathcal{S}_A} \{\mu(F)\} \leq \inf_{F \in \mathcal{S}_B} \{\mu(F)\} = \mu^*(B)
\]  

as desired.

(iii) Finally I wish to verify that \(\mu^*(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)\) whenever \(\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)\). To this end, fix \(\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)\) and let \(\varepsilon > 0\). For each \(n \in \mathbb{N}\) there exists a set \(F_n \in \mathcal{A}\) such that \(A_n \subseteq F_n\) and \(\mu(F_n) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}\). Observe that \(\bigcup_{n=1}^k A_n \subseteq \bigcup_{n=1}^k F_n\) for all \(k \in \mathbb{N}\) from whence it follows

\[
\mu^*(\bigcup_{n=1}^k A_n) \leq \mu\left(\bigcup_{n=1}^k F_n\right) \leq \sum_{n=1}^k \mu(F_n) \leq \sum_{n=1}^k \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \varepsilon + \sum_{n=1}^k \mu^*(A_n).
\]  

Letting \(n \to \infty\) in Equation 64 yields

\[
\mu^*(\bigcup_{n \in \mathbb{N}} A_n) \leq \varepsilon + \sum_{n \in \mathbb{N}} \mu^*(A_n)
\]  

and since Equation 65 holds for every \(\varepsilon > 0\) we have that \(\mu^*(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)\), as desired. It finally follows that \(\mu^*\) is an outer measure, as desired.