Points
Have no parts or joints
How then can they combine
To form a line?

— J. A. Lindon [1]
This therefore is Mathematics:
She reminds you of the invisible forms of the soul;
She gives life to her own discoveries;
She awakens the mind and purifies the intellect;
She brings light to our intrinsic ideas;
She abolishes oblivion and ignorance which are ours by birth…

— Proclus Lycaeus [2]

**ACKNOWLEDGEMENTS**

The \LaTeX\ formatting used in this document is thanks to a template written by André Miede and posted on the LaTeX Templates website [6]. Though I edited it quite a bit, the core of the document still follows that template.

The images in this document were made with the \texttt{tikz} package. Documentation can be found on the CTAN website [7].

The content and examples in this document are my own but the subject matter is meant to reflect some of the material covered in *Functions Modeling Change: A Preparation for Calculus for the University of Oregon* by Connally et al. [4].

Something needs to be said of my students. I have been using this document in place of a textbook since the Fall quarter of 2013. My students have been both patient and helpful.
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INTRODUCTION

This course covers an introduction to trigonometry, an introduction to vectors, and the operations on functions. The University of Oregon course catalog describes it as follows:

MATH 112 discusses trigonometric functions on the unit circle and mathematical applications of trigonometric functions. Stress is placed on sine and cosine over other trigonometric functions. Other topics include solving right and non-right triangles, periodic functions, a more thorough treatment of inverse functions, and vectors from mathematical and physical science perspectives. The course covers content from chapters 7 through 10 and chapter 12 from *College Algebra and Trigonometry for the University of Oregon*, by Connally et al.

The course text is a custom edition of *Functions Modeling Change: A Preparation for Calculus* by Connally et al. [3] Students using the national version should know that page numbers will not match up between the national and custom versions of the text.

Elementary functions is a calculus preparation course and as such has a strong algebraic focus. Students should primarily take MATH 112 to fulfill the MATH 246 or MATH 251 prerequisites, although it does also satisfy two courses of the Bachelor of Science mathematics requirement. Students solely seeking credit toward the Bachelor of Science mathematics requirement should consider MATH 105, 106, or 107.

Prior to taking this class it is strongly recommended that students have recently taken a college algebra course such as MATH 111. Students should expect to take a readiness quiz during the first week of classes to demonstrate preparedness for MATH 112.

The material covered by chapters 7 through 10 and chapter 12 in Connally et al.’s text is what is covered by these notes. Emphasis is
different, some things are missing, and we do not always go in the same order as that text but roughly the same material is covered. The exercises in the readiness check below are designed to test your knowledge of the prerequisite material.
1.1 READINESS CHECK

1.1.1. Find $\frac{1}{3} - \frac{2}{7} + 1$. Leave your answer in exact form and simplify your answer.

1.1.2. Find $\sqrt{50} + \sqrt{72}$. Leave your answer in exact form and simplify your answer.

1.1.3. Simplify the expression $\frac{1}{x^2 + 7x + 12} + \frac{1}{x+4} + \frac{1}{x+3}$.

1.1.4. Find all values of $x$ such that $x^2 + 7x + 1 = 0$. Leave your answer in exact form.

1.1.5. Find all values of $x$ such that $(x + 2)^2 = x^2 - 2(x - 8)$.

1.1.6. Find all values of $x$ such that $1 + 9 \log(x - 3) = 19$.

1.1.7. Let $f(x) = x^2 - 4x - 11$. Find all values of $t$ such that $f(t) = 10$.

1.1.8. Find the equation of a line with slope $\frac{3}{2}$ which goes through the point $(2, -1)$.

1.1.9. What is the largest possible domain of the function $g(x) = \frac{x+3}{x^2+3x+2}$? Write your answer in interval notation.

1.1.10. Find the x-intercepts and y-intercepts of the function $f(x) = x^2 - 3x - 10$.

1.1.11. Let $f(x) = \frac{1}{2} e^x - 3$.
   
   a.) Find $f(3)$. Round your answer to two decimal places.
   
   b.) Find a value $x$ such that $f(x) = 2$.
       Round your answer to two decimal places.
   
   c.) Graph $y = f(x)$. Label the y-intercept and $f(3)$.

1.1.12. A software manufacturer is developing a particular computer game. They are doing market research to determine at which price they should sell this game. They find that if they sell the game for $x$ dollars each then their expected profit per month from this game’s sales is $P(x) = -10x^2 + 700x - 2250$ dollars.
   
   a.) At what price should they sell the game to maximize their profit?
   
   b.) What is the maximum profit they can make in a month from the sale of this game?

1.1.13. Three consecutive positive integers are such that twice the smallest integer minus the largest integer plus the square of the middle integer is $153$. What are the three integers?

1.1.14. The graph of $f(x)$ is drawn below. Draw the graph of $g(x) = \frac{1}{2} f(x + 3) - 4$ on the same plot. (Assume that the scale on the axes is in increments of one.)
NEW FUNCTIONS FROM OLD FUNCTIONS

The bulk of MATH 111 covers the definition of a function as well some important examples of functions including linear functions, polynomials, power functions, exponential functions, and logarithms. This chapter examines how one can use those functions and combine them to form new functions. This chapter covers the content of Chapter 10 in *College Algebra and Trigonometry for the University of Oregon*, by Connally et al. [4]
2.1 COMBINATIONS OF FUNCTIONS

The first example of creating new functions from old functions is to use arithmetic operations to combine two functions. For example, the functions \( f(x) = x^3 \) and \( g(x) = e^{-x} \) and in this section we venture to use them to understand functions like \( p(x) = x^3 + e^{-x} \) and \( q(x) = x^3 e^{-x} \).

**Motivation**

The Seahawks organization has done some market research. During a home football game their two sources of revenue are ticket sales and purchases inside the stadium such as concessions and souvenirs. They find that the average ticket price is $65 per person and that each person spends an average of $42 while in the stadium. Thus if \( p \) people come to the game then the Seahawks organization will see an average of 65p dollars in revenue from ticket sales and an average of 42p dollars in revenue from purchases inside of the stadium. Their total revenue, then, will be \( 65p + 42p = 107p \) dollars.

It is valuable to write the above situation in terms of functions. Consider a function \( T \) which describes revenue from ticket sales and a function \( C \) which describes revenue from purchases inside the stadium. All of the revenue expressions above are written in terms of \( p \) so the independent variable for these functions will be the number of people that attend the game; that is, \( p \) will be the input of these functions. Based on the information above we have that \( T(p) = 65p \) and \( C(p) = 42p \). Now consider a function \( R \) which describes the total revenue on the day of the game as a function of the number of people that come to the game. Since the only two sources of revenue are ticket sales and purchases inside of the stadium it follows that

\[
\]

This is one example of using arithmetic operations to combine known functions into new functions.

**Theory**

We start by introducing some notation. It is likely that, until now, students are accustomed to using single letters to denote functions like \( f \) and \( g \). Functions work like machines that input a number and output another number. If \( f \) is a function and \( x \) is in the domain of \( f \) we write \( f(x) \) to denote the output corresponding to the input \( x \).
There is no good reason that functions need to be denoted with single letters. Mathematicians often use subscripts in their notation. We may define a function \( F_t \) whose outputs look like \( F_t(x) \). We could also use whole words for functions if it were convenient. We often use the letter \( f \) for “function” but instead of writing \( f(x) \) we could just as easily write function\((x)\) so that the function is simply notated by function instead of \( f \).

Definition 2.1.1 below defines several new functions given known functions \( f \) and \( g \) and a real number \( c \). For example, it defines a function whose name is \( f + g \). That means that the outputs of \( f + g \) will look like \((f + g)(x)\) where \( x \) is an input. We put the parentheses around \( f + g \) in the outputs to avoid confusion; when writing \( f + g(x) \) it is unclear as to whether \( x \) is an input into \( g \) or \( f + g \). Writing it as \((f + g)(x)\) avoids this confusion. Definition 2.1.1 attempts to define the function \( f + g \). In order to define a function, one must define outputs that correspond to inputs. That is, given an input \( x \) we must prescribe the value of \((f + g)(x)\). Keep in mind that \( f \) and \( g \) are known functions and \( f + g \) is a new function that we are defining.

(2.1.1) Definition. Let \( f \) and \( g \) be known functions and let \( c \) be a real number.

- Define the function \( f + g \) such that \((f + g)(x) = f(x) + g(x)\) for values of \( x \) in the domain of both \( f \) and \( g \).

- Define the function \( fg \) such that \((fg)(x) = (f(x))(g(x))\) for values of \( x \) in the domain of both \( f \) and \( g \).

- Define the function \( cf \) such that \((cf)(x) = c(f(x))\) for values of \( x \) in the domain of \( f \).

- Define the function \( 1/f \) such that \((1/f)(x) = 1/(f(x))\) for values of \( x \) in the domain of \( f \) such that \( f(x) \neq 0 \).

Definition 2.1.1 should be somewhat intuitive. The notation \( f + g \) suggests that this new function is the sum of the functions \( f \) and \( g \) and it turns out that the outputs of \( f + g \) are formed by the sum of the outputs of \( f \) and \( g \). Similarly, the outputs of the product of \( f \) and \( g \) - written \( fg \) - are the product of the outputs of \( f \) and \( g \) and the outputs of the reciprocal of \( f \) - written \( 1/f \) - are the reciprocal of the outputs of \( f \).

In the notation of Definition 2.1.1, the thing that students often find the most confusing is the difference between \( fg \) and \( cf \). The notation for these two functions look the same but remember that \( f \) and \( g \) are functions whereas \( c \) is a number. Since \( c \) is not a function it makes no sense to write \((cf)(x) = (c(x))(f(x))\) because there is no way to plug \( x \) into \( c \). Additionally, when we write \((cf)(x) = c(f(x))\) we mean that

If \( f \) is a function and we talk about the function \( kf \) then we need to have already defined \( k \) as either a function or a constant.
f(x) is being multiplied by c; we do not mean that the value f(x) is being plugged into c since c is not a function.

**Example 2.1.2** should help clear up some of these definitions.

<table>
<thead>
<tr>
<th>(2.1.2) Example</th>
<th>Consider the functions f(x) = x^2 + 2 and g(x) = 3x − 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Evaluate (f + g)(−2).</td>
<td>A. Evaluate (f + g)(−2) = f(−2) + g(−2) = (6) + (−7) = −1.</td>
</tr>
<tr>
<td>B. Evaluate (1/f)(3).</td>
<td>B. Again, according to Definition 2.1.1 we have ( (1/f)(3) = \frac{1}{(f(3))} = \frac{1}{(3^2 + 2)} = \frac{1}{11}. )</td>
</tr>
<tr>
<td>C. Find and simplify (fg)(x).</td>
<td>C. Here we want to find the outputs of fg for arbitrary inputs instead of for a specific input. That is, if we want to find (fg)(x) for inputs x. According to Definition 2.1.1 we have ( (fg)(x) = (f(x))(g(x)) = (x^2 + 2)(3x − 1) = 3x^3 − x^2 + 6x − 2. )</td>
</tr>
<tr>
<td>D. Find and simplify (7g)(x).</td>
<td>D. Again, we wish to find (7g)(x) for arbitrary values of x. Obviously 7 is a real number so 7 is the value of c in Definition 2.1.1. According to the definition we have that (7g)(x) = 7(g(x)) for every value of x so that ( (7g)(x) = 7(g(x)) = 7(3x − 1) = 21x − 7. )</td>
</tr>
</tbody>
</table>

_Solution:_

A. According to Definition 2.1.1 we have

\[
(f + g)(−2) = f(−2) + g(−2) = (6) + (−7) = −1.
\]

**Answer:** \( (f + g)(−2) = −1 \)

B. Again, according to Definition 2.1.1 we have

\[
(1/f)(3) = \frac{1}{(f(3))} = \frac{1}{(3^2 + 2)} = \frac{1}{11}.
\]

**Answer:** \( (1/f)(3) = \frac{1}{11} \)

c. Here we want to find the outputs of fg for arbitrary inputs instead of for a specific input. That is, if we want to find (fg)(x) for inputs x. According to Definition 2.1.1 we have

\[
(fg)(x) = (f(x))(g(x)) = (x^2 + 2)(3x − 1) = 3x^3 − x^2 + 6x − 2.
\]

**Answer:** \( (fg)(x) = 3x^3 − x^2 + 6x − 2 \)

D. Again, we wish to find (7g)(x) for arbitrary values of x. Obviously 7 is a real number so 7 is the value of c in Definition 2.1.1. According to the definition we have that (7g)(x) = 7(g(x)) for every value of x so that

\[
(7g)(x) = 7(g(x)) = 7(3x − 1) = 21x − 7.
\]

**Answer:** \( (7g)(x) = 21x − 7 \)
If we can add and multiply functions it makes sense that we should be able to subtract and divide them, as well. **Definition 2.1.3** defines these functions by appealing to **Definition 2.1.1**. That is, we already know how to define \((-1)g\) and we know how to add functions together so \(f - g\) is defined simply as \(f - g = f + (-1)g\).

**Definition.** Let \(f\) and \(g\) be known functions.

- Define the function \(f - g\) such that \(f - g = f + (-1)g\). That is, if \(x\) is in the domain of both \(f\) and \(g\) then \((f - g)(x) = f(x) - g(x)\).
- Define the function \(f/g\) such that \(f/g = (f)(1/g)\). That is, if \(x\) is in the domain of both \(f\) and \(g\) and \(g(x) \neq 0\) then \((f/g)(x) = (f(x))/(g(x))\).

**Example.** Consider the functions \(f(x) = x^2 + 2\) and \(g(x) = 3x - 1\).

A. Evaluate \((f/g)(1)\).

B. Find and simplify \((f - g)(x)\).

**Solution:**

A. According to **Definition 2.1.3** we have

\[
(f/g)(1) = \frac{f(1)}{g(1)} = \frac{1^2 + 2}{3(1) - 1} = \frac{3}{2}.
\]

**Answer:** \((f/g)(1) = \frac{3}{2}\)

B. Again using the definition, we have

\[
(f - g)(x) = f(x) - g(x)
= (x^2 + 2) - (3x - 1)
= x^2 - 3x + 3.
\]

**Answer:** \((f - g)(x) = x^2 - 3x + 3\)

Each of **Definition 2.1.1** and **Definition 2.1.3** discuss the domain of the functions that they define. In each case, the domain of the function is essentially any input for which the output is well-defined. For example, consider a function \(f + g\). The outputs of \(f + g\) look like...
10 NEW FUNCTIONS FROM OLD FUNCTIONS

so as long as both \( f(3) \) and \( g(3) \) are well-defined (that is, as long as 3 is in the domain of both \( f \) and \( g \)) then \( f(3) + g(3) \) is well-defined so \((f + g)(3)\) makes sense and 3 is in the domain of \( f + g \). Thus the domain of \( f + g \) is every value of \( x \) which is in the domain of both \( f \) and \( g \).

Quotients of functions are a little more complicated. In order for 3 to be in the domain of \( f/g \) we need the output of \( f/g \) to be well-defined. If 3 were in the domain of \( f/g \) then its output would be \( \frac{f(3)}{g(3)} \). For this value to be well-defined we first need \( f(3) \) and \( g(3) \) to be well-defined (that is, 3 needs to be in the domain of \( f \) and \( g \)) but we also need the fraction to be well-defined which means that the denominator cannot be zero. In order for \( \frac{f(3)}{g(3)} \) to make sense mathematically we need \( f(3) \) to be defined, \( g(3) \) to be defined, and \( g(3) \) to be non-zero. Thus the domain of \( f/g \) is every value of \( x \) which is in the domain of both \( f \) and \( g \) such that \( g(x) \neq 0 \).

Some of the more difficult examples of sums and products of functions involve rational functions. These can often be challenging for students and may require extra practice depending on the student's level of previous experience.

\[
\text{(2.1.5) Example.} \quad \text{Let } f(x) = \frac{1}{x^2 + 3x + 2} \text{ and let } g(x) = \frac{x+1}{x+2}.
\]

\begin{enumerate}
\item Find and simplify \((f + g)(x)\).
\item Find and simplify \((fg)(x)\).
\item Find and simplify the function \((f/g)(x)\).
\end{enumerate}

\textbf{Solution:}

\begin{enumerate}
\item Observe that
\[
(f + g)(x) = f(x) + g(x) = \frac{1}{x^2 + 3x + 2} + \frac{x+1}{x+2} = \frac{1}{(x+1)(x+2)} + \frac{(x+1)(x+1)}{(x+2)(x+1)} = \frac{1 + (x^2 + 2x + 1)}{(x+1)(x+2)} = \frac{x^2 + 2x + 2}{x^2 + 3x + 2}.
\]

In problems like these, be careful with your algebra and make sure that you cancel any common factors.

\textbf{Answer:} \((f + g)(x) = \frac{x^2 + 2x + 2}{x^2 + 3x + 2}\)
b. Observe that

\[(fg)(x) = (f(x))(g(x))\]

\[= \left( \frac{1}{(x+1)(x+2)} \right) \left( \frac{x+1}{x+2} \right)\]

\[= \frac{(x+1)}{(x+1)(x+2)^2}\]

\[= \frac{1}{(x+2)^2}\]

\[= \frac{1}{x^2 + 4x + 4}\]

**Answer:** \((fg)(x) = \frac{1}{x^2 + 4x + 4}\)

c. Observe that

\[(f/g)(x) = \frac{f(x)}{g(x)}\]

\[= \left( \frac{1}{(x+1)(x+2)} \right) \left( \frac{x+2}{x+1} \right)\]

\[= \frac{1}{(x+1)^2(x+2)}\]

\[= \frac{1}{(x+1)^2}\]

\[= \frac{1}{x^2 + 2x + 1}\]

**Answer:** \((f/g)(x) = \frac{1}{x^2 + 2x + 1}\)

(2.1.A) **Practice Exercise.** Let \(f(x) = \frac{1}{x+1}\), \(g(x) = \frac{1}{x+3}\), and \(h(x) = \frac{1}{x+1}\).

(i) Find and simplify \((f - g)(x)\).

(ii) Find and simplify \((f/h)(x)\).

The last thing to be discussed in this section is how to combine functions graphically. Consider the graphs of two functions \(f\) and \(g\) shown in Figure 2.1.6. Suppose that we want to draw the graph of \(f + g\).
For each function \( f \) and \( g \) in Figure 2.1.6, the height of the function at an \( x \)-value corresponds to the output of the function at that \( x \)-value. In order to graph \( f + g \) we need to determine its height at each \( x \)-value. For a particular value of \( x \) we know that \((f + g)(x) = f(x) + g(x)\). Well, \( f(x) \) corresponds to the height of the graph of \( f \) and \( g(x) \) corresponds to the height of the graph of \( g \). Thus the height of the graph of \( f + g \) at a particular \( x \)-value should be the height of the graph of \( f \) with the height of the graph of \( g \) added on to it.

This gives us a method to determine points on the graph of \( f + g \). Consider \( x = 9 \). In order to find the height of the graph of \( f + g \) when \( x = 9 \) we determine the height of the graph of \( g \) when \( x = 9 \) and place it “on top” of the the graph of \( f \) when \( x = 9 \). Figure 2.1.7 demonstrates this process. The height of the vertical blue line represents the value of \( g(9) \). It is shown between the \( x \)-axis and the graph of \( g \) as one expects and it is also shown starting at the graph of \( f \) when \( x = 9 \). The end of this second segment is the value of \((f + g)(9)\) which is shown as a point in red.
This process certainly can be repeated. In Figure 2.1.8 the process is performed at every integer x-value and a pattern emerges that allows us to sketch the graph of \( f + g \). In practice one should draw in as many points as needed until it is clear how to sketch the graph of \( f + g \). Sometimes it only takes a few points to get enough detail to sketch the graph but at other times it takes several.

Notice in Figure 2.1.8 that the blue vertical line segments are drawn below the graph of \( f \) on the left of \( x = 5 \) but are drawn above the graph of \( f \) on the right of \( x = 5 \). That is because when \( x \) is less than 5 the graph of \( g \) is below the x-axis. Since \( g(x) \) is a negative number when \( x \) is less than 5 the graph of \( f + g \) should be below the graph of \( f \) for those \( x \)-values. This is because \((f + g)(x) < f(x)\) when \( g(x) < 0 \).

![Graph of functions](image)

**Figure 2.1.8:** The graph of the function \( f + g \).

A similar process can allow us to graph products, quotients, and differences as well. However, in this course students will only be asked to graph sums of functions.

**Applications**

As one can imagine, there are several applications of combining functions. We start with a financial application.

Suppose a small company is examining its quarterly reports. It finds that last quarter they brought in an income of \$10,000\ from sales and it had to spend \$4000\ to make that money. We call that income of \$10,000\ their **REVENUE** and we call the \$4000\ that they spent their **COST**. Of course, when calculating the amount of money that they actually earned during the quarter we must subtract the cost from the revenue. This last amount is called their **PROFIT** and it is always calculated by subtracting cost from revenue. In this case, the company’s profit is \$6000\.
(2.1.9) **Example.** Vandelay Industries sells tablet PC’s. They have done some market research and found two different functions. Their cost function, $C$, describes their costs in a given month. In order to sell $x$ tablets in a month they must spend $C(x)$ dollars where

$$C(x) = 0.0143x^2 + 183x + 156000.$$ 

Their demand function, $D$, describes the price that they need to charge in order to sell a certain number of tablets. In order to sell $x$ tablets in a month they must charge $D(x)$ dollars per tablet where

$$D(x) = -0.002x + 290.$$ 

**A.** The company’s revenue from selling $x$ tablets in a month is given by $R(x) = xD(x)$. Find and simplify the revenue function.

**B.** The company’s profit from selling $x$ tablets in a month is given by $P = R - C$ (that is, profit is equal to the revenue less the cost). Find and simplify the profit function.

**C.** How many tablets should they sell in a month in order to maximize their profit? How much profit do they make at that production level? What price do they need to charge for each tablet in order to sell the optimal number of tablets?

**D.** The break-even point is a value $n$ such that if the company sells fewer than $n$ tablets they will lose money and if they sell more than $n$ tablets (up to a point) they will make money. The break-even point, in this case, is the smallest positive root of the profit function. Find the break-even point.

**Solution:**

**A.** Observe that

$$R(x) = xD(x) = x(-0.002x + 290) = -0.002x^2 + 290x$$

**Answer:** $R(x) = -0.002x^2 + 290x$
b. Since $P = R - C$ we have that

$$P(x) = R(x) - C(x)$$

$$= (-0.002x^2 + 290x) - (0.0143x^2 + 183x + 156000)$$

$$= -0.0163x^2 + 107x - 156000.$$

**Answer:** $P(x) = -0.0163x^2 + 107x - 156000$

c. This question asks us to maximize the profit function. The profit function is a second degree polynomial and the maximum of such a polynomial occurs at its vertex. Recall that the first coordinate of the vertex is found at

$$x = \frac{-b}{2a}.$$ 

In this case we have that

$$x = \frac{-107}{2(-0.0163)} = 3282.21.$$ 

The value above is an input for the profit function. That is, in order to attain the maximum possible profit in a month the company must sell 3282.21 tablets. The profit that is attained at that level of production is given by

$$P(3282.21)$$

$$= -0.0163(3282.21)^2 + 107(3282.21) - 156000$$

$$= 19598.16.$$ 

Hence the maximum total profit attainable in a month is $\$19,598.16$.

The final part of the question asks for the price at which they must sell tablets in order to sell the optimal number of tablets. We know that the optimal number of tablets is 3282.21 and we know that $D(3282.21)$ should tell us the price they must charge to sell 3282.21 tablets. Finally,

$$D(3282.21) = -0.002(3282.21) + 290 = 283.44.$$ 

Indeed, each tablet should be sold for $\$283.44$ in order to sell the optimal number of tablets.

**Answer:** The company should sell 3282.21 tablets to maximize their profit. The maximum obtainable profit is $\$19,598.16$ and they must sell their tablets for $\$283.44$ each to sell the optimal number of tablets.
d. The last question asks us to find the smallest positive root of the profit function. The roots are found by using the quadratic formula:

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ = \frac{-107 \pm \sqrt{107^2 - 4(-0.0163)(-15600)}}{2(-0.0163)} \]

\[ = 3282.21 \pm 1096.51 \]

It follows that the roots of the profit function occur when \( x = 2185.70 \) and when \( x = 4378.72 \). Since we want the smallest positive root we see that the break-even point is \( n = 2185.70 \)

**Answer:** 2185.70 units

There are many other applications of this type of function construction. Example 2.1.10 explores one of them.

(2.1.10) Example. A small university sells parking permits to its students. The parking office classifies the students in two categories: “residents” are those students who live on (or near) campus and “commuters” are those students who live far from the university and travel to campus every day. The functions \( R \) and \( C \) describe the number of each type of student. That is, if \( t \) is the number of years after the year 2000 then \( R(t) \) is the number of residents attending the university that year and \( C(t) \) is the number of commuters attending the university that year. The university estimates that

\[ R(t) = 500 + 40t \quad \text{and} \quad C(t) = 3000(1.02)^t. \]

The university also estimates that 40% of residents will need parking permits whereas 90% of commuters will need parking permits. Find a function, \( P \), which describes the total number of parking permits that the university should expect to sell. That is, if \( t \) is the number of years after the year 2000 then \( P(t) \) should be the approximate number of parking permits that the university will sell in that year.

**Solution:**

The function \( R \) tells us the number of residents at the university. Since 40% of them need parking permits, the function \( 0.4R \) describes the number of permits that residents will need.
Similarly, since 90% of the commuters need parking permits, the function $0.9C$ describes the number of permits that commuters will need. The total number of permits is then simply the sum of the number of permits from each group. Thus $P = 0.4R + 0.9C$ and

$$P(t) = 0.4(500 + 40t) + 0.9(3000(1.02)^{t})$$
$$= 200 + 16t + 2700(1.02)^{t}.$$  

**Answer:** $P(t) = 200 + 16t + 2700(1.02)^{t}$

---

**2.1.3 Practice Exercise.** Ken is mowing his lawn. Ken’s lawn is rectangular and when Ken mows it he starts in the center and mows outward so that as he continues to mow, the space that has already been mowed is a rectangle which is increasing in size. Ken’s wife is watching him mow and as she watches she keeps track of his progress. She finds that $t$ minutes after he started mowing, the length of the space that he’s mowed is $L(t)$ feet and the width of the space that he’s already mowed is $W(t)$ feet where

$$L(t) = 7t^{1/3} \quad \text{and} \quad W(t) = 3t^{2/3}.$$

(i) Find a function $A$ such that $t$ minutes after Ken started mowing, the area of the space that he’s already mowed is $A(t)$ square feet.

(ii) Ken’s entire yard has an area of 567 ft$^2$. How long does it take him to mow the entire yard?

(iii) Find the dimensions (the length and the width) of Ken’s yard. *Hint: Use the functions for length and width.*
2.1.1. Imagine two functions, \( p \) and \( q \), such that the domain of \( p \) is \((-\infty, 0) \cup (0, \infty)\) and the domain of \( q \) is \((-\infty, 2) \cup (2, \infty)\). What is the domain of \( p + q \)?

2.1.2. Let \( f(x) = \frac{1}{x^2 + 6x + 5} \), let \( g(x) = x + 5 \), and let \( h(x) = \frac{1}{x+1} \).
   
   a.) Find \((f + h)(x)\).
   
   b.) Find \((fg)(x)\).
   
   c.) Find \((f/h)(x)\).

2.1.3. Let \( f(x) = \frac{1}{2x-7} \), let \( g(x) = \frac{1}{x+7} \), and let \( h(x) = 3 - 5x \). Find and simplify \((\frac{f}{g} + 2h)(x)\).

2.1.4. Let \( f(x) = \frac{1}{x^2-6} \), let \( g(x) = \frac{x}{x+7} \), and let \( h(x) = 3x + 1 \). Find and simplify \( fh + g \).

2.1.5. Let \( f(x) = \frac{1}{x+2} \), let \( g(x) = \frac{x}{x^2-x-2} \), and let \( h(x) = x \). Find and simplify \( gh - f \).

2.1.6. Let \( f(x) = \frac{x^3}{x-1} \), let \( g(x) = \frac{1}{3x^2-x-7} \), and let \( h(x) = \frac{1}{x-7} \). Find and simplify the function \((fg + h)(x)\).

2.1.7. Let \( f(x) = \frac{x}{x+3} \), let \( g(x) = x^2 + 4x + 3 \), and let \( h(x) = x + 1 \).
   
   a.) Find and simplify \((f + \frac{g}{f})(x)\).
   
   b.) Find and simplify \((fg - h)(x)\).

2.1.8. Let \( f(x) = \frac{x^2 + x - 6}{x^2 + 5x + 4} \), let \( g(x) = \frac{1}{x+1} \), and let \( h(x) = \frac{1}{x-2} \).
   
   a.) Find and simplify \((fh)(x)\).
   
   b.) Find and simplify \((f/g)(x)\).
   
   c.) Find and simplify \((f + g)(x)\).
   
   d.) Find and simplify \((f + \frac{g}{h})(x)\).

2.1.9. Define functions \( f \) and \( g \) such that

\[
f(x) = \frac{x}{x-3} \quad \text{and} \quad g(x) = \frac{12}{x^2 - 2x - 3}.
\]

A.) Find and simplify \((f - g)(x)\).

b.) Find and simplify \((g/f)(x)\).

2.1.10. Consider the following functions:

\[
f(x) = 1 - x^2 \quad \text{and} \quad h(x) = \frac{x^2 - x - 12}{x^2 + 7x + 10}.
\]

Find and simplify the following:

A.) \((f + pq)(x)\)

b.) \(\left(\frac{hr}{q}\right)(x)\)

c.) \((ph)(x)\)

d.) \(\frac{p(x) - 2}{h(x)}\)

2.1.11. The functions \( f \) and \( g \) are graphed below. Sketch the graph of \( f + g \). (Assume that \( f \) has a vertical asymptote at \( x = 0 \).)

A.) Find and simplify \((f - g)(x)\).

2.1.12. The functions \( f \) and \( g \) are graphed below. Sketch the graph of \( f + g \).
2.1.13. The functions \( f \) and \( g \) are graphed below. Sketch the graph of \( f + g \).

2.1.14. The functions \( f \) and \( g \) are graphed below. Sketch the graph of \( f + g \).

2.1.15. A friend of mine knits and sells blankets. After doing some market research she found a demand function, \( D \). That is, if she charges \( x \) dollars for each blanket then she will be able to sell \( D(x) \) blankets in a month where \( D(x) = 25 - x/2 \). In such a month her costs are given by \( C(x) = 150 - 2x \) when she charges \( x \) dollars for each blanket.

A.) The revenue function \( R \) is defined such that \( R(x) = xD(x) \). Find and simplify the revenue function.

B.) Find and simplify the profit function, \( P \). Hint: How do we find profit from cost and revenue?

C.) How much should she charge for each blanket to maximize her profit? Hint: How do we find the highest point on a quadratic polynomial?

D.) What is the maximum profit that she can make in a month? Hint: What is the profit that she earns if she charges the price determined in part (c)?

**Challenge Problems**

2.1.c1 Challenge Problem. The graphs of two functions, \( y = f(x) \) and \( y = g(x) \), are shown below.
A. Sketch the graph of \( y = (f + g)(x) \).

B. Sketch the graph of \( y = (f - g)(x) \).

C. Sketch the graph of \( y = (fg)(x) \). Briefly justify your answer.

D. Sketch the graph of \( y = (f/g)(x) \). Briefly justify your answer.

*Hint: Be sure to investigate carefully what happens to each of these graphs when \( x \) is an even integer or an odd integer.*

**(2.1.c2) Challenge Problem.** Consider the following two piecewise functions:

\[
\begin{align*}
f(x) &= \begin{cases} 
1 & \text{if } x < -2 \\
-x & \text{if } -2 \leq x \leq 2 \\
1 - x & \text{if } x > 2
\end{cases} \\
g(x) &= \begin{cases} 
-\frac{1}{3}x & \text{if } x < 0 \\
5 & \text{if } 0 \leq x \leq 2 \\
\frac{1}{x-2} & \text{if } x > 2
\end{cases}
\end{align*}
\]

A. Find and simplify \((f + g)(x)\).

B. Find and simplify \((fg)(x)\).

*Hint: Both answers should be piecewise functions. Graphing \( f \) and \( g \) may be helpful for some people.*

**(2.1.c3) Challenge Problem.** The population of the fictional city of Kakariko is growing rapidly and its government is examining the local economy. A city planner has developed a model for its population. If \( t \) is the number of years after 2010 then \( P(t) \) is the projected population of the city where

\[
P(t) = 12000e^{0.035t}.
\]

The city’s financial director has a model for the number of jobs that the city can support. If \( t \) is the number of years after 2010 then the
financial director expects there to be $J(t)$ jobs available within the city limits where

$$J(t) = 10t^2 - 10t + 13000.$$

These models are predicted to be reasonable approximations through the year 2030.

A. At any given moment, approximately 75% of the city’s population is employable and expects to have a job within the city’s limits. Find a function $E$ such that $t$ years after 2010, $E(t)$ is the number of Kakariko residents that expect to have a job in the city.

B. One measurement of Kakariko’s economy is the number of surplus jobs in the city. Let $S(t)$ be the number of jobs in Kakariko that cannot be filled by Kakariko’s citizens $t$ years after 2010. That is, $t$ years after 2010 there should be $S(t)$ more jobs than employable citizens. Find a formula for $S(t)$.

C. Sketch a graph of $y = S(t)$. Make sure to use an appropriate domain.

D. Use a calculator or some other type of software to find any roots of $S$. Explain what these roots mean to the city of Kakariko.
2.2 COMPOSITION OF FUNCTIONS

This section corresponds to section 10.1 in Functions Modeling Change [4]

Section 2.1 describes how to use arithmetic to create new functions by combining the outputs of old functions. Functions are like machines which input a number and then output another number. This section describes how to create a new function by using two functions consecutively.

Motivation

Imagine two machines. The first machine will process truffula trees into thneeds and the second machine processes thneeds into socks. Now imagine that you have truffula trees but you need socks. There isn’t a machine available that will process your truffula trees into socks, but that doesn’t mean you can’t make socks. If you put the truffula trees into the first machine you will have thneeds. If you then put the thneeds into the second machine you will get the socks that you wanted.

The preposterous example above is the rough idea behind function composition. Functions are analogous to machines; a function inputs certain values and outputs values which correspond to the input. A toy store is developing a new toy and they find a cost function for this toy in terms of the number of toys sold and produced. That is, if they want to sell \( n \) of these toys then it will cost the company \( C(n) \) dollars to produce them. Then the function \( C \) is a machine which inputs a number, \( n \), and outputs the cost required to produce \( n \) toys.

A convenient visual representation for the function \( C \) uses boxes and arrows as in Figure 2.2.1

![Figure 2.2.1: A visual representation of the cost function, C.](image)

The left-to-right arrow in the diagram signifies that the function \( C \) inputs things in the box on the left and outputs things in the box on the right.

Now consider a demand function, \( D \), which describes the number of toys they can sell in terms of the price for which they sell each toy. That is, if the company sells the toys for \( p \) dollars each then they will be able to sell \( D(p) \) toys. A visualization of this demand function is shown in Figure 2.2.2.

Suppose that you’re working for this company and your job is to analyze cost as a function of price. What you need is a function, \( K \),
which inputs the unit price of the toy and outputs the production cost. You have a function which inputs the unit price and you have another function which outputs the production cost but neither function does both. In Figure 2.2.3, C and D are the functions that you already have and K is the function that you want.

In the same way that we were (theoretically) able to make socks out of truffula trees, we can make the function K. If we start with a unit price, p, we want to find the production cost. The domain of D is unit price and D(p) tells us the number of toys the company can sell at a unit price of p dollars. Fortunately the domain of C is the number of toys that they sell so C(D(p)) should be the production cost of selling D(p) toys. If we define K(p) = C(D(p)) then K(p) is the cost of producing toys at a unit price of p dollars. This is exactly what we needed K to be.

The construction of the function K above is an example of function composition which is the focus of this section.

**Theory**

We start with a definition.

(2.2.4) **Definition.** Let f and g be real functions. Define the function f \( \circ \) g such that

\[
(f \circ g)(x) = f(g(x))
\]

for all values of x such that x is in the domain of g and g(x) is in the domain of f. The process of forming the function f \( \circ \) g from the functions f and g is called **FUNCTION COMPOSITION.**

It is straightforward to compute the outputs of composition functions from the definition.
(2.2.5) Example. Consider the functions \( f(x) = x^2 + 1 \) and \( g(x) = 2x - 2 \). Compute \((f \circ g)(3)\).

Solution:
From Definition 2.2.4 we have that
\[
(f \circ g)(3) = f(g(3)).
\]
First, \( g(3) = 2(3) - 2 = 6 - 2 = 4 \). Thus
\[
(f \circ g)(3) = f(g(3)) = f(4).
\]
Then \( f(4) = 4^2 + 1 = 16 + 1 = 17 \) so that
\[
(f \circ g)(3) = f(g(3)) = f(4) = 17.
\]
Answer: \((f \circ g)(3) = 17\)

Be mindful of the difference between \( f \circ g \) and \( g \circ f \). That is,
\[
(f \circ g)(x) = f(g(x)) \quad \text{and} \quad (g \circ f)(x) = g(f(x)).
\]
Note that to compute \((f \circ g)(x)\) we first plug \( x \) into \( g \) and then plug \( g(x) \) into \( f \) but to compute \((g \circ f)(x)\) we first plug \( x \) into \( f \) and then plug \( f(x) \) into \( g \). To help remember this difference, the notation \( f \circ g \) is sometimes read as “\( f \) after \( g \)” since computing \((f \circ g)(x)\) requires us to plug \( g(x) \) into \( f \) after we plug \( x \) into \( g \). This difference is highlighted in Example 2.2.6.

(2.2.6) Example. Consider the functions \( f(x) = x^2 + 1 \) and \( g(x) = 2x - 2 \). Compute \((g \circ f)(3)\).

Solution:
From Definition 2.2.4 we have
\[
(g \circ f)(3) = g(f(3)) = g(3^2 + 1) = g(10) = 2(10) - 2 = 18.
\]
Remember from Example 2.2.5 that \((f \circ g)(3) = 17\).
Answer: \((g \circ f)(3) = 18\)

In addition to performing computations we need to be able to find and simplify outputs for arbitrary inputs. Before doing that, though, let us recall how we input variables and expressions into functions.
(2.2.7) Example. Let \( f(x) = x^2 - 2x + 3 \).

A. Compute \( f(4) \).

B. Find and simplify \( f(t+3) \).

C. Find and simplify \( f(x^2 - 1) \).

Solution:

A. As expected, \( f(4) = 4^2 - 2(4) + 3 = 16 - 8 + 3 = 11 \).

Answer: \( f(4) = 11 \)

B. To find \( f(t+3) \) we replace every occurrence of \( x \) with the expression \( t + 3 \) and then simplify.

\[
\begin{align*}
f(t+3) &= (t+3)^2 - 2(t+3) + 3 \\
&= t^2 + 6t + 9 - 2t - 6 + 3 \\
&= t^2 + 4t + 6.
\end{align*}
\]

Answer: \( f(t+3) = t^2 + 4t + 6 \)

C. Again, to find \( f(x^2 - 1) \) we replace every occurrence of \( x \) with \( x^2 - 1 \).

\[
\begin{align*}
f(x^2 - 1) &= (x^2 - 1)^2 - 2(x^2 - 1) + 3 \\
&= x^4 - 2x^2 + 1 - 2x^2 + 2 + 3 \\
&= x^4 - 4x^2 + 6.
\end{align*}
\]

Answer: \( f(x^2 - 1) = x^4 - 4x^2 + 6 \)

Now that we remember how to plug expression into functions, finding and simplifying composition functions is straightforward.

(2.2.8) Example. Let \( f(x) = x^2 - 2x + 3 \) and let \( g(x) = x^2 - 1 \).

A. Compute \( (f \circ g)(1) \).

B. Find and simplify \( (f \circ g)(x) \).

C. Compute \( (f \circ g)(0) \).
Solution:

A. From the definition, \((f \circ g)(1) = f(g(1))\). We first plug 1 into \(g\) and see that \(g(1) = 0\). We then plug \(g(1) = 0\) into \(f\) and see that \(f(0) = 3\). It follows that
\[
(f \circ g)(1) = f(g(1)) = f(0) = 3.
\]

**Answer:** \((f \circ g)(1) = 3\)

B. Following a similar process, in order to find \((f \circ g)(x)\) we first plug \(x\) into \(g\) and see that \(g(x) = x^2 - 1\). We then plug \(g(x) = x^2 - 1\) into \(f\) to obtain
\[
(f \circ g)(x) = f(g(x)) = f(x^2 - 1).
\]
We can then simplify \(f(x^2 - 1)\) using the same process as in Example 2.2.7.

\[
(f \circ g)(x) = f(x^2 - 1) \\
= (x^2 - 1)^2 - 2(x^2 - 1) + 3 \\
= x^4 - 2x^2 + 1 - 2x^2 + 2 + 3 \\
= x^4 - 4x^2 + 6.
\]

**Answer:** \((f \circ g)(x) = x^4 - 4x^2 + 6\)

C. It would be simple to compute \((f \circ g)(0)\) in the same way that we computed \((f \circ g)(1)\). However, we found an equation for \(f \circ g\) for arbitrary inputs which makes the computation a little easier this time.

\[
(f \circ g)(0) = (0)^4 - 4(0)^2 + 6 = 6.
\]

**Answer:** \((f \circ g)(0) = 6\)

Example 2.2.9 demonstrates a more complicated example.

\((2.2.9)\) **Example.** Let \(f(x) = 2x + 3\) and let \(g(x) = \frac{x+1}{x-4}\).

A. Find and simplify \((f \circ f)(x)\).

B. Find and simplify \((g \circ f)(x)\).

C. Find and simplify \((f \circ g)(x)\).
Solution:

A. From Definition 2.2.4 we have that $f \circ f(x) = f(f(x))$. That is we first plug $x$ into $f$ and we plug the output again into $f$.

$$(f \circ f)(x) = f(f(x)) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 9$$

**Answer:** $(f \circ f)(x) = 4x + 9$

b. Observe that

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3) + 1 = (2x + 3) - 4 = 2x + 4$$

**Answer:** $(g \circ f)(x) = \frac{2x + 4}{2x - 1}$

c. Observe that

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x + 1}{x - 4}\right) = 2\left(\frac{x + 1}{x - 4}\right) + 3 = \frac{2(x + 1)}{x - 4} + \frac{3(x - 4)}{x - 4} = \frac{2x + 2 + 3x - 12}{x - 4} = \frac{5x - 10}{x - 4}$$

**Answer:** $(f \circ g)(x) = \frac{5x - 10}{x - 4}$

(2.2.A) **Practice Exercise.** Let $f(x) = 3x - 2$, let $g(x) = x^2 + 1$, and let $h(x) = \frac{x + 1}{x - 1}$.

(i) Find and simplify $(g \circ f)(x)$.

(ii) Find and simplify $(h \circ g)(x)$. 

Remember that an equation is not the only way to present a function. Example 2.2.11 and Example 2.2.10 below explore composition of functions presented by a table and by a graph. Of course, the definition of function composition doesn’t change in this case; only the execution of the computations changes.

(2.2.10) Example. The functions $f$ and $g$ are defined in the table below. Fill in the rest of the table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$(f \circ g)(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(g \circ f)(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(f \circ f)(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Solution:
Let’s start by calculating $(f \circ g)(0)$. We know that $(f \circ g)(0) = f(g(0))$. From the table we know that $g(0) = 0$ and we know that $f(0) = 2$. Hence

$$(f \circ g)(0) = f(g(0)) = f(0) = 2.$$ 

The computations for the entries when $x = 0$ and $x = 1$ are shown below and the complete table is shown in the end. You can verify the rest of the entries for yourself. We can compute the rest of the entries similarly:

$$(f \circ g)(1) = f(g(1)) = f(4) = 3$$
$$(g \circ f)(0) = g(f(0)) = g(2) = 1$$
$$(g \circ f)(1) = g(f(1)) = g(5) = 5$$
$$(f \circ f)(0) = f(f(0)) = f(2) = 0$$
$$(f \circ f)(1) = f(f(1)) = f(5) = 1$$

Answer: In the table below:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$(f \circ g)(x)$</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$(g \circ f)(x)$</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$(f \circ f)(x)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
(2.2.11) Example. The graphs of \( y = h(x) \) and \( y = r(x) \) are shown below.

A. Find \((h \circ r)(1)\).

B. Find \((h \circ r)(2)\).

C. Find \((h \circ r)(3)\).

D. Find \((r \circ h)(0)\).

E. Find \((r \circ h)(5)\).

F. Find \((h \circ h)(4)\).

Solution:

A. From the definition, \((h \circ r)(1) = h(r(1))\). From the graph of \( y = r(x) \) we see that \( r(1) = 5 \) so that \((h \circ r)(1) = h(5)\). From the graph of \( y = h(x) \) we see that \( h(5) = 2 \). Hence \((h \circ r)(1) = h(r(1)) = h(5) = 2 \). We can solve the rest of these similarly.

\[ \text{Answer: } (h \circ r)(1) = 2 \]

B. \((h \circ r)(2) = h(r(2)) = h(5) = 2 \)

\[ \text{Answer: } (h \circ r)(2) = 2 \]

C. \((h \circ r)(3) = h(r(3)) = h(5) = 2 \)

\[ \text{Answer: } (h \circ r)(3) = 2 \]

D. \((r \circ h)(0) = r(h(0)) = r(2) = 5 \)

\[ \text{Answer: } (r \circ h)(0) = 5 \]

E. \((r \circ h)(5) = r(h(5)) = r(2) = 5 \)

\[ \text{Answer: } (r \circ h)(5) = 5 \]
\[ (h \circ h)(4) = h(h(4)) = h(1) = 4 \]

**Answer:** \( (h \circ h)(4) = 4 \)

The last theoretical topic to discuss is the domain of composition functions. **Definition 2.2.4** suggests that \((f \circ g)(x)\) is defined whenever \(x\) is in the domain of \(g\) and \(g(x)\) is in the domain of \(f\). More simply, \(x\) is in the domain of \(f \circ g\) whenever the computation of \((f \circ g)(x) = f(g(x))\) is well-defined.

Consider the functions \(f(x) = \frac{1}{x^2}\) and \(g(x) = \sqrt{x}\). In this section we’ve developed the tools to compute

\[ (f \circ g)(9) = f(g(9)) = f(3) = 1. \]

Since the computation of \((f \circ g)(9)\) makes sense, 1 is in the domain of \(f \circ g\). Consider, however, \((f \circ g)(-3)\). If \(-3\) were in the domain of \(f \circ g\) then we would have

\[ (f \circ g)(-3) = f(g(-3)), \]

but \(g(-3) = \sqrt{-3}\) is undefined. Since there is no way to make sense of \(g(-3)\), there is certainly no way to make sense of \(f(g(-3))\) and hence \(-3\) is not in the domain of \(f \circ g\). Now consider \((f \circ g)(4)\). Again, we should be able to calculate

\[ (f \circ g)(4) = f(g(4)). \]

Certainly \(g(4) = 2\) but then \(f(2) = \frac{1}{2}\) which is undefined. Since \(f(g(4)) = f(2)\) is not defined it cannot be the case that 4 is in the domain of \(f \circ g\).

**Applications**

Applications of function composition often present themselves similarly to the cost example at the beginning of this section. Oftentimes the difficulty with these types of problems is deciding the direction of the composition. Recall that in the beginning of the section we had a cost function \(C\) and a demand function \(D\). The input of the demand function was unit price while its output was the number of units produced and the input of the cost function was the number of units produced while its output was the company’s production costs. In the context of the application, the functions \(C\) and \(D\) have meaningful inputs and outputs. However, from a theoretical standpoint both \(C\) and \(D\) input and output real numbers. This means that, from a mathematical perspective, both \(C \circ D\) and \(D \circ C\) make sense. In the context of the application, though, only one of those functions has any meaning.
First consider the function \( C \circ D \) and the meaning of \( C(D(x)) \). First, \( x \) is plugged into \( D \). This means that \( x \) must be in the domain of \( D \) which is a unit price. Then \( D(x) \) is the number of toys produced at a unit price of \( x \) dollars. The domain of \( C \) should be a number of units produced and fortunately \( D(x) \) is such a quantity. Then \( C(D(x)) \) is the cost of producing \( D(x) \) toys which is perfectly logical. Earlier in the section we defined the function \( K \) such that \( K(x) = C(D(x)) \). Now that we have function composition notation we could simply write \( K = C \circ D \). The diagram in Figure 2.2.3 helps clarify how these function work together.

Now consider \( D \circ C \) and the meaning of \( D(C(x)) \). Since, in this case, \( x \) is first plugged into \( C \) it follows that \( x \) must be a quantity of toys produced. Then \( C(x) \) is the cost of producing \( x \) items. The domain of \( D \), however, is unit price so \( D(C(x)) \) is meaningless in the context of the problem. Note that if we know the functions \( C \) and \( D \) it is possible to compute the function \( D \circ C \) but, as we just demonstrated, the function \( D \circ C \) has no meaning in the context of the application.

(2.2.12) Example. The Duck Store is evaluating the amount of money that it spends on textbooks for MATH 111/112. After contacting the registrar and looking over their own sales records, they find a relationship between the number of students enrolled in MATH 111/112 in a term and the number of textbooks that they sell. In a term where there are \( n \) students enrolled in MATH 111/112 the Duck Store sells approximately \( S(n) \) copies of the textbook where

\[
S(n) = 0.45n + 50.
\]

Additionally, their distributor gives a discounted rate when they buy larger quantities of books. If the Duck Store purchases \( n \) books from their distributor then they are charged \( C(n) \) dollars for that order where

\[
C(n) = 50n - e^{0.01n}.
\]

Find and simplify a function \( f \) such that when there are \( n \) students enrolled in MATH 111/112 the Duck Store must spend \( f(n) \) dollars on purchasing textbooks for MATH 111/112 from their distributor.

Solution:

It can be helpful (even in problems that don’t involve composition) to make a diagram of the inputs and outputs involved in this problem. The function \( S \) inputs the number of students enrolled in MATH 111/112 and outputs the number
of textbooks that the Duck Store needs to purchase. The function C inputs the number of textbooks that the Duck Store needs to purchase and outputs the cost of purchasing those textbooks. Those three quantities are shown below:

<table>
<thead>
<tr>
<th>Students Enrolled</th>
<th>Textbooks Purchased</th>
<th>Cost of Purchase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We then draw the functions which relate those quantities. S and C are the functions given in the problem and f is the function that we are trying to find.

The function f should input a quantity of students enrolled in MATH 111/112, n, and f(n) should be the cost of purchasing textbooks for MATH 111/112. Given such a value, n, we have that S(n) is the number of textbooks that the Duck Store needs to purchase. Then S(n) is in the domain of C and C(S(n)) is the cost of producing those textbooks. It follows that f(n) = C(S(n)).

All that remains is to find and simplify the function f = C o S.

\[
f(n) = C(S(n)) \\
= C(0.45n + 50) \\
= 50(0.45n + 50) - e^{0.01(0.45n+50)} \\
= 22.5n + 2500 - e^{0.0045n+0.5}.
\]

**Answer:** \[f(n) = 22.5n + 2500 - e^{0.0045n+0.5}\]

The solution to Example 2.2.12 presented above includes box and arrow diagrams demonstrating the relationship between the functions and quantities involved. While these diagrams can help understand the application, they are not a required part of the answers. If you can figure out the proper composition without the diagrams then feel free to leave them out.

(2.2.B) **Practice Exercise.** Scott works for Kettle Potato Chips. The rate at which the plant makes chips varies throughout the year. This rate of production is expressed by a percent-
age; for example, if the plant is running at 60% for a given week then it is producing 60% of its maximum possible output. If the plant is running at a percentage of $p$ for the week then Scott works $H(p)$ hours that week where

$$H(p) = \frac{25p - 70}{p - 2}.$$ 

Additionally, when Scott works $h$ hours in a week he is paid $C(h)$ dollars where $C(h) = 27h + 500$. Find a function $S$ such that if the plant is running at a percentage of $p$ for a given week then Scott makes $S(p)$ dollars that week. Be sure to simplify your answer.
2.2.1. Fill in the table below.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>g(x)</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(f + g)(x)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(g ∘ f)(x)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.2.2. Let \( f(x) = 2x - 1 \) and \( g(x) = \frac{1}{3x-4} \).

A.) Find \( (f \circ g)(x) \).
B.) Find \( (g \circ f)(x) \).
C.) Find \( (g \circ f)(\frac{3}{2}) \).

2.2.3. Let \( f(x) = x^2 + 2 \), let \( g(x) = \frac{1}{x+1} \), and let \( h(x) = e^x \). Simplify your answers.

A.) Find \( (f \circ g)(x) \)
B.) Find \( (g \circ f)(x) \)
C.) Find \( (f \circ h)(x) \)
D.) Find \( (h \circ f)(x) \)
E.) Find \( (g \circ h)(x) \)
F.) Find \( (h \circ g)(x) \)

2.2.4. Let \( f(x) = \frac{1}{x-3} \), let \( g(x) = x + 5 \), and let \( h(x) = x^2 - 1 \). Find and simplify

A.) \((f \circ g)(x)\)
B.) \((g \circ f)(x)\)
C.) \((f \circ h)(x)\)
D.) \((h \circ f)(x)\)
E.) \((g \circ h)(x)\)
F.) \((h \circ g)(x)\)

2.2.5. The pressure of 0.029 mol of air in a 1 L container is \( P(k) \) atm (a unit of pressure) where

\[ P(k) = 0.00476k \]

and \( k \) is the temperature of the air in Kelvin. Temperature in Kelvin is \( T(f) \) where

\[ T(f) = \frac{5}{9}f + 255.372 \]

and \( f \) is the temperature in Fahrenheit. Use function composition to find a function \( Q(f) \) for the pressure of 0.029 mol of air in a 1 L container where \( Q(f) \) is in atm and \( f \) is the temperature in Fahrenheit. Be sure to simplify your answer completely and, when necessary, round to five decimal places.

2.2.6. The number of guitars that a particular store sells in a week is a function of the price for which they sell them. If they sell the guitars for \( p \) dollars then they sell \( g(p) \) guitars where

\[ g(p) = \frac{5(360 - p)}{p - 80} \]

The store’s costs in a week are a function of the number of guitars that they sell that week. If they sell \( n \) guitars then their costs for that week are \( c(n) \) dollars where

\[ c(n) = \frac{(n + 2)^2 + 40000}{500} \]

Find a function \( f \) such that if the store decides to sell their guitars for \( p \) dollars then the store’s costs for the week are \( f(p) \) dollars. Be sure to simplify your answer completely.

2.2.7. Alicia works as a waitress in a restaurant. She uses the functions \( T \) and \( P \) to approximate the money she makes in an evening. If she waits on a table whose total bill is \( p \) dollars then she will get a tip of approximately \( T(p) \) dollars where

\[ T(p) = \frac{3}{20}(p + 10) \]

Also, if she waits on a table seating \( n \) people then the total bill at that table is approximately \( P(n) \) dollars where

\[ P(n) = 15n - 5\sqrt{n} + 10 \]
Find a function $f$ such that if she waits on a table seating $n$ people then she gets a tip of approximately $f(n)$ dollars.

2.2.8. Ned, the park ranger, monitors animal populations in a particular park. He finds that in $m$ months into the year there are approximately $W(m)$ thousand wolves in the park where

$$W(m) = e^{-0.01(m-6)^2}.$$ 

Additionally, the wolf population influences the rabbit population since wolves feed on rabbits. Ned finds that whenever there are $w$ thousand wolves in the park there are approximately $R(w)$ thousand rabbits in the park where

$$R(w) = 2 - 10 \ln w.$$ 

Find and simplify a function $F$ such that in $m$ months into the year there are $F(m)$ thousand rabbits in the park. In your final answer you may use decimals rounded to two decimal places. In this context, $m = 0$ corresponds to the first moment of January and $m = 12$ corresponds to the last moment of December but this is not really important to the problem.

2.2.9. The function $V(r) = \frac{4}{3} \pi r^3$ gives the volume (in in$^3$) of a sphere with radius $r$ (in inches). If I am inflating a spherical balloon at a rate such that its radius after $t$ seconds is $R(t) = 7 \log(t+1)$ (in inches) then find a function $S$ such that after $t$ seconds the volume of the balloon is $S(t)$ (in in$^3$).

2.2.10. A farmer is producing corn. The function $R$ gives his revenue in terms of the amount of corn he produces; that is, if he produces $p$ pounds of corn in a year then his revenue is $R(p)$ dollars for that year. The function $C$ gives the amount of corn he expects to produce in terms of the average temperature for the year; that is, if the average temperature for the year is $t$ (in °F) then he expects to produce $C(t)$ pounds of corn that year. Find a function $r$ which describes his revenue for the year in terms of the expected average temperature.

2.2.11. If a computer manufacturer sells $u$ computers in a month then his revenue for the month (in dollars) is approximated by $R(u) = 1000 + 1200u$. If he spends $d$ dollars on advertising in a month then he will sell $C(d) = 0.7\sqrt{d} - 3$ computers in that month. Find a function $f$ such that if he spends $d$ dollars on advertising in a given month then he will make $f(d)$ dollars in revenue.

2.2.12. The demand function for a particular product is given by $D(p) = 16000 - 80p$; that is if the company charges $p$ dollars per unit then they will sell $D(p)$ units of that product. The cost function for this particular product is given by $C_0(q) = 20000 + 30q$; that is if they sell $q$ units then it will cost them $C_0(q)$ dollars to manufacture those units. Round to two decimal places when necessary.

A.) If the revenue function is given by $R(p) = p D(p)$, find and simplify the revenue function.

B.) Find a function $C$ which describes the cost as a function of the price for which they sell the product; that is, if the company charges $p$ dollars per unit then it will cost them $C(p)$ dollars to manufacture the units that they sell.

C.) Find a profit function $M$ in terms of the price for which they sell the product; that is, if the company charges $p$ dollars per unit then they will make a profit of $M(p)$.

D.) Find the price per unit that the company should charge for this particular product in order to maximize their profit. Then find the maximum profit. Round to two decimal places.
(2.2.c1) Challenge Problem. To this point we’ve been starting with functions \( f \) and \( g \) and finding the function \( f \circ g \). However, it is often convenient to be able to “undo” composition. As an example, suppose that \( f(x) = x^2 + 3x + 2 \) and \( h(x) = x^4 - 3x^2 + 2 \). It is sometimes useful to be able to find a function \( g \) such that \( f \circ g = h \). It is easy to check that \( g(x) = x^2 \) is such a function.

\[
(f \circ g)(x) = (x^2)^2 + 3(x^2) + 2 = x^4 + 3x^2 + 2 = h(x)
\]

In a sense we are examining the equation \( f \circ g = h \) where \( f \), \( g \), and \( h \) are functions. Throughout this section we discussed how to find \( h \) if \( f \) and \( g \) are known. In this exercise we explore how to find \( f \) if \( g \) and \( h \) are known and how to find \( g \) if \( f \) and \( h \) are known.

A. Suppose that \( f(x) = x^2 + 3x + 2 \) and \( h(x) = x^4 - 3x^2 + 2 \). Find a function \( g \) such that \( f \circ g = h \). \textit{Hint: It is not} \( g(x) = x^2 \).

b. Suppose that \( g(x) = e^{3x+4} \) and \( h(x) = 3x + 4 \). Find a function \( f \) such that \( f \circ g = h \).

c. Suppose that \( f(x) = 2x + 5 \) and \( h(x) = 6x - 3 \). Find a function \( g \) such that \( f \circ g = h \).

d. Suppose that \( f(x) = x^2 - 5x + 4 \) and \( h(x) = x^4 - 2x^3 + x - 2 \). Find a function \( g \) such that \( f \circ g = h \).

e. Suppose that \( g(x) = 5x - 4 \) and \( h(x) = \frac{10x - 9}{15x + 15} \). Find a function \( f \) such that \( f \circ g = h \).

f. Let \( f(x) = x^2 \) and \( h(x) = \ln(x) \). Does there exist a function \( g \) such that \( f \circ g = h \)? Explain your answer.

(2.2.c2) Challenge Problem. Consider the following two piecewise functions:

\[
f(x) = \begin{cases} 
  x - 1 & \text{if } x < -1 \\
  2x + 2 & \text{if } -1 \leq x \leq 1 \\
  3 + x^2 & \text{if } x > 1 
\end{cases}
\]

\[
g(x) = \begin{cases} 
  \frac{1}{x} & \text{if } x < 0 \\
  \sqrt{x + 4} & \text{if } x \geq 0 
\end{cases}
\]

A. Find and simplify \((f \circ g)(x)\).

b. Find and simplify \((g \circ f)(x)\).

\textit{Hint: Both answers should be piecewise functions. Graphing} f \textit{and} g \textit{may be helpful for some people.}
2.3 INVERSE FUNCTIONS

We have discussed how functions work like machines. When we input a number into a function it outputs a different number according to some rule. In some cases, when given an output it is possible to determine the unique input which will produce that output. In these cases we can define a new function which is the same as the original function except with the inputs and outputs reversed. These new functions are called inverse functions.

Motivation

Suppose that Aaron invests $10,000 in an account on which interest is compounded continuously at an annual rate of 2.5%. From MATH 111 we know that \( t \) years after this investment is made, the account has a value of \( A(t) \) dollars where

\[
A(t) = 10000e^{0.025t}.
\]

That is, since

\[
A(5) = 10000e^{0.025(5)} = 11331.48
\]

it follows that after 5 years Aaron’s account is worth $11,331.48.

Now, suppose Aaron wants to know how long it will take for his investment to be worth $12,000. From a mathematical standpoint he wants to know a value of \( t \) such that \( A(t) = 12000 \). If \( t \) exists then

\[
12000 = 10000e^{0.025t}.
\]

In MATH 111 we also learned how to solve the above equation.

\[
12000 = 10000e^{0.025t} \\
1.2 = e^{0.025t} \\
\ln(1.2) = 0.025t \\
t = \frac{\ln(1.2)}{0.025} \\
t = 7.29
\]

Thus we see that the account will be worth $12,000 after 7.29 years.

We can actually perform the above calculation in a more general sense. Instead of considering a particular value of the investment (like $12,000), suppose we want to know how long it takes for the account to reach a value of \( a \) dollars. That is, given a value \( a \) we want to know a value of \( t \) such that \( a = A(t) \). Again, we are looking for a value of \( t \) such that

\[
a = 10000e^{0.025t}.
\]
Using the same technique we can solve this equation for $t$ in terms of $a$.

\[
\begin{align*}
a &= 10000e^{0.025t} \\
\frac{a}{1000} &= e^{0.025t} \\
\ln\left(\frac{a}{1000}\right) &= 0.025t \\
t &= 40\ln\left(\frac{a}{1000}\right)
\end{align*}
\]

The calculation in Section 2.3 tells us that in order for the account to reach a value of $a$ dollars the investment must last $t = 40\ln\left(\frac{a}{1000}\right)$ years. Hence we can define a new function, $T$, such that

\[T(a) = 40\ln\left(\frac{a}{1000}\right)\]

Note that the functions $A$ and $T$ relate the same quantities: the value of the investment and the length of the investment. However, the roles of these quantities are switched. The function $A$ inputs the length of the investment and outputs the value of the investment while the function $T$ inputs the value of the investment and outputs the amount of time required for the investment to reach that value. Figure 2.3.1 relates these quantities and functions.

Figure 2.3.1: A visual representation of the functions $A$ and $T$.

In this section we will come to call the functions $A$ and $T$ inverses of one another. This is a useful way of creating a new function from a known function. Now, consider a fixed value of $t$ years. By definition, $A(t)$ is the value of the investment after $t$ years. Since $A(t)$ is a value of the investment, it follows that $T(A(t))$ is the number of years required for the account to reach a value of $A(t)$ dollars. However, we already know that it takes $t$ years for the account to reach a value of $A(t)$ dollars. It stands to reason, then, that $T(A(t))$ should be equal to $t$. This is certainly something that we can check using the techniques developed in Section 2.2. Observe that

\[
\begin{align*}
T(A(t)) &= 40\ln\left(\frac{10000e^{0.025t}}{10000}\right) \\
&= 40\ln\left(e^{0.025t}\right) \\
&= 40(0.025t) \\
&= t
\end{align*}
\]
so that $T(A(t)) = t$ as expected. Similarly, $A(T(a))$ should be the value of the investment after $T(a)$ years which is, of course, $a$ dollars so we would expect that $A(T(a)) = a$ which turns out to be the case.

As we will see soon, it is not a coincidence that $A(T(a)) = a$ and that $T(A(t)) = t$. This is a defining property of inverse functions.

*Theory*

If $f$ is a function then its inverse should be a function which contains the same information as $f$ but the inputs and outputs should be switched. Using the machine analogy, the inverse of $f$ should be the same machine but running in the opposite direction. The inverse of $f$ will be a function which we will denote with $f^{-1}$ which will be defined (in Definition 2.3.2 below) such that $f(x) = y$ is equivalent to $f^{-1}(y) = x$. That is, if $x$ is an input for $f$ whose output is $y$ then $y$ is an input for $f^{-1}$ whose output is $x$. For instance, if $f(2) = 4$ then $f^{-1}(4) = 2$.

It happens that it is not always possible to make an inverse function. Remember that functions are required to have a unique output for each input. However, for arbitrary functions it is acceptable that an output corresponds to multiple inputs. That is, it is perfectly reasonable that $f(2) = 4$ and $f(3) = 4$ but it is forbidden for $f(4) = 3$ and $f(4) = 2$. We require that the inverse of a function must also be a function. Now, if $f$ is a function such that $f(2) = 4$ and $f(3) = 4$ then how should we define $f^{-1}(4)$? Since $f(2) = 4$ it must be that $f^{-1}(4) = 2$ and since $f(3) = 4$ it must also be that $f^{-1}(4) = 3$. But this is a problem as $f^{-1}$ should be a function.

The problem above arises because there are two inputs of $f$ that correspond to an output of 4. This shows that a function can only have an inverse when every output corresponds to a unique input. A great example of a function which does not have an inverse is the function $f(x) = x^2$. Since $f(2) = 4$ and $f(-2) = 4$ we cannot define $f^{-1}(4)$. Definition 2.3.2 below gives a precise definition of an inverse function and a criterion to determine if an inverse function exists.

(2.3.2) **Definition.** A function $f$ is called **invertible** if every output of $f$ corresponds to exactly one input which yields that output. More precisely, $f$ is invertible if whenever $y = f(x)$ there is no other value $t$ such that $y = f(t)$. In the case that $f$ is invertible there exists a function denoted $f^{-1}$ such that $f^{-1}(y) = x$ if and only if $f(x) = y$. The function $f^{-1}$ is called the **inverse** of $f$.

Consider the function $f(x) = 2x - 1$. For now, assume that this function is, in fact, invertible. We want to investigate the function $f^{-1}$.
Observe that
\[ f(2) = 2(2) - 1 = 3. \]

**Definition 2.3.2** stipulates that \( f^{-1}(y) = x \) if and only if \( f(x) = y \).
Since \( f(2) = 3 \) we have that \( f^{-1}(3) = 2 \). Similarly,
\[ f(-6) = 2(-6) - 1 = -13 \]
so that \( f^{-1}(-13) = -6 \). Suppose that we want to find \( f^{-1}(5) \). According to **Definition 2.3.2** we have that \( f^{-1}(5) = x \) if and only if \( f(x) = 5 \).
Thus we want a value of \( x \) such that \( f(x) = 5 \). We can use the definition of \( f \) to solve for \( x \) as follows:
\[
5 = f(x) \\
5 = 2x - 1 \\
6 = 2x \\
3 = x
\]
and hence \( f^{-1}(5) = 3 \). We can check our work by computing \( f(3) \). If \( f^{-1}(5) = 3 \) then we expect \( f(3) = 5 \) and we see that
\[ f(3) = 2(3) - 1 = 5, \]
as desired.

It is clear that the previous calculation can be performed to compute \( f^{-1}(y) \) for any value of \( y \). If we’re given a value of \( y \) and we want to figure out an \( x \)-value such that \( f^{-1}(y) = x \) then we set up the equation \( f(x) = y \) and solve it for \( x \) in terms of \( y \). For this particular function, we can do this for arbitrary values of \( y \) as follows:
\[
f(x) = y \\
2x - 1 = y \\
2x = y + 1 \\
x = \frac{1}{2}y + \frac{1}{2}
\]
Thus \( f^{-1}(y) = \frac{1}{2}y + \frac{1}{2} \). In fact, this is the general procedure for finding equations for the inverses of invertible functions.

**Example.** The following functions are invertible. Find an equation for their inverses.

A. \( f(x) = \frac{1}{3}x - \frac{5}{3} \)
b. \( g(x) = 3\ln(x - 1) \)

\[
\begin{align*}
\frac{1}{3}x - \frac{5}{3} &= y \\
\frac{1}{3}x &= y + \frac{5}{3} \\
x &= 3y + 5
\end{align*}
\]

It follows that \( g^{-1}(y) = 3y + 5 \). You can check this if you want by picking an \( x \)-value. For example, consider \( x = 11 \). Observe that \( f(11) = 2 \). If our equation for \( f^{-1}(y) \) is correct we should get that \( f^{-1}(2) = 11 \) and that is, indeed, the case.

**Answer:** \( f^{-1}(y) = 3y + 5 \)

b. In order to find an equation for \( g^{-1}(y) = x \) we solve the equation \( g(x) = y \) for \( x \).

\[
\begin{align*}
3\ln(x - 1) &= y \\
\ln(x - 1) &= \frac{y}{3} \\
x - 1 &= e^{y/3} \\
x &= 1 + e^{y/3}
\end{align*}
\]

It follows that \( g^{-1}(y) = 1 + e^{y/3} \).

**Answer:** \( g^{-1}(y) = 1 + e^{y/3} \)

c. In order to find an equation for \( h^{-1}(y) = x \) we solve the equation \( h(x) = y \) for \( x \).

\[
\begin{align*}
\frac{1}{2x + 5} &= y \\
1 &= y(2x + 5) \\
\frac{1}{y} &= 2x + 5 \\
\frac{1}{y} - 5 &= 2x \\
\frac{1}{2} \left( \frac{1}{y} - 5y \right) &= x \\
\frac{1 - 5y}{2y} &= x
\end{align*}
\]
It follows that \( h^{-1}(y) = \frac{1-5y}{2y} \).

**Answer:** \( h^{-1}(y) = \frac{1-5y}{2y} \)

### 2.3 Practice Exercise

The function \( f(x) = x^3 - 8 \) is invertible. Find a formula for \( f^{-1}(y) \).

As we saw earlier in the section, not all functions are invertible. The criterion developed in **Definition 2.3.2** for determining if a function is invertible is intuitive but there is a more practical way. We will now develop that method.

**Theorem 2.3.4** Suppose that \( f \) and \( g \) are functions such that

- \( (f \circ g)(x) = x \) for all \( x \) in the domain of \( g \) and
- \( (g \circ f)(x) = x \) for all \( x \) in the domain of \( f \).

Then \( f \) and \( g \) are both invertible with \( f = g^{-1} \) and \( g = f^{-1} \).

**Theorem 2.3.4** gives a couple of useful properties of inverse functions. First, if \( g \) is the inverse of \( f \) - that is if \( g = f^{-1} \) - then \( f \) is also the inverse of \( g \). This is straightforward. The function \( f^{-1} \) is the same as \( f \) but with the inputs and outputs swapped. The inverse of \( f^{-1} \) should again swap the inputs and outputs so the inverse of \( f^{-1} \) is \( f \). The second property in **Theorem 2.3.4** gives a useful way to check if two functions are inverses of one another.

### 2.3 Example

In each part below, two functions, \( f \) and \( g \), are given. For each pair of functions determine if \( f \) and \( g \) are inverses of one another.

**A.** \( f(x) = 3x - 6 \) and \( g(x) = \frac{1}{3}x + 2 \)

**B.** \( f(x) = \sqrt{x + 1} - 2 \) and \( g(x) = (x + 2)^3 - 1 \)

**C.** \( f(x) = 5 + e^{0.1x} \) and \( g(x) = 10 \ln(x + 5) \)

**D.** \( f(x) = \frac{x-1}{3x+2} \) and \( g(x) = \frac{-2x-1}{3x-1} \)
Solution:
For each of the parts of this problem we will check if \((f \circ g)(x) = x\) and \((g \circ f)(x) = x\). According to Theorem 2.3.4, if both of these conditions hold then \(f\) and \(g\) are inverses of one another. If either one of them fails, however, the functions are not inverses.

A. First observe that
\[
(f \circ g)(x) = f(g(x))
\]
\[
= 3 \left( \frac{1}{3}x + 2 \right) - 6
\]
\[
= x + 6 - 6
\]
\[
= x.
\]
Now observe that
\[
(g \circ f)(x) = g(f(x))
\]
\[
= \frac{1}{3}(3x - 6) + 2
\]
\[
= x - 2 + 2
\]
\[
= x.
\]
Since both \((f \circ g)(x) = x\) and \((g \circ f)(x) = x\) it follows that \(f\) and \(g\) are inverses of one another.

**Answer:** Yes, \(f\) and \(g\) are inverses of one another.

B. First observe that
\[
(f \circ g)(x) = f(g(x))
\]
\[
= \sqrt[3]{(x + 2)^3} - 1 + 1 - 2
\]
\[
= \sqrt[3]{(x + 2)^3} - 2
\]
\[
= x + 2 - 2
\]
\[
= x.
\]
Now observe that
\[
(g \circ f)(x) = g(f(x))
\]
\[
= \left( \sqrt[3]{x + 1} - 2 + 2 \right)^3 - 1
\]
\[
= \left( \sqrt[3]{x + 1} \right)^3 - 1
\]
\[
= x + 1 - 1
\]
\[
= x.
\]
Since both \((f \circ g)(x) = x\) and \((g \circ f)(x) = x\) it follows that \(f\) and \(g\) are inverses of one another.

**Answer:** Yes, \(f\) and \(g\) are inverses of one another.
c. First observe that
\[(f \circ g)(x) = f(g(x))\]
\[= 5 + e^{0.1(10 \ln(x+5))} = 5 + e^{\ln(x+5)} = 5 + x + 5 = x + 10.\]
Hence we see that \((f \circ g)(x) \neq x\) so \(f\) and \(g\) cannot be inverses of one another. There is no need to check if
\[(g \circ f)(x) = x\] since, in order for \(f\) and \(g\) to be inverses of one another, it must be the case that both \((f \circ g)(x) = x\) and \((g \circ f)(x) = x\).

[Answer:] No, \(f\) and \(g\) are not inverses of one another.

d. First observe that
\[(f \circ g)(x) = f(g(x))\]
\[= \frac{-2x - 1}{3x - 1} - 1 = \frac{3}{x-1} + 2 = \frac{-2x - 1}{3x - 1} - \frac{3x - 1}{3x - 1} + \frac{6x - 2}{3x - 1}\]
\[= \frac{-5x}{3x - 1}(\frac{3x - 1}{3x - 1}) = \frac{-5x}{3x - 1} = x.\]
Now observe that
\[(g \circ f)(x) = g(f(x))\]
\[= \frac{-2}{3} \left( \frac{x + 1}{3x + 2} - 1 \right) = \frac{-2}{3} \left( \frac{x + 1}{3x + 2} - 1 \right) = \frac{-2x + 2}{3x + 2} - \frac{3x + 2}{3x + 2} - \frac{3x + 2}{3x + 2}
\[= \frac{-5x}{3x + 2}\]
\[= \frac{-5x}{3x + 2} \left( \frac{3x + 2}{-5} \right) = \frac{-5x}{3x + 2} = x.\]
Since both \((f \circ g)(x) = x\) and \((g \circ f)(x) = x\) it follows that \(f\) and \(g\) are inverses of one another.

[Answer:] Yes, \(f\) and \(g\) are inverses of one another.
Questions like the ones in Example 2.3.5 can be tricky. For example, consider the function \( f(x) = x^2 \) and \( g(x) = \sqrt{x} \). To start with, we know that these two functions cannot be inverses of each other. We know this because we know that \( f \) is not invertible. However, checking to see if they’re inverses of each other can be misleading. First of all, 

\[
(f \circ g)(x) = f(g(x)) = (\sqrt{x})^2 = x
\]

so that \( (f \circ g)(x) = x \). On the other hand, 

\[
(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|.
\]

The last equality in the above equation can often be missed. Whenever \( x \) is positive we have that \( \sqrt{x^2} = x \) which is what students often expect for all values of \( x \). However if \( x \) is negative then \( x^2 \) is positive and \( \sqrt{x^2} \) is still positive so \( \sqrt{x^2} = -x \). Take, for example, \( x = -3 \):

\[
\sqrt{(-3)^2} = \sqrt{9} = 3 = -(-3).
\]

These things combined yield that \( \sqrt{x^2} = |x| \). Because \( (g \circ f)(x) \neq x \) it cannot be the case that \( f \) and \( g \) are inverses of each other even though \( (f \circ g)(x) = x \). This shows that in order to prove that two functions are inverses of each other, one must check the composition in both directions.

We can finally address the question of invertibility. That is, we can now determine if a function is invertible. Consider the function \( f(x) = 3x + 4 \). If it were invertible then we could find its inverse. To do so we would solve \( f(x) = y \) for \( x \) (as in Example 2.3.3) which would yield

\[
3x + 4 = y \\
3x = y - 4 \\
x = \frac{1}{3}y - \frac{4}{3}.
\]

It follows that if \( f \) is invertible then \( f^{-1}(y) = \frac{1}{3}y - \frac{4}{3} \). Define \( g(x) = \frac{1}{3}x - \frac{2}{3} \) (this function is defined to be equal to the possible inverse that we just found). We know that in the case that \( f \) is invertible, the only possible inverse it could have is \( g \). That means that the only way \( f \) can be invertible is if \( f \) and \( g \) are inverses of each other and we know how to check for that. Observe that

\[
(f \circ g)(x) = 3\left(\frac{1}{3}x - \frac{4}{3}\right) + 4 = x - 4 + 4 = x
\]

and that

\[
(g \circ f)(x) = \frac{1}{3}(3x + 4) - \frac{4}{3} = x + \frac{4}{3} - \frac{4}{3} = x
\]
so that \( f \) and \( g \) are inverses of one another. Hence \( f \) is invertible and
\[
\begin{align*}
f^{-1}(y) &= \frac{1}{3}y - \frac{4}{3}.
\end{align*}
\]
The above procedure is the one that we will use to determine if a function, \( f \), is invertible. We first assume that \( f \) is invertible and try to solve the equation \( y = f(x) \) for \( x \). We then come up with a possible inverse and check if it is actually the inverse using Theorem 2.3.4.

(2.3.6) Example. Check if the following functions are invertible. If so, find their inverse.

A. \( f(x) = 6x + 4 \)
B. \( g(x) = \frac{1}{x+1} \)
C. \( h(x) = 3 - |x| \)

Solution:

A. We first solve the equation \( y = f(x) \) for \( x \) which will provide us with a potential inverse function.

\[
\begin{align*}
y &= 6x + 4 \\
y - 4 &= 6x \\
\frac{1}{6}y - \frac{2}{3} &= x
\end{align*}
\]
The last line in the above equation suggests that if \( f \) has an inverse then \( f^{-1}(y) = \frac{1}{6}y - \frac{2}{3} \). Since we don’t know yet if \( f \) is invertible we define \( p(x) = \frac{1}{6}y - \frac{2}{3} \). If \( f \) has an inverse then it must be \( p \) so we now check if \( f \) and \( p \) are inverses as we did in Example 2.3.5.

\[
\begin{align*}
(f \circ p)(x) &= 6 \left( \frac{1}{6}x - \frac{2}{3} \right) + 4 = x - 4 + 4 = x \\
(p \circ f)(x) &= \frac{1}{6}(6x + 4) - \frac{2}{3} = x + \frac{2}{3} - \frac{2}{3} = x
\end{align*}
\]

It follows that \( f \) and \( p \) are inverses and hence \( f^{-1} = p \) so that \( f^{-1}(y) = \frac{1}{6}y - \frac{2}{3} \).

**Answer:** Yes, \( f \) is invertible and \( f^{-1}(y) = \frac{1}{6}y - \frac{2}{3} \).

B. We first solve the equation \( y = g(x) \) for \( x \).

\[
\begin{align*}
y &= \frac{1}{x+1} \\
y(x+1) &= 1 \\
x + 1 &= \frac{1}{y} \\
x &= \frac{1}{y} - 1
\end{align*}
\]
It follows that if \( g \) has an inverse it must be \( p(x) = \frac{1}{x} - 1 \).

\[
(g \circ p)(x) = \frac{1}{\left(\frac{1}{x} - 1\right) + 1} = \frac{1}{x} = x
\]

\[
(p \circ g)(x) = \frac{1}{x + 1} - 1 = x + 1 - 1 = x
\]

It follows that \( g \) and \( p \) are inverses and hence \( g^{-1} = p \).

We then simplify to find

\[
g^{-1}(y) = \frac{1}{y} - 1 = \frac{1 - y}{y}
\]

**Answer:** Yes, \( g \) is invertible and \( g^{-1}(y) = \frac{1 - y}{y} \).

c. We first try to solve the equation \( y = h(x) \).

\[
y = 3 - |x|
\]

\[
3 - y = |x|
\]

Unfortunately, we get stuck; we cannot isolate the \( x \) because of the absolute value bars. Consider \( y = 1 \). Then 
\( 2 = |x| \) and \( x = 2 \) or \( x = -2 \). Hence we cannot solve for \( x \) in terms of \( y \) and \( h \) is not invertible.

**Answer:** No, \( h \) is not invertible.

When following the procedure shown in Example 2.3.6 there are two things that can happen which will signify that the function is not invertible. First, if we try to solve the equation \( y = f(x) \) for \( x \) and it proves impossible then \( f \) is not invertible. Second, if we come up with a possible inverse, \( g \), and it turns out that either \((f \circ g)(x) \neq x\) or \((g \circ f)(x) \neq x\) then \( f \) is not invertible.

We now look at the graphs of inverse functions. Let \( f \) be an invertible function. The graph of \( y = f(x) \) consists of points in the \( xy \)-plane. Consider a point \((a, b)\) on this graph. Then, obviously, \( b = f(a) \). It follows, of course, that \( a = f^{-1}(b) \) so that the point \((b, a)\) is on the graph of \( y = f^{-1}(x) \). This shows that every point \((a, b)\) on the graph of \( y = f(x) \) corresponds to a point \((b, a)\) on the graph of \( y = f^{-1}(x) \).

Consider the function \( y = f(x) \) in Figure 2.3.7. There are four points shown on the graph: \((-3, -4.5)\), \((0, -1)\), \((1, 2)\), and \((3, 3)\). It follows that the points \((-4.5, -3)\), \((-1, 0)\), \((2, 1)\), and \((3, 3)\) are on the graph of \( y = f^{-1}(x) \). If we reverse the coordinates of every point on the graph of \( y = f(x) \) in this manner we obtain the graph of \( y = f^{-1}(x) \), which is also shown in Figure 2.3.7.
If given the graph of an invertible function, the way that we graph its inverse is by a reflection about the line $y = x$ as was done in Figure 2.3.7. However, how can we know that a function is invertible just by its graph? Well, in order for a function to be invertible it must be the case that every output corresponds to one input. On a graph, the requirement that every output corresponds to one input is equivalent to the requirement that for every $y$-value, the graph of the function contains (at most) one point with that $y$-value. Theorem 2.3.8 formalizes this idea.

(2.3.8) **Theorem (Horizontal Line Test).** A function $f$ is invertible if and only if the graph of $y = f(x)$ intersects every horizontal line at most once.

There are two ways to interpret Theorem 2.3.8. From a theoretical standpoint it says that a function is invertible if the line $y = k$ and the graph $y = f(x)$ intersect in at most one place for every value of $k$, or that the equation $k = f(x)$ has at most one solution for each value of $k$, which should sound familiar. From a graphical standpoint, the line $y = k$ contains every point of the form $(x, k)$ and the theorem says that, in order for $f$ to be invertible, the graph of $y = f(x)$ can only contain one of those points for each $k$. Theorem 2.3.8 is straightforward to apply in practice.
Example. The graphs of four functions are shown below. First, decide which of them are invertible. Then, for each that is invertible, sketch its inverse.

\[ y = f(x) \quad \text{for} \quad a. \]
\[ y = g(x) \quad \text{for} \quad b. \]
\[ y = p(x) \quad \text{for} \quad c. \]
\[ y = q(x) \quad \text{for} \quad d. \]

Solution:

A. The function \( f \) is invertible by the Horizontal Line Test as any horizontal line passes through the graph of \( y = f(x) \) in at most one place. To draw the graph of \( y = f^{-1}(x) \) we simply reflect the graph about the line \( y = x \) (which is drawn for reference) as shown below.
50 NEW FUNCTIONS FROM OLD FUNCTIONS

ANSWER: Yes, \( f \) is invertible and its inverse is shown above.

b. The function \( g \) is not invertible. There are many horizontal lines which intersect the graph of \( y = g(x) \) in two places - one of which is shown on the graph below - hence \( g \) cannot be invertible by the Horizontal Line Test.

ANSWER: No, \( g \) is not invertible.

c. This one looks tricky as it is in two pieces, but every horizontal line that you can draw will pass through the graph of \( y = p(x) \) in only one place and hence \( p \) is invertible by the Horizontal Line Test. Again, to draw the inverse of \( p \) we reflect \( p \) about the line \( y = x \).
The last theoretical point to mention in this section is the idea of restricting a domain. It has already been discussed that the function $f(x) = x^2$ is not an invertible function. This, again, becomes clear when we look at its graph shown in Figure 2.3.10.

Now define a new function, $g$, such that $g(x) = x^2$ for values of $x$ in the interval $[0, \infty)$. That is, $g$ is the same as $f$ but it is defined on a smaller domain. It turns out that $g$ is an invertible function. Suppose we try to solve $y = g(x)$ for $x$ in terms of $y$. We get $y = x^2$ and $x = \sqrt{y}$. When we attempted to do the same thing with $f(x) = x^2$ we came to a problem where had $x = \pm \sqrt{y}$ but the only valid $x$-values
for our function $g$ are positive so in this case we only end up with
the positive value of $\sqrt{y}$. The inverse of $g$ is then $g^{-1}(y) = \sqrt{y}$. This
becomes more clear from the graph of $y = g(x)$ in Figure 2.3.11

It is often the case that a function is not invertible but it becomes
invertible when we only consider a portion of the domain. This idea
of restricting a function to some “reasonable” domain in order to
force invertibility becomes rather important in later sections.

Applications

Oftentimes in an application one can gather some sort of intuition as
to whether a function should turn out to be invertible.

Consider, for example, a stone which is dropped from a height
of 200 feet. Let $h$ be a function such that $t$ seconds after the stone is
dropped it is at a height of $h(t)$ feet. It seems reasonable that $h$ should
be invertible. Outputs of $h$ corresponds to the height of the stone and
the stone is always falling. Until the stone hits the ground it is only
at each height for one value of $t$. Now consider a function, $N$, such
that if $s$ is a University of Oregon student’s social security number
then $N(s)$ is their student ID number. We certainly would expect $N$
to be invertible. If you choose an output - a student ID number - one
would hope that it would correspond to exactly one student’s social
security number.
Consider a function, $D$, such that in year $y$ the Oregon Duck’s football team had $D(y)$ wins. This function is not invertible. For example, the Ducks had 12 wins in both 2010 and 2011 so $D(2010) = 12$ and $D(2011) = 12$. Since two inputs correspond to an output of 12, $D$ cannot be invertible. Suppose you’re watching a basketball game and keeping track of your team’s score so that $t$ minutes after tipoff they have $P(t)$ points. There will be stretches of time where their score stays the same and all of those values of $t$ have the same output so $P$ is not going to be invertible.

Once you’ve realized that an application is asking for an inverse function, finding the inverse is straightforward using the techniques developed earlier in this section.

(2.3.12) Example. John goes to a party and, after a night of drinking, his blood alcohol concentration (BAC) is 0.1 when he stops drinking. Then $t$ hours after he stops drinking his BAC is given by $B(t)$ where

$$B(t) = \frac{1}{5t + 10}.$$  

The function $B$ is not based on any facts; it was invented for this problem.

A. John needs to go home but he doesn’t want to drive until his BAC is 0.02 or lower. How long does he need to wait (from the time that he stops drinking) before he can drive?

B. Find a function $T$ such that John’s BAC will be at a level of $b$ if he waits $T(b)$ hours after he stops drinking.

Solution:

A. Since John wants his BAC to be 0.02 we need to find a value of $t$ such that $B(t) = 0.02$.

$$0.02 = B(t)$$

$$0.02 = \frac{1}{5t + 10}$$

$$5t + 10 = 50$$

$$5t = 40$$

$$t = 8$$

It follows that $B(8) = 0.02$ so that it will take 8 hours for John’s BAC to be 0.02.

Answer: 8 hours
b. We have a function, $B$, which inputs time and outputs BAC and we want a function, $T$, which inputs BAC and outputs time.

\[
\begin{array}{c}
\text{Time} \quad \xrightarrow{\text{B}} \quad \text{BAC} \\
\end{array}
\]

It follows that $T$ should be the inverse of $B$. To find $B^{-1}$ we first solve the equation $b = B(t)$ for $t$.

\[
b = B(t) \]
\[
b = \frac{1}{5t + 10} \]
\[
b(5t + 10) = 1 \]
\[
5t + 10 = \frac{1}{b} \]
\[
5t = \frac{1 - 10b}{b} \]
\[
t = \frac{1 - 10b}{5b} \]

Thus $B^{-1}(b) = \frac{1 - 10b}{5b}$ and $T(b) = \frac{1 - 10b}{5b}$. In an application like this it is not necessary to use composition to ensure that $B$ is invertible. Since you were asked to find an inverse function it is safe to assume that the inverse exists; you will not be asked to find nonexistent functions.

\[\text{Answer: } T(b) = \frac{1 - 10b}{5b}\]

At times an application requires a restricted domain without mention of any domain. Consider the height function, $h$, described earlier. That is, if a stone is dropped from a height of 200 feet then $t$ seconds later its height is given by $h(t)$. It does not make sense for the domain of $h$ to be all real numbers. For instance, when $t$ is negative $h(t)$ has no meaning. We explore this more in Example 2.3.13.

(2.3.13) Example. A stone is dropped from a height of 200 feet. After $t$ seconds its height is $h(t)$ feet where

\[h(t) = 200 - 16t^2.\]

a. How long will it take for the stone to hit the ground?
b. Find and simplify a function $T$ such that the ball is $y$ feet off the ground $T(y)$ seconds after it was dropped.

**Solution:**

A. How high is the ball when it hits the ground? It is $0$ feet off of the ground when it hits the ground. Hence we want a value of $t$ such that $h(t) = 0$.

\[
0 = h(t) \\
0 = 200 - 16t^2 \\
200 = 16t^2 \\
12.5 = t^2 \\
t = \pm 3.54
\]

The values of $t$ such that $h(t) = 0$ are $t = 3.54$ and $t = -3.54$. However, a negative value of $t$ doesn’t make any sense in this application since $h$ only knows the height of the stone after it was dropped. Hence it takes approximately $3.54$ seconds for the ball to hit the ground.

**Answer:** $3.54$ seconds

b. The function $T$ should be the inverse of $h$. We can see that because $h$ inputs time and outputs height whereas $T$ should input height and output time. To find the inverse of $h$ we solve $y = h(t)$ for $t$.

\[
y = h(t) \\
y = 200 - 16t^2 \\
y - 200 = 16t^2 \\
\frac{y - 200}{16} = t^2 \\
\pm \sqrt{\frac{y - 200}{16}} = t \\
\pm \frac{1}{4} \sqrt{y - 200} = t
\]

It seems, at this point, that we have a problem since there are two values of $t$. However, as mentioned before, only positive values of $t$ work in this application so we only use the positive value of $t$ above. Hence $h^{-1}(y) = \frac{1}{4} \sqrt{y - 200}$ and $T(y) = \frac{1}{4} \sqrt{y - 200}$.

**Answer:** $T(y) = \frac{1}{4} \sqrt{y - 200}$
(2.3.c) Practice Exercise. Temperature in Kelvin is given by $T(f) \text{ where}$

$$T(f) = \frac{5}{9} f + 255.372$$

and $f$ is the temperature in Fahrenheit. Find a function $F$ such that temperature in Fahrenheit is given by $F(k)$ where $k$ is the temperature in Kelvin.
2.3.1. Consider \( f \) and \( g \) in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>0</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>7</td>
<td>6</td>
<td>2</td>
<td>7</td>
<td>9</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

A.) Is \( f \) invertible?

B.) Is \( g \) invertible?

2.3.2. The functions \( p \) and \( q \) are defined such that

\[
p(x) = x^2 + 4x \quad \text{and} \quad q(x) = 6x + 1.
\]

A.) Find and simplify \((p \circ q)(x)\).

B.) Find and simplify \((pq)(x)\).

C.) The function \( q \) is invertible. Find its inverse.

D.) Explain briefly why \( p \) is not invertible.

2.3.3. The following functions are invertible. Find their inverses:

A.) \( f(x) = -\frac{2}{3}x - 6 \)

B.) \( r(x) = \frac{1}{3}x - 1 \)

C.) \( t(x) = -\frac{4}{3}x + 3 \)

D.) \( w(x) = 3x - 1 \)

E.) \( g(x) = x^3 + 4 \)

F.) \( h(x) = \frac{2x+3}{x+1} \)

G.) \( p(x) = 1 - 2e^x \)

H.) \( q(x) = \ln(x - 3) \)

I.) \( s(x) = 3 + \sqrt{x - 2} \)

J.) \( v(x) = 1 - \sqrt{7x + 4} \)

K.) \( z(x) = 3 + 3\log(x + 5) \)

2.3.4. Sketch the inverse of the function \( y = f(x) \) graphed below:

2.3.5. A function \( y = h(x) \) is graphed below. Draw the graph of \( h^{-1} \).

2.3.6. Is the function \( f(x) = \frac{2}{2x+3} \) invertible? If so, find its inverse.

2.3.7. Is the function \( g(x) = 3x - 1 \) invertible? If so, find its inverse.

2.3.8. Is the function \( h(x) = |2x + 3| - 1 \) invertible? If so, find its inverse.

2.3.9. Is the function \( p(x) = 2\log(x) + 3 \) invertible? If so, find its inverse.

2.3.10. Are the following functions invertible? If so, find their inverses.

A.) \( f(x) = 3x + 7 \)

B.) \( g(x) = 1 - \frac{3}{x} \)

C.) \( h(x) = x^2 - 4 \)
D.) \( p(x) = \frac{2}{3}x - \frac{1}{7} \)
E.) \( q(x) = \frac{5 - 3x}{2x + 1} \)

f.) The function \( y = r(x) \) is graphed below. You may draw the inverse if it is invertible.

\[
\begin{array}{c}
\text{y = r(x)} \\
\end{array}
\]

2.3.11. True or False: The functions \( f(x) = 1 + 7x^3 \) and \( g(x) = \sqrt[3]{\frac{x - 1}{7}} \) inverses of each other.

2.3.12. True or False: The functions \( f(x) = 1 - \frac{1}{x - 1} \) and \( g(x) = 1 + \frac{1}{x - 1} \) inverses of each other.

2.3.13. For each of the following pairs of functions, decide whether or not they are inverses for one another.

A.) \( f(x) = x^3 - 2 \) and \( g(x) = \sqrt[3]{x - 2} \)
B.) \( f(x) = \frac{x^4 + 4}{2x - 5} \) and \( g(x) = \frac{4 + 5x}{2x - 1} \)
C.) \( f(x) = \sqrt[3]{2x - 4} + 3 \) and \( g(x) = \frac{1}{2}(x - 3)^3 + 2 \)
D.) \( f(x) = 1 + 4e^{3x} \) and \( g(x) = \frac{1}{3\ln(\frac{x - 1}{4})} \)

2.3.14. Find any two functions \( f \) and \( g \) such that \( (f \circ g)(x) = x \) for all \( x \) but there exists a number \( t \) such that \( (g \circ f)(t) \neq t \).

2.3.15. Hint: The two parts of this question have different answers. Why is that?

A.) Are the functions \( f(x) = 3 + \sqrt{x - 1} \) and \( g(x) = (x - 3)^3 + 1 \) inverses of one another?
B.) Are the functions \( f(x) = 3 + \sqrt{x - 1} \) and \( g(x) = (x - 3)^2 + 1 \) inverses of one another?

2.3.16. The graphs of two functions, \( y = f(x) \) and \( y = g(x) \), are shown below.

\[
\begin{array}{c}
\text{y = f(x)} \\
\text{y = g(x)} \\
\end{array}
\]

A.) Is \( f \) an invertible function?
B.) Is \( g \) an invertible function?
C.) Sketch the graph of \( y = (f + g)(x) \).
D.) Is \( f + g \) an invertible function?

2.3.17. The airspeed velocity of a European swallow is proportional to its heart rate. That is, if a European swallow’s heart rate is \( h \) bpm then its airspeed is \( A(h) = 0.01h \) m/s.

A.) The average heart beat of a European swallow is 900 bpm. What is the airspeed velocity of such a swallow?
B.) If a European swallow flies at 11.5 m/s, find its heart rate.
2.3.18. Cindy works at an hourly job where her pay is determined by a function \( P \). If Cindy works an average of \( t \) hours a week over the course of a year then she makes \( P(t) \) dollars in that year where
\[
P(t) = 600t + 800.
\]
Additionally, the amount that Cindy puts into savings depends on the amount of money that she makes in a year according to the function \( S \). That is, if she makes \( d \) dollars in a year then she will put \( S(d) \) dollars into her savings account that year where
\[
S(d) = \frac{3d - 8000}{20}.
\]
A.) She wants to know how many hours a week that she needs to work in order to make a given amount of money. Find a function \( f \) such that if she wants to make \( d \) dollars in a year then the average number of hours she needs to work per week during that year is \( f(d) \).

b.) Cindy’s mom is worried about her and wants to know how much she will save depending on how many hours she works. Find a function \( g \) such that if she works an average of \( t \) hours a week over the course of a year then she put \( g(t) \) dollars into savings in that year.

c.) Cindy also wants to know how many hours a week that she needs to work in order to save a given amount of money. Find a function \( h \) such that if she wants to save \( m \) dollars in a year then the average number of hours she needs to work per week during that year is \( h(m) \).

---

**Challenge Problems**

(2.3.c1) **Challenge Problem.** Consider the following two piecewise functions:

\[
f(x) = \begin{cases} 
\frac{1}{2}x + 1 & \text{if } x < 0 \\
2x - 1 & \text{if } x \geq 0
\end{cases} \quad g(x) = \begin{cases} 
3x & \text{if } x \leq -1 \\
\frac{1}{2}x + \frac{1}{2} & \text{if } -1 < x \leq 1 \\
x^2 & \text{if } x > 1
\end{cases}
\]

A. Is \( f \) an invertible function? If so, find its inverse. If not, explain why not.

B. Is \( g \) an invertible function? If so, find its inverse. If not, explain why not.

*Hint: Graphing \( f \) and \( g \) may be helpful for some people.*

(2.3.c2) **Challenge Problem.** In this exercise we investigate what is sometimes called the **socks and shoes property**, which is a property that relates function inversion to function composition.

A. Let \( f(x) = 4x - 7 \) and let \( g(x) = \frac{1}{x-1} \). Note that these are both invertible functions.

(i) Find \( f^{-1}(x) \) and \( g^{-1}(x) \).
(ii) Find \((f \circ g)(x)\) and \((g \circ f)(x)\).

(iii) Find \((f^{-1} \circ g^{-1})(x)\) and \((g^{-1} \circ f^{-1})(x)\).

(iv) Find \((f \circ g)^{-1}(x)\) and \((g \circ f)^{-1}(x)\). You can assume that these functions are invertible.

What do you notice? At this point, it should be the case that

\[
(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x) \quad \text{and} \quad (g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x).
\]

b. Let \(f\) and \(g\) be any arbitrary invertible functions. (Do not worry about their domains; assume that their domains are appropriately compatible.)

(i) Explain why the function \(f \circ g\) is invertible using the definition of invertibility. For a fixed number, \(y\), how do you know that there is at most one solution to \((f \circ g)(x) = y\)?

(ii) It is the case, in general, that \((f \circ g)^{-1} = g^{-1} \circ f^{-1}\). Explain why this is the case. There are two ways to do this: (1) The first way is to show that \(g^{-1} \circ f^{-1}\) is the inverse of \(f \circ g\). You know that \((f \circ g)^{-1}\) is the inverse of \(f \circ g\) by definition and you also know that inverses are unique. You can then show that \(g^{-1} \circ f^{-1}\) and \(f \circ g\) are inverses of one another (there is a theorem which tells you how to show that) which tells you (since inverses are unique) that \(g^{-1} \circ f^{-1}\) must equal \((f \circ g)^{-1}\). (2) You can also simply solve for the inverse of \(f \circ g\) and describe it as a composition function. Using either of these two methods is sufficient.
Periodic functions are, roughly speaking, functions which repeat along the x-axis. They are an important class of functions that will be used extensively in later chapters. Periodic functions can be difficult to analyze, but by looking at these functions only on a small interval and realizing that they repeat, we can often use techniques that we already understand to find information about them.

Motivation

Our motivating example requires a rudimentary understanding of angles. These don’t get introduced until Section 3.1 but it is likely that students have a basic understanding of how degrees are used to measure angles. If not, then either skip the motivating example or read a little bit in Section 3.1 about angles.

Suppose someone is sitting on a Ferris wheel. Let’s say that its center is 100 ft off the ground, its radius is 80 ft, and it rotates 1° every second. When this ride starts the rider is sitting level with the center and is moving upward. Define a function $h$ such that $h(t)$ is the rider’s height off the ground in feet $t$ seconds after the ride starts. This situation is shown in Figure 2.4.1.

Let’s explore some of the outputs of this function. It is easy to see that $h(0) = 100$ since the rider starts level with the center. After 90 seconds the Ferris wheel has rotated 90° so the rider is at the top of the Ferris wheel and $h(90) = 180$. As we go around the circle, $h(180) = 100$ and $h(270) = 20$. After 360 seconds, however, the carriage is back to where it started and $h(360) = 100$.

If the ride continues, the second 360 seconds will look exactly like the first 360 seconds as the carriage takes the same path. Figure 2.4.2 shows a graph of some of the values of $h$. This pattern will then continue indefinitely as long as the ride continues. Filling in the graph
Figure 2.4.2: Some of the points on the graph of $y = h(t)$ shown in green.

shown in Figure 2.4.2 is complicated and we will cover that later. However, if we can fill in the graph of $h$ on the interval $[0, 360]$ then certainly we can fill in the rest of the graph since the function is repetitive.

The function $h$ demonstrated above is an example of what will be called a periodic function; periodic functions are functions that repeat on a regular interval. As you can probably imagine, these types of functions are extremely common.

**Theory**

We now need to be precise with the definition of a periodic function.

(2.4.3) **Definition.** Let $f$ be a non-constant function. We say that $f$ is a **periodic** function if there exists a positive number $p$ such that $f(x + p) = f(p)$ for all $x$ in the domain of $f$. The **period** of $f$ is the smallest positive value of $p$ such that $f(x + p) = f(x)$ for all $x$ in the domain of $f$.

Figure 2.4.4 shows a portion of the graph of a periodic function $f$ with period 2. The portion in red is a section of the graph on an interval whose width is one period.

We notice that the entire graph is made by horizontally translating the red section of the graph in either direction. However, we ought to check that this function satisfies the definition of a periodic function. **Definition 2.4.3** states that $f$ is periodic with period 2 if $f(x) = f(x + 2)$ whenever $x$ is in the domain of $f$. This means that the function $g$ defined by $g(x) = f(x + 2)$ is the same as the function $f$. Recall that the graph of $g$ is the graph of $f$ shifted to the left by 2 units. Hence saying that “$f$ is periodic with period 2” is the same as saying that “when we shift the graph of $f$ to the left by 2 units we get the same graph as before the shift.” We can see from Figure 2.4.4 that shifting
the graph of \( f \) to the left by 2 units results in the same exact graph which is why \( f \) is periodic.

We note that it seems, at first glance, that the function \( f \) in Figure 2.4.4 could be of period 4. After all, if you shift the graph of \( f \) to the left by 4 units you will obtain the same graph. However, Definition 2.4.3 requires that the period of \( f \) be the smallest number \( p \) such that a horizontal shift by \( p \) does not change the graph of \( f \). Thus the period of \( f \) cannot be \( p = 4 \) since \( p = 2 \) works just fine.

\[(2.4.5) \text{ Example.} \text{ The graphs of four functions are shown below. Which are periodic functions?}

A. The graph of \( y = f(x) \).

\[(0.75,1)\]  
\[(-1.25,1)\]

B. The graph of \( y = g(x) \).

C. The graph of \( y = p(x) \).

D. The graph of \( y = q(x) \).
Solution:

A. The function $f$ is a periodic function. If you shift the graph of $f$ to the left by 1 unit then you get the same exact graph. It follows that $f$ is periodic with period 1.

**Answer:** $f$ is periodic.

B. The function $g$ is not a periodic function. You can see that if you shift the graph of $g$ to the left by anything it can’t be the same graph. This function looks like it could be periodic because it is visually repetitive, but it is not.

**Answer:** $g$ is not periodic.

C. The function $p$ is not a periodic function. Imagine any horizontal shift of the graph of $p$; certainly it will be different than the original graph.

**Answer:** $p$ is not periodic.

D. Functions with gaps in their domain can be tricky. However, be sure to note that periodic functions only require that $q(x) = q(x + p)$ when $x$ is in the domain of $q$. Thus domain gaps don’t preclude a function from being periodic. Notice that if the graph of $q$ is shifted to the left by 1 unit then the graph is unchanged, hence $q$ is periodic with period 1.

**Answer:** $q$ is periodic.

There is one thing about which we need to be careful when deciding if functions are periodic from their graphs. Consider the graph of $y = f(x)$ shown in Figure 2.4.6. Whenever we graph a function whose domain is all real numbers we must truncate the graph somewhere and the choice of where to truncate the graph can make a function look periodic even if it is not periodic. In Figure 2.4.6 we see two different views of the same function. When we look at $f$ on the interval $[-2.5, 2.5]$ then $f$ looks periodic but when we look at $f$ on the interval $[-7, 7]$ it looks like $f$ is not periodic. In fact it is impossible to tell without any knowledge of the rest of the function.

It is important to understand that this can be a tricky issue, but you shouldn’t worry out about it while reading these notes. When you are asked to use the graph of a function to determine whether or not it is periodic, you can assume that the graph continues in either direction as expected.
Figure 2.4.6: A function that looks periodic in one view but does not look periodic in another view.

Our approach to this point has been largely graphical. Example 2.4.7 shows that periodic functions can be expressed without a graph. It is often very difficult to express a periodic function with an equation. However, in order to define a periodic function it is sufficient to define it on one period. That is, in the event that $f$ is a periodic function with period $p$ and we can use an equation to express $f$ on some interval of the form $[a, a + p)$ then that single period defines the rest of the function. Such a function is described in Example 2.4.7.

(2.4.7) Example. A function $f$ is periodic with period 5. Additionally, whenever $x \in [-2, 3)$ we have

$$f(x) = -x^2 - 2x + 3 \quad (2.4.8)$$

A. Find $f(1)$.

B. Find $f(-6)$.

C. Find $f(5)$.

D. Find all of the roots of $f$.

Solution:

A. Since $1 \in [-2, 3)$ we can use Equation 2.4.8 to calculate $f(1) = -(1)^2 - 2(1) + 3 = 0$.

Answer: $f(1) = 0$

B. It is tempting to use Equation 2.4.8 to calculate $f(-6)$ but this can’t be done as $-6 \notin [-2, 3)$. However, since $f$ is periodic with period 5 we have that $f(x) = f(x + 5)$ for any value of $x$. If we let $x$ be $-6$ then $f(-6) = f(-1)$. This is useful because $-1 \in [-2, 3)$ which means we can use Equation 2.4.8 to calculate $f(-1)$ and hence $f(-1) = -(-1)^2 - 2(-1) + 3 = 4$. It follows that $f(-6) = f(-1) = 4$. 
Answer: \( f(-6) = 4 \)

c. Again, we cannot use Equation 2.4.8 since \( 5 \notin [-2, 3) \). However, \( f(0) = f(0+5) = f(5) \) since \( f \) is periodic of period 5. Since \( 0 \in [-2, 3) \) we can use Equation 2.4.8 to calculate \( f(0) = 3 \) and hence \( f(5) = f(0) = 3 \).

Answer: \( f(5) = 3 \)

d. To find all of the roots of \( f \) we first find the roots which lie in \([-2, 3) \). On that interval \( f \) is given by Equation 2.4.8. By setting \( f(x) = 0 \) on that interval and factoring we get

\[
0 = -x^2 - 2x + 3 \\
0 = -(x + 3)(x - 1)
\]

and hence \( x = -3 \) or \( x = 1 \). But Equation 2.4.8 is only valid on \([-2, 3) \) and \(-3 \notin [-2, 3) \) which means that \((1, 0)\) is a root but \((-3, 0)\) is not a root. Thus the only root in the interval \([-2, 3) \) is the point \((1, 0)\).

Now \( f \) has period 5 which means that the rest of the graph is obtained by shifting the graph on \([-2, 3) \) to the left and the right indefinitely. Hence all of the roots of \( f \) are just shifts of \((1, 0)\) by multiples of 5 in the \( x \)-direction. That is, \((1+5, 0)\) is a root, \((1+2\cdot5, 0)\) is a root, and so on. Finally, all of the roots of \( f \) are of the form \((1+5n, 0)\) for integers \( n \).

Answer: \((1+5n, 0)\) for integers \( n \).

In order to clarify, the graph of the function \( y = f(x) \) is shown below on the interval \([-10, 10] \).

Example 2.4.7 states that the roots of this function are points \((1+5n, 0)\) for integers, \( n \). Remember that integers refers to both positive and negative whole numbers.
There is an important point in Example 2.4.7 that needs to be reinforced. Reconsider the function \( f \) defined in Figure 2.4.4. Suppose we want to solve the equation \( f(x) = k \) for some real number \( k \). This is the same as asking the \( x \)-values of the points at which the graphs of \( y = k \) and \( y = f(x) \) intersect. Consider specifically the solutions to \( f(x) = 0.5 \). The graphs of \( y = f(x) \) and \( y = 0.5 \) are both shown in Figure 2.4.9. You can see that these two lines do not intersect at all so \( f(x) = 0.5 \) has no solutions.

![Figure 2.4.9: The function \( y = f(x) \) in Figure 2.4.4 and the horizontal lines \( y = 0.5 \) and \( y = 1.5 \).](image)

Now consider the solutions to \( f(x) = 1.5 \). As you can see in Figure 2.4.9, this equation is much more interesting. The solutions are the \( x \)-values of the points at which \( y = f(x) \) and \( y = 1.5 \) intersect. In the figure, we see that there are plenty of these. In fact, there are an infinite number of them. It seems like a daunting task to find them all, but luckily we can handle it precisely because \( f \) is periodic.

Look at any one period of \( f \). That is, consider any interval of the form \([a, a + 2)\) for a real number \( a \). For convenience, we’re going to use \( a = 0 \) and look at \([0, 2)\) since that was the period we were examining earlier. In Figure 2.4.9 we see that the only two solutions to \( f(x) = 1.5 \) on \([0, 2)\) are \( x = 0.19 \) and \( x = 1.81 \). Now remember that the entire graph of \( y = f(x) \) is determined by the graph of \( y = f(x) \) on the interval \([0, 2)\). It follows that \( f(0.19 + 2n) = 1.5 \) for every integer, \( n \), and that \( f(1.81 + 2n) = 1.5 \) for every integer, \( n \). Furthermore, these can be the only solutions because otherwise there would be more on \([0, 2)\). We’ve now determined that all of the values of \( x \) such that \( f(x) = 1.5 \) are

\[
x = 0.19 + 2n \quad \text{and} \quad x = 1.81 + 2n \quad \text{for integers, } n.
\]

You can see in Figure 2.4.9 that every solution to \( f(x) = 1.5 \) is a multiple of 2 units away from one of the two that are in \([0, 2)\). That shift of \( 2n \) for some integer, \( n \), is what accounts for the \( 2n \) in our list of solutions.

We can generalize this procedure to any periodic function. Let \( f \) be some periodic function with period \( p \). Suppose we want to solve
f(x) = k for some real number k. Then there are either zero solutions or an infinite number of solutions. If there are an infinite number of solutions and we know all of the solutions on a single period, then all of the solutions are obtained by adding \( pn \) to those solutions for integers, \( n \).

(2.4.10) **Example.** Let \( f \) be a periodic function with period 10 such that

\[
f(x) = x^3 - x + 5
\]

whenever \(-5 \leq x < 5\). Find all of the values of \( x \) such that \( f(x) = 5 \).

**Solution:**

We first need to find all of the solutions on a single period of \( f \). This will be easiest if we look at the period \([-5, 5]\) since we know the definition of the function on that period. If \( x \) is a value in \([-5, 5]\) such that \( f(x) = 5 \) then we know that \( f(x) = x^3 - x + 5 \) so

\[
5 = x^3 - x + 5 \\
0 = x^3 - x \\
0 = x(x+1)(x-1)
\]

and hence \( x = -1, x = 0, \) or \( x = 1 \). These account for all of the solutions to \( f(x) = 5 \) on the interval \([-5, 5]\). Since this interval accounts for one whole period of \( f \) and \( f \) has a period of 10, we know that all of the solutions to \( f(x) = 5 \) can be found by adding \( 10n \) to the ones that we already have for integers, \( n \). Thus all the solutions are

\[
-1 + 10n, \quad 10n, \quad \text{and} \quad 1 + 10n \quad \text{for integers, } n.
\]

**Answer:** \(-1 + 10n, 10n, \) and \( 1 + 10n \) for integers, \( n \).

The answer in Example 2.4.10 was, unfortunately, an infinitely long list of numbers. We certainly cannot list all of these numbers, so we are forced to include a variable, \( n \), in our answer. There simply is no better way to write this list of numbers.

(2.4.a) **Practice Exercise.** The function \( Q \) is periodic with period 30 and

\[
Q(t) = 100e^{-0.01t^2}
\]

for all values of \( t \) where \(-15 \leq t \leq 15\).
(i) Find \( Q(5) \).

(ii) Find \( Q(-50) \).

(iii) Find \( Q(25) \).

(iv) Find all real numbers \( t \) such that \( Q(t) = 10 \).

Round all answers to two decimal places.

We finish the theory by introducing two definitions.

(2.4.11) Definition. Let \( f \) be a periodic function. If \( M \) is the maximum of \( f \) and \( m \) is the minimum of \( f \) then

- the **midline** of \( f \) is the horizontal line defined by \( y = \frac{1}{2}(M + m) \) and
- the **amplitude** of \( f \) is \( \frac{1}{2}(M - m) \).

From a graphical standpoint, the midline of a function is the horizontal line which is half way between the top and bottom of the graph of the function. The amplitude of the function is then the vertical distance between the midline and the maximum of the function.

Let’s revisit the example with which we started the section. Figure 2.4.12 shows the graph of the function \( y = f(x) \) that is also shown in Figure 2.4.4. Recall that the function \( f \) is periodic with period 2.

![Figure 2.4.12](image)

Figure 2.4.12: A periodic function \( y = f(x) \).

We see that the maximum value of \( f \) is 3 and that the minimum value of \( f \) is 1. According to the definition we then have that the midline of \( f \) is the line \( y = \frac{1}{2}(3 + 1) \) or \( y = 2 \) and the amplitude is \( \frac{1}{2}(3 - 1) = 1 \). The line \( y = 2 \) is graphed in Figure 2.4.12. From the graph we see that the midline is the horizontal line which is half way between the maximum and the minimum of the graph. Notice that
the midline is both 1 unit from the maximum and 1 unit from the minimum and that the amplitude is precisely 1.

(2.4.8) **Practice Exercise.** A periodic function \( y = T(x) \) is graphed below. Find the period, amplitude, and midline of \( T \).

![Graph of \( y = T(x) \)](image)

Applications

To this point we have only defined periodic functions and have not yet addressed any mathematical questions. Hence we can only look at some examples of periodic functions.

(2.4.13) **Example.** An aircraft carrier in the pacific ocean has been observing a lighthouse on the coast. The spotlight in the lighthouse has a luminosity of 1,500,000 lm (the unit lm stands for lumens). However, since the spotlight is spinning its luminosity appears to fluctuate from the carrier’s point of view. The carrier finds that \( t \) seconds into a day the lighthouse’s brightness appears to be \( B(t) \) percent of its maximum brightness. Since the spotlight spins at a constant rate, \( B \) is a periodic function. On one period of \( B \) – say the interval \([0, p]\) where \( p \) is the period – the graph of \( y = B(t) \) looks like the following:
The value of $p$ depends on the speed at which the spotlight is spinning. When $t = p$ we have that the spotlight is facing directly away from the carrier and when $t = \frac{p}{2}$ we have that the spotlight is facing directly toward the carrier.

(2.4.14) Example. A cardiologist finds a function $f$ which describes the velocity of blood flow in a certain artery when a patient’s heart rate is 60 bpm. The cardiologist turns on the monitoring equipment and $t$ seconds later the blood flow is $f(t)$ cm/s. The velocity of the blood peaks shortly after every heart beat so the function $f$ is periodic with period 1. The graph of $y = f(t)$ on $[0,2)$ is shown below:

The maximum value of $f$ is 200 and the minimum value of $f$ is 50. It follows that the midline is the line $y = \frac{(200 + 50)}{2}$ or $y = 125$ and the amplitude of $f$ is $\frac{(200 - 50)}{2} = 75$. We see that the midline is vertically centered on the velocity function (although it is not exactly the average) and that the amplitude describes how far away from the midline the velocity function can get.
(2.4.15) Example. This example describes a function $f$ which is important in theoretical mathematics. For each real number $x$ let $n$ be the largest integer which is smaller than or equal to $x$. For example if $x = 5.6$ then $n = 5$. Then define $f$ such that $f(x) = x - n$. This function is not as complicated as it sounds. For example, $f(5.6) = 5.6 - 5 = 0.6$, $f(1.7) = 0.7$, and $f(11/5) = 1/5$. We sometimes call $f(x)$ the fractional part of $x$. Then $f$ is periodic with period 1. The graph of $y = f(x)$ on $[0,4)$ is shown below:

We will see many more examples of periodic functions as the text progresses.
2.4.1. True or False: The graph below is the graph of a periodic function.

2.4.2. True or False: The graph below is the graph of a periodic function.

2.4.3. True or False: The graph below is the graph of a periodic function.

2.4.4. True or False: The graph below is the graph of a periodic function.

2.4.5. True or False: The graph below is the graph of a periodic function.

2.4.6. True or False: The graph below is the graph of a periodic function.

2.4.7. True or False: The graph below is the graph of a periodic function.
2.4.8. True or False: The graph below is the graph of a periodic function.

2.4.13. A function $g$ is periodic of period 100. Whenever $0 < x \leq 100$ we have that $g(x) = \log(x)$.
   a.) Find $g(0)$.
   b.) Find $g(200)$.
   c.) Find $g(-10)$.
   d.) Find all of the zeros of $g$.

2.4.14. Find the midline, amplitude, and period of the function $y = f(x)$ graphed below.

2.4.15. Below is the graph of a periodic function, $y = f(x)$.
   a.) What is the midline of $f$?
   b.) What is the amplitude of $f$?
   c.) What is the period of $f$? 
      Hint: Look at points that lie on the midline.
Adding periodic functions is a curious business. Let \( f \) and \( g \) be two periodic functions with period, \( p \). If \( x \) is any value in the domain of \( f + g \) then

\[
(f + g)(x + p) = f(x + p) + g(x + p) = f(x) + g(x) = (f + g)(x)
\]

since both \( f \) and \( g \) have period \( p \). We have just shown that

\[
(f + g)(x + p) = (f + g)(x)
\]

for all values of \( x \) in the domain of \( f + g \) which makes it seem like \( f + g \) is periodic with period \( p \). Not so fast. It is actually possible to add two periodic functions and get a function which is not periodic. For example, if it happens that \( g = -f \) then \( f + g \) is a constant function which is not periodic. However, if \( f + g \) is not a constant function then it is always periodic. It is not the case, though, that \( f + g \) needs to have a period of \( p \). The graphs below show two periodic functions, \( f \) and \( g \), with period 2 whose sum has period 1.

If \( f \) and \( g \) are periodic functions with a period of \( p \) and \( f + g \) is periodic function (that is, not a constant function) with period \( q \) then we know two things: \( q \) is less than \( p \) and \( \frac{p}{q} \) is an integer. This is all very complicated and is, quite frankly, an unnecessary pain. It is usually good enough to know that if \( f \) and \( g \) are periodic functions with a period of \( p \) then \( p \) is the largest possible period of \( f + g \).

A. Let \( f \) be a periodic function with a period of 1 and let \( g \) be a periodic function with a period of 2. What is the largest possible period of \( f + g \)?

B. Let \( f \) be a periodic function with a period of 2 and let \( g \) be a periodic function with a period of 3. What is the largest possible period of \( f + g \)?
c. Let $f$ be a periodic function with a period of 4 and let $g$ be a periodic function with a period of 6. What is the largest possible period of $f + g$?

D. Let $f$ be a periodic function with a period of $p$ and let $g$ be a periodic function with a period of $q$. If $p$ and $q$ are both integers, what is the largest possible period of $f + g$? There is a mathematical term for this but it is enough to describe it in words.

E. It is not the case that the sum of two periodic functions must either be a periodic function or a constant function. That is, if $f$ and $g$ are periodic then $f + g$ might be neither periodic nor constant. Find an example of such a thing; find a periodic function $f$ and a periodic function $g$ such that $f + g$ is neither periodic nor constant. Hint: There are a lot of answers to this question. The functions that you use for $f$ and $g$ do not need to be terribly complicated.
Trigonometry is the area of geometry that is concerned with the study of angles and triangles. In this chapter we introduce the foundations of trigonometry and some of its introductory applications. With the exception of Section 3.1 this chapter covers the content of Chapter 7 in *College Algebra and Trigonometry for the University of Oregon*, by Connally et al. [4]
3.1 BASIC GEOMETRY

The material does not correspond to a section in Functions Modeling Change [4].

This section does not correspond to a section in Connally et al.’s text. Most of this material is meant to be a rigorous introduction to the basic terminology of trigonometry. It is suggested that students read this section but it is provided mostly as a reference for the rest of the chapter.

Definitions

The most common way to measure the position of something is by using a distance. The height of a building might be measured in feet, the diameter of a car tire might be measured in inches, and the distance between two cities might be measured in miles. However, in order to describe how to aim a model rocket, the incline of a road, or perhaps the way that a Ferris wheel spins, distance is not enough; for these things we need to use angles.

At this point it is likely that you have a good, intuitive understanding of an angle. Mathematics, however, demands precision in its definitions. It takes a couple of steps to define angles and how to measure them.

(3.1.1) Definition. If $p$ is a point in the plane, a ray emanating from $p$ is a segment of a straight line which starts at $p$ and goes off indefinitely in one direction.

![Figure 3.1.2: Three examples of a ray.](image)

Figure 3.1.2 shows some examples of rays. Note that the rays shown in Figure 3.1.2 are drawn as arrows but a ray actually extends off indefinitely in the direction that the arrow points. Unfortunately there is no better way to draw a ray. This is similar to the way we often draw lines except that lines go off indefinitely in both directions.
(3.1.3) **Definition.** An **angle** is the geometric object that is formed by two rays emanating from the same point. Specifically an angle is the rotation about the point required to move one of the rays to the other. The point from which the rays emanate is called the **vertex** of the angle and the rays are called the **legs** of the angle.

It is important to be careful when using rays to indicate an angle. **Figure 3.1.4a** shows two rays emanating from the same point which forms an angle. However, those two rays actually form two different angles which are shown in **Figure 3.1.4b** and **Figure 3.1.4c**. Sometimes a small arc of a circle will be used to indicate which angle is being considered when there is confusion. Other times, when the angle is labeled, we simply put the label where the arc would go as with angles A and B in **Figure 3.1.7b**.

![Figure 3.1.4: Different angles can look the same.](image)

(3.1.5) **Definition.** We define a **right angle** to be the smaller angle formed by two perpendicular rays.

(3.1.6) **Definition.** Two angles are said to be **adjacent** if they share a vertex and a leg.

![Figure 3.1.7: Examples of angle types.](image)

**Figure 3.1.7a** shows an example of a right angle while the angles A and B in **Figure 3.1.7b** will be used to indicate right angles.
and B in Figure 3.1.7b demonstrate two adjacent angles. In order to use and analyze angles we need to be able to measure them.

The unit most commonly used to measure angles is the degree. Definition 3.1.9 gives the precise definition below but when thinking about measuring angles in degrees, it is easier to think about a protractor. A protractor is generally a circular, transparent piece of plastic with measurement markings on it and it is used to measure angles which are made at the vertex of two rays.

Figure 3.1.8 shows a protractor measuring an angle A which measures 58 degrees; we write this as $58^\circ$. This is done by first placing the center of the protractor on the vertex of the angle and then by lining up the $0^\circ$ marking with one of the legs. We then look at where the other leg falls on the protractor and that tells us the degree measure of the angle. In Figure 3.1.8 we see that the second ray falls under the $58^\circ$ marking which tells us that the angle is $58^\circ$.

![Figure 3.1.8: A protractor measuring a 58° angle.](image)

As mentioned before, mathematics requires precision. Below is a precise definition of degree measure.

(3.1.9) **Definition.** A **degree** is a unit of measure for angles. An angle which corresponds to no rotation at all has a degree measure of zero which we write $0^\circ$ and a right angle has a degree measure of 90 which we write as $90^\circ$. We then define all other angle measures linearly from those two measurements.

The last sentence in the definition of degrees simply means that you can **add** angle measures as you might expect. If something first rotates by $35^\circ$ and then rotates by $25^\circ$ it seems natural that the total rotation would be an angle of $60^\circ$ and this is indeed the case. More precisely, two adjacent angles form a larger angle and the degree measure of the larger angle is obtained by adding the degree measures of the smaller angles; this is shown in Figure 3.1.10. You shouldn’t think too
hard about this definition; there is nothing deep or complicated about degree measure.

![Figure 3.1.10: How angles add.](image)

**Figure 3.1.10** shows a few examples of angles and their degree measures. Definition 3.1.3 stipulates that the legs of an angle should be rays but using line segments instead of rays will suffice. We now proceed to develop some language with which we can refer to some important types of angles.

(3.1.11) **Definition.**

- An angle whose measure is larger than 0° but smaller that 90° is called an **ACUTE ANGLE**.
- An angle whose measure is larger than 90° but smaller than 180° is called an **OBSTUSE ANGLE**.
- An angle whose measure is larger than 180° but smaller than 360° is called a **REFLEX ANGLE**.
- Two angles are called **COMPLEMENTARY** when their measures sum to 90°.
- Two angles are called **SUPPLEMENTARY** when their measures sum to 180°.

**Figure 3.1.12c** gives an example of a 60° angle which, by definition, is an acute angle and **Figure 3.1.12d** gives an example of a 135° angle which is an obtuse angle. In each of those examples the angle that is not indicated - a 300° angle in **Figure 3.1.12c** and a 225° angle in **Figure 3.1.12d** - is a reflex angle. **Figure 3.1.12a** is an example of two complementary angles and **Figure 3.1.12b** shows an example of supplementary angles.

Note that, when drawn adjacent as in **Figure 3.1.12b**, two supplementary angles together make a straight line. When two rays emanating from a vertex form a 180° angle they look like a straight line. As a rotation we think of an angle of 180° as **turning to face the opposite direction**. Hence an angle of 360° is two 180° angles in a row so that, as a rotation, 360° is **turning around until you’re back where you started**.
Later in the chapter we will make sense of angles larger than $360^\circ$ but for now we assume that all angles are between $0^\circ$ and $360^\circ$.

Example 3.1.14 uses the letter $\theta$. This is one of 24 letters in the greek alphabet. Greek letters are often used to denote angles. The entire greek alphabet is shown in Table 3.1.13.

<table>
<thead>
<tr>
<th>LETTER</th>
<th>NAME</th>
<th>LETTER</th>
<th>NAME</th>
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<tbody>
<tr>
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<td>alpha</td>
<td>I</td>
<td>iota</td>
<td>P</td>
<td>rho</td>
</tr>
<tr>
<td>B</td>
<td>beta</td>
<td>K</td>
<td>kappa</td>
<td>Sigma</td>
<td>sigma</td>
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<tr>
<td>Eta</td>
<td>Omicron</td>
<td>Psi</td>
<td>Omega</td>
<td>OMEGA</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1.13: The Greek alphabet.

Solution (3.1.14) Example. Find the angle $\theta$ below.
Solution:
As drawn, the two angles in the diagram are supplementary since they form a straight line when drawn with a shared leg. That means that $135^\circ + \theta = 180^\circ$ so that $\theta = 45^\circ$.

\textbf{Answer:} $\theta = 45^\circ$

Before we move on from angles there is one last type of example that we need to explore. These are simple calculations but it is important that we are proficient in them as they are crucial throughout the rest of the notes.

(3.1.15) \textbf{Example.} Find each of the six angles shown below. Assume that angles which look like right angles are, in fact, right angles.

\begin{itemize}
\item $\theta$: From the diagram we see that $90^\circ + 60^\circ = \theta$ so that $\theta = 150^\circ$.
\textbf{Answer:} $\theta = 150^\circ$
\item $\phi$: From the diagram we see that $180^\circ + 45^\circ = \phi$ so that $\phi = 225^\circ$.
\textbf{Answer:} $\phi = 225^\circ$
\item $\gamma$: From the diagram we see that $\gamma + 70^\circ = 360^\circ$ so that $\gamma = 290^\circ$.
\end{itemize}
\textbf{Answer: }\gamma = 290^\circ

$\delta$: From the diagram we see that $180^\circ + \delta = 235^\circ$ so that $\delta = 55^\circ$.

\textbf{Answer: }\delta = 55^\circ

$\alpha$: From the diagram we see that $310^\circ + \alpha = 360^\circ$ so that $\alpha = 50^\circ$.

\textbf{Answer: }\alpha = 50^\circ

$\beta$: From the diagram we see that $140^\circ + \beta = 180^\circ$ so that $\beta = 40^\circ$.

\textbf{Answer: }\beta = 40^\circ

We now introduce triangles which, along with angles, will form the basis of the study of trigonometry. Note that \textbf{Definition 3.1.16} introduces a lot of terminology but in reality the definition of a triangle is exactly what one expects it to be.

\begin{center}
\textbf{(3.1.16) Definition.} A \textbf{triangle} is a polygon with three sides, i.e. a shape which is made up of three straight lines. Every two of these sides intersect in one place which we call a \textbf{vertex}. At each vertex, the smaller of the two angles made by the intersection of the sides is called an \textbf{interior angle} of the triangle. For each vertex (or interior angle), the \textbf{opposite side} to that vertex (or interior angle) is the side which does not intersect at that vertex.

- A triangle is called an \textbf{obtuse triangle} if one of its interior angles is an obtuse angle.

- A triangle is called an \textbf{acute triangle} if all of its interior angles are acute.

- A triangle is called a \textbf{right triangle} if one of its interior angles is a right angle. In a right triangle, the opposite side to the right angle is called its \textbf{hypotenuse} and the other two sides are called its \textbf{legs}.

- A triangle is called an \textbf{isosceles triangle} if two of its angles have the same measure.

- A triangle is called an \textbf{equilateral triangle} if all of its angles have the same measure.
\end{center}
Figure 3.1.17 demonstrates some of the types of triangles in Definition 3.1.16.

(a) A right triangle.  (b) An acute triangle.  (c) An obtuse triangle.

Figure 3.1.17: Examples of different types of triangles.

The last thing we are going to introduce at this point is the unit circle. The unit circle introduced in Definition 3.1.18 will provide us with a valuable tool in the study of angles.

(3.1.18) Definition. The unit circle is the circle of radius 1 centered at the origin of the xy-plane.

We are interested in the unit circle because it allows us to do calculations with angles. Inscribed angles are the devices that are used for those calculations. Figure 3.1.20 demonstrates the terms in Definition 3.1.19. The inscribed angles are shown in red and the inscription points are shown in blue.

(3.1.19) Definition. An angle is called an inscribed angle if its vertex is at the center of the unit circle, one of its legs is along the positive x-axis, and the other leg is oriented counterclockwise from the positive x-axis. The leg which is not along the positive x-axis is called the terminal leg and the point at which the terminal leg intersects the unit circle is called the inscription point.

Figure 3.1.20: Some examples of inscribed angles (in red) according to Definition 3.1.19. Inscription points are shown in blue.

Normally angles are drawn with two rays emanating from a vertex as in Figure 3.1.20 but when drawing inscribed angles we will make a
few adjustments. We only need to draw the terminal leg as the other leg is assumed to be along the x-axis and the terminal leg is drawn as a line segment ending in the inscription point. Also, when drawing angles we often use a small arc of a circle to indicate the angle of interest. To indicate that we are actually using an inscribed angle we attach a small arrow to the end of the arc. These differences are shown in Figure 3.1.21.

![Figure 3.1.21: The same inscribed angles as in Figure 3.1.20 but with the appropriate decoration.](image)

Inscribed angles, by definition, are drawn in the unit circle. However, we will be working with other angles that are drawn in the unit circle and it is important to be able to recognize the difference between those that are inscribed angles and those that are not. Remember that according to Definition 3.1.19 an inscribed angle must have one leg along the positive x-axis and the other oriented counterclockwise from the positive x-axis.

![Figure 3.1.22: Various angles in the unit circle. Inscribed angles are in red and angles which are not inscribed angles are in black.](image)

Figure 3.1.22 shows some inscribed angles and some angles that are not inscribed angles. In Figure 3.1.22a the 60° angle is not an inscribed angle because neither of its legs are along the positive x-axis. In Figure 3.1.22b the 135° angle is not an inscribed angle because while one of its legs is along the positive x-axis, the second leg is oriented clockwise from the positive x-axis.

Not all angles drawn in the unit circle are inscribed angles but given any angle we can draw it in the unit circle as an inscribed angle.
if desired. This could be done by using a protractor to draw an angle whose vertex is at the center of the unit circle and whose rotation is the desired degree measure oriented counterclockwise from the positive x-axis. Figure 3.1.23 shows a 58° inscribed angle. Compare this with Figure 3.1.8.

![Figure 3.1.23: A 58° inscribed angle.](image)

### Early Results

At this point we have defined a large portion of the terminology required to talk about triangles and angles. We now turn to finding some basic relationships in triangles.

**Lemma.** The internal angles of a triangle sum to 180°.

**Example.** Find θ below

![Example triangle](image)

**Solution:**
Let Λ be the angle supplementary to θ; that is, the third interior angle in the triangle. Then, by Lemma 3.1.24, we have that

$$\Lambda + 63° + 71° = 180°$$

and hence $\Lambda = 46°$. Then, since $\theta$ and $\Lambda$ are supplementary, we have that

$$180° = \Lambda + \theta = 46° + \theta.$$

It follows that $\theta = 134°$.

**Answer:** $\theta = 134°$

The result in Lemma 3.1.24 is simple but it is quite lucrative. Con-
Consider a right triangle whose interior angles measure $90^\circ$, $B$, and $C$. Then Lemma 3.1.24 tells us that $90^\circ + A + B = 180^\circ$ so that $A + B = 90^\circ$. Since no angles in a triangle can be $0^\circ$ it must be that both $A$ and $B$ are smaller than $90^\circ$. This means that in a right triangle the two angles which are not right angles must be acute.

An obtuse triangle has one angle which is larger than $90^\circ$ which means that if the other two angles are $A$ and $B$ then $A + B < 90^\circ$ so that both $A$ and $B$ are acute angles. You will note that in the definition of obtuse triangles we were careful to say that a triangle is obtuse if one of its angles are obtuse. That definition is motivated by the fact that if one angle in a triangle is obtuse then the other two must be acute.

**Lemma 3.1.26.** Consider a triangle and fix a given side; call it the base. The height of the triangle relative to that base is the shortest distance between the line containing the base and the vertex opposite to it. Then the area of the triangle is $\frac{1}{2}bh$ where $b$ is the length of the base and $h$ is the height of the triangle.

Lemma 3.1.26 allows us to calculate area based on the choice of a base. However, there are three sides so which base do we choose? The answer is that it doesn’t matter. Any choice of a base and a corresponding altitude will give the same area calculation. The next consideration is then how to find the height. Once a base has been chosen, find the vertex opposite the base. To find the shortest distance from the vertex to the line containing the base, draw a line from that vertex which is perpendicular to the line containing the base; it might not be inside of the triangle. The length of the line is then the height.

Figure 3.1.27 shows the three choices of base and height. Each side is a different color and the corresponding height is drawn in the same color.
(3.1.28) **Example.** A farmer is planting crops in a square area of field which is 200 m wide on each side. She wants to dedicate a triangular section of this area for corn. One side of the triangle will be one entire side of the square. She plans on putting the vertex opposite that side somewhere along the side of the square which is parallel to the first side of the triangle. This layout is shown below.

![Diagram of a square garden with a triangular section dedicated to corn.](image)

The large square is the entire garden and the yellow triangle is the section dedicated to corn. What is the area of the section to be dedicated to corn?

**Solution:**
At first it seems as though not enough information is given in the problem since the precise location of one of the vertices is not given. However, it turns out that it doesn’t matter. We designate the base of the triangle as the side which is shared with the square. From a theoretical standpoint it doesn’t matter which side is chosen as the base but oftentimes a clever choice will make the problem significantly easier. The length of the base is 200 m so all that’s left to do is to find the height. Call the vertex which is not on the base \( p \). Then the height is the shortest distance from \( p \) to the base which is certainly 200 m regardless of where \( p \) is placed along the top edge of the square. It follows that the area of the triangle is given by

\[
\frac{1}{2} bh = \frac{1}{2} (200)(200) = 20000.
\]

Thus the area of the triangle is 20,000 m\(^2\).  

**Answer:** 20,000 m\(^2\)

**Lemma 3.1.26** was introduced because it is required in the proof of **Theorem 3.1.29** and because we will use area calculations in a few examples. The Pythagorean Theorem below, however, is essential to our study of trigonometry. It is a beautiful theorem whose proof is included because of its elegance. Students can skip the proof if they’d like, but it is very interesting.
(3.1.29) **Theorem** (*The Pythagorean Theorem*). In a right triangle whose legs have length \(a\) and \(b\) and whose hypotenuse has length \(c\),
\[
a^2 + b^2 = c^2. \tag{3.1.30}
\]

**Proof.** Consider an arbitrary right triangle as shown below.

I now wish to arrange four copies of the above triangle into a square.

Now consider the angle \(\theta\) as drawn above. Since the edges of the outer-most square are straight lines,
\[
\theta + A + B = 180^\circ \tag{3.1.31}
\]
But then, by considering one of the individual triangles, we have that
\[
A + B = 90^\circ \tag{3.1.32}
\]
by Lemma 3.1.24. Combining Equation 3.1.31 and Equation 3.1.32 we have that \(\theta = 90^\circ\) so that the blue shape is a square.

Let \(S\) be the area of the outermost square. On one hand the area of a square is the square of one of its sides so
\[
S = (a + b)^2 = a^2 + 2ab + b^2. \tag{3.1.33}
\]
On the other hand we can find \(S\) by adding the areas of the shapes inside it. The outermost square consists of four yellow triangles which have area \(\frac{1}{2}ab\) by Lemma 3.1.26 and the blue square which has area \(c^2\). Therefore
\[
S = 4 \left( \frac{1}{2}ab \right) + c^2 = 2ab + c^2. \tag{3.1.34}
\]
By combining Equation 3.1.33 and Equation 3.1.34 we have that
\[
a^2 + 2ab + b^2 = 2ab + c^2. \tag{3.1.35}
\]
After subtracting $2ab$ from each side of Equation 3.1.35 we have Equation 3.1.30.

The Pythagorean Theorem is a beautiful and somewhat surprising result. There are records indicating that the Pythagorean Theorem was known to the Chinese in the 500s BC, the Indians in the 800s BC, the Babylonians in the 1000s BC, and even by the Sumarians as early as the 2000s BC. However, mathematical legend suggest that it was not proven until Pythagoras in the late 500s BC. Unfortunately Pythagoras refused to allow any of his teachings and results to be recorded in writing so we can never know for sure, but it is suspected that the proof Pythagoras produced was very geometrical in nature, much like the one provided above.

Example 3.1.36 mentions in the solution that the triangle involved is a right triangle. It is important that the Pythagorean Theorem is only applied to right triangles. Notice, also, that all three sides of the right triangle in Example 3.1.36 have integer lengths. Any set of three integers which satisfy Equation 3.1.30 are called Pythagorean Triples.

Example 3.1.37 Example. Find $b$ below.
Solution:
Since the triangle is a right triangle we can apply Theorem 3.1.29. Thus $2^2 + b^2 = 4^2$ and $b^2 = 12$. Finally $b = \pm \sqrt{12} = \pm 2\sqrt{3}$ so $b = 2\sqrt{3}$ since $b$ is the length of a line.

**Answer:** $b = 2\sqrt{3}$

(3.1.A) Practice Exercise. Find the area of the outermost triangle shown below. Hint: You will need the Pythagorean Theorem. What should your base and height be?

Lemma 3.1.38 and Lemma 3.1.40 are presented solely for reference; no examples will be done with them and there will be no exercises using them. They are technical results that will be useful in passing later in the chapter.

(3.1.38) Lemma. In any triangle, the largest side is opposite the largest angle and the smallest side is opposite the smallest angle.

(3.1.39) Definition. Two triangles are called **similar** if their internal angles are the same.
Lemma 3.1.40. Suppose two triangles are similar. If one of the triangles has side lengths $a$, $b$, and $c$ with $a \leq b \leq c$ and the other has side lengths $x$, $y$, and $z$ with $x \leq y \leq z$ then

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}.$$ 

We conclude our results in this section with Lemma 3.1.41. This is the result that will allow us to use inscribed angles to study trigonometry. It is recommended that students read the proof of this lemma as it is an interesting application of the Pythagorean Theorem.

Lemma 3.1.41. A point $(x, y)$ is on the unit circle if and only if

$$x^2 + y^2 = 1.$$ \hspace{1cm} (3.1.42)

That is, the graph of Equation 3.1.42 is the unit circle.

Proof. Pick an arbitrary point $(x, y)$ on the unit circle like the one shown in Figure 3.1.43.

![Figure 3.1.43: A point on the unit circle.](image)

Examine the triangle highlighted in yellow. Since the point is supposed to be arbitrary it may be that $x$ and $y$ are either positive or negative but, in either case, the lengths of the legs are $|x|$ and $|y|$. Of course, since the circle is a unit circle it has radius 1 and hence the hypotenuse of the triangle is length 1. By the Pythagorean Theorem it then follows that

$$|x|^2 + |y|^2 = 1^2$$

but, of course, $|x|^2 = x^2$ and $|y|^2 = y^2$ so

$$x^2 + y^2 = 1,$$
as desired.

In the proof of Lemma 3.1.41 we develop a triangle from the point that was chosen. We will come to call this triangle a reference triangle and is important to realize that the point \((x, y)\) is a point whose coordinates can be either positive or negative but that the legs of the triangle must have positive lengths.

(3.1.44) Example. The line \(y = 2x + 1\) and the unit circle intersect at two points. Find them.

Solution:

This problem can be solved without the use of a graph but a picture is always illustrative. The line \(y = 2x + 1\) and the unit circle are shown below. Their points of intersection are shown in red.

From a visual inspection it looks as if one of the intersection points is the point \((0, 1)\) but the other is more difficult to determine. We can find them both algebraically. Suppose the point \((x, y)\) is one of the intersection points. Then it satisfies both

\[
x^2 + y^2 = 1 \quad \text{and} \quad y = 2x + 1.
\]

By plugging the second equation into the first we have that

\[
x^2 + (2x + 1)^2 = 1
\]
\[
x^2 + 4x^2 + 4x + 1 = 1
\]
\[
5x^2 + 4x = 0
\]
\[
x(5x + 4) = 0
\]

and hence we have that either \(x = 0\) or \(x = -0.8\). These are the \(x\)-coordinates of the two points. In order to find the \(y\)-coordinates we need only plug the \(x\)-coordinates into the equation \(y = 2x + 1\). When \(x = 0\) we have

\[
y = 2x + 1 = 2(0) + 1 = 1
\]
and hence \((0,1)\) is one of the two intersection points. That is the point that we already expected to find. When \(x = -0.8\) we have

\[
y = 2x + 1 = 2(-0.8) + 1 = -1.6 + 1 = -0.6
\]

and hence \((-0.8,-0.6)\) is the second of the two intersection points.

**Answer:** \((0,1)\) and \((-0.8,-0.6)\)

**(3.1.b) Practice Exercise.** The line \(y = \frac{1}{2}x - \frac{2}{3}\) and the unit circle intersect at two points. Find them.

---

**Angles in Practice**

We discussed the use of protractors to measure angles. A protractor is a convenient tool to measure or draw angles on a piece of paper. In industry, however, every profession has its own tool for measuring angles.

A carpenter is working on a house and wants to know the dimensions of the trusses used to hold up the roof. Trusses have the shape of an isosceles triangle whose base is the width of the house. The carpenter uses a tool called a pitch finder to find that the pitch of the roof is at an angle of \(32^\circ\) and the floor plans tell him that the house is 125 ft wide. These dimensions are shown in Figure 3.1.45. How do we find the dimensions of the truss?

![Figure 3.1.45: The dimensions of a truss on a house.](image)

A surveyor is standing 60 ft away from a building and uses a theodolite to determine that his line of sight with the top of the building makes an angle \(65^\circ\) with the ground as shown in Figure 3.1.46. How tall is the building?

Two pilots are planning flights to two different cities from the same airport. They use a navigational plotter to determine that their flight paths are \(60^\circ\) apart. One of the planes flies 140 mi before landing in
the first city and the other flies 100 mi before landing in the second city as depicted in Figure 3.1.47. How far apart are they after they’ve both landed?

We cannot answer these questions just yet but they are only a few of the examples of applications of the study of trigonometry.
3.1.1. The line \( y = \frac{1}{2}x \) intersects the unit circle at two points. What are they? Round to two decimal places.

3.1.2. The line \( y = -\frac{1}{2}x \) intersects the unit circle in two places. What are they? Round to two decimal places.

3.1.3. The function \( f(x) = 2|x| \) intersects the unit circle in two places. What are they? Round to two decimal places.

3.1.4. A right triangle has legs with lengths 4 in and 6 in. What is its area?

3.1.5. A right triangle has a hypotenuse of length 10 cm. If one of its legs is twice as long as the other, find its perimeter. Leave your answer in exact form.

3.1.6. Angles \( \theta \) and \( \phi \) are supplementary. If \( \phi \) is 10° larger than \( \theta \), find both angles.

3.1.7. Mark goes jogging one morning. He leaves his house and jogs 3 mi north and he then turns right and jogs 5 mi east before he stops to stretch. At that point, how far away from his house is he (in a straight line)? Round your answer to two decimal places.

3.1.8. Dale and Betty go through a traffic light at the same time but Dale goes straight and Betty turns right. After two minutes Dale is 2000 yd from the intersection and Betty is 750 yd from the intersection. Assuming the roads met at a right angle and both were perfectly straight, how far are Dale and Betty away from each other after two minutes? Round your answer to two decimal places.

3.1.9. Find the angle \( \theta \) below.

3.1.10. Find the value of \( a \) below. Round your answer to two decimal places.

3.1.11. What is the linear distance between the points \((-3, 1)\) and \((2, -4)\) in the xy-plane? That is, what is the length of the line in the image shown below? Leave your answer in exact form. Hint: Try drawing a right triangle with the line whose length you want as the hypotenuse.

3.1.12. A television set has a rectangular screen (assume that the corners are not rounded). If it is 48 in wide and 27 in tall, how far does the screen measure from corner to corner? Round your answer to two decimal places.

3.1.13. Consider a right triangle whose hypotenuse has a length of 53 in. If one leg is 17 in longer than the other leg, find the perimeter of the triangle. Round your answer to two decimal places.
3.1.14. The hypotenuse of a right triangle has a length of 200 and the longer leg is three times as long as the shorter leg. Find the perimeter of the triangle. Round to two decimal places.

3.1.15. A pizza box is rectangular. If it is 16 in wide and 18 in tall, how far does the box measure from corner to corner? Round your answer to two decimal places.

3.1.16. A particular township is in the shape of a rectangle which is 4 mi on one side and 6 mi on the other side. A hiker wants to hike the longest possible distance in a straight line without leaving the township so she decides to hike from the northeast corner to the southwest corner. How far is her hike? Round to two decimal places.

**Challenge Problems**

(3.1.c1) **CHALLENGE PROBLEM.** This is meant to be an exercise in using similar triangles.

a. Two posts of heights 20 ft and 30 ft are sticking straight out of the ground (that is, they both make right angles with the ground). Stretch from the top of each post to the bottom of the opposite post as shown in the diagram below. How high is point where these two wires touch? That is, find \( h \) in the diagram.

\[
\begin{align*}
& \text{\[20 \text{ ft}\]} \\
& \text{\[30 \text{ ft}\]} \\
& h
\end{align*}
\]

b. Explain why \( xy = h^2 \) in the triangle below. Note that the angle at the top of the outermost triangle is a right angle and the line whose length is \( h \) is a height of the triangle.

\[
\begin{align*}
& \text{\[h\]} \\
& \text{\[x\]} \\
& \text{\[y\]}
\end{align*}
\]
3.2 THE SINE AND COSINE FUNCTIONS

In this section we start with the trigonometry of right triangles. Recall from Section 3.1 that a right triangle is a triangle whose largest interior angle is a right angle.

Motivation

Certainly right triangles do not account for all of the triangles that arise naturally but they are a very important special case. Also, a mastery of right triangles will allow us to study other triangles as well.

We have already seen a result that is specific to right triangles: the Pythagorean Theorem in Theorem 3.1.29. Given a right triangle where two of the side length are known, the Pythagorean Theorem allows us to calculate the length of the third side. Suppose that instead we know the length of the hypotenuse and one of the interior angles of this right triangle. In this section we will develop techniques to find the lengths of the legs.

For example, consider a television. In the technical specifications for this television it says that the screen’s diagonal measures 40 in and that the diagonal makes a $30^\circ$ angle with the bottom of the screen as shown in Figure 3.2.1.

Figure 3.2.1: The measurements on a television.

In this case we might want to know the length and width of the television screen. These are the types of questions that we wish to address in this section.

Theory

Take a moment now to review Definition 3.1.19. This includes the definition of inscribed angles and inscription points.
(3.2.2) Definition. Let $\theta$ be an angle. Let $(x, y)$ be the inscription point resulting from drawing $\theta$ as an inscribed angle in the unit circle. Define the cosine function, $\cos$, such that $\cos(\theta) = x$ and define the sine function, $\sin$, such that $\sin(\theta) = y$.

It is important to realize that $\cos$ and $\sin$ are functions whose inputs are angles. If $\theta$ is an angle then to find $\cos(\theta)$ and $\sin(\theta)$ we place $\theta$ in the unit circle as an inscribed angle; the resulting inscription point has coordinates $(\cos(\theta), \sin(\theta))$. Figure 3.2.3 shows the values of $\cos(60^\circ)$ and $\sin(60^\circ)$ on the unit circle.

![Figure 3.2.3: The point $(\cos(60^\circ), \sin(60^\circ))$ drawn as an inscription point on the unit circle.](image)

In general it is difficult to calculate values of $\cos(\theta)$ and $\sin(\theta)$. A few special cases, however, can be computed easily. For example, consider an angle of $0^\circ$. When we put $0^\circ$ on the unit circle as an inscription angle it falls on the far right-most edge of the circle which has coordinates $(1, 0)$. It follows that $(1, 0) = (\cos(0^\circ), \sin(0^\circ))$ so that $\cos(0^\circ) = 1$ and $\sin(0^\circ) = 0$. This, along with some other inscription angles, is shown in Figure 3.2.4. Similarly, since the inscription point for a $180^\circ$ is at $(-1, 0)$ we have that $\cos(180^\circ) = -1$ and $\sin(180^\circ) = 0$.

The inscription points in Figure 3.2.4 allow us to calculate a few outputs of the sin and cos functions which are shown in Table 3.2.5.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$0^\circ$</th>
<th>$90^\circ$</th>
<th>$180^\circ$</th>
<th>$270^\circ$</th>
<th>$360^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos(\theta)$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\sin(\theta)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.2.5: Values of trigonometric functions on the axes.

Observe that the inscription points corresponding to $0^\circ$ and $360^\circ$ are the same points. Consider this for a moment. An angle of $0^\circ$ means no rotation at all. Hence when we draw an angle of $0^\circ$, both legs are in the same place. On the other hand, an angle of $360^\circ$ corresponds to a complete rotation, so when we draw an angle of $360^\circ$ both
legs are also in the same place. Thus the inscription angles for both $0^\circ$ and $360^\circ$ have terminal legs on the positive x-axis which means that both of their inscription points are the point $(1, 0)$.

As mentioned earlier, it is difficult to calculate $\cos(\theta)$ and $\sin(\theta)$ for arbitrary values of $\theta$. There are a few that we know exactly, but most of the time we need a calculator to approximate these values. You should stop now, turn on your calculator, and attempt to do some calculations. The only thing that might be tricky about your calculator is the mode that it is in. There are two common ways to measure angles: in degrees and in radians. We have already discussed degrees and we will discuss radians later. Your calculator has a separate mode for each of these types of angle measures. In degree mode your calculator expects angles to be in degrees and in radian mode your calculator expects angles to be in radians.

Go to your calculator and try to calculate $\cos(90)$. If you get that $\cos(90) = 0$ then your calculator is in degree mode and if you get $\cos(90) = -0.4481$ then your calculator is in radian mode. For now it should be in degree mode but eventually it will need to be in radian mode so, one way or another, you should learn how to switch between the two modes. Example 3.2.6 below is meant to help you check your calculator.

(3.2.6) Example. Use your calculator to find the following values:

A. $\cos(35^\circ)$
B. $\cos(200^\circ)$
C. $\sin(312^\circ)$

Not all calculators can evaluate trigonometric functions, but most modern calculators can.
D. \( \sin(122^\circ) \)

**Solution:**

A. Directly from a calculator,

\[ \textbf{Answer: } \cos(35^\circ) = 0.8192 \]

B. Directly from a calculator,

\[ \textbf{Answer: } \cos(200^\circ) = -0.9397 \]

C. Directly from a calculator,

\[ \textbf{Answer: } \sin(312^\circ) = -0.7431 \]

D. Directly from a calculator,

\[ \textbf{Answer: } \sin(122^\circ) = 0.8480 \]

Fix an angle \( \theta \). Even if we cannot calculate \( \cos(\theta) \) and \( \sin(\theta) \) there is a convenient way to relate them. Since \((\cos(\theta), \sin(\theta))\) is an inscription point it lies on the unit circle which means it satisfies Equation 3.1.42 in Lemma 3.1.41. This is the result in Theorem 3.2.7.

\[(3.2.7) \textbf{ Theorem (The Pythagorean Identity).} \text{ Fix an angle } \theta. \text{ Then}
\[
(cos(\theta))^2 + (sin(\theta))^2 = 1 \quad (3.2.8)
\]

Before we look at some examples, there is an issue with notation that we need to address. There are many times where quantities like \((\cos \theta)^k\) show up. Instead of writing \((\cos \theta)^k\) we will henceforth write \(\cos^k \theta\). This is simply a matter of convenience that mathematicians have adopted over the years. We adopt the same convention with the \(\sin\) function so that \(\sin^k(\theta) = (\sin(\theta))^k\). In this notation, Equation 3.2.8 becomes

\[\cos^2(\theta) + \sin^2(\theta) = 1.\]

\[(3.2.9) \textbf{ Example.} \text{ Suppose that } \theta \text{ is some angle such that } \cos(\theta) = 0.6. \text{ There are two possible values of } \sin(\theta). \text{ What are they?}

**Solution:**
We can answer this question without any reference to geometry. This is a bad strategy in general, however it will demonstrate how to use Theorem 3.2.7.

Combining the equation in Theorem 3.2.7 with the fact that \( \cos(\theta) = 0.6 \) we have that

\[
\cos^2(\theta) + \sin^2(\theta) = 1 \\
(0.6)^2 + \sin^2(\theta) = 1 \\
0.36 + \sin^2(\theta) = 1 \\
\sin^2(\theta) = 0.64.
\]

Remembering that \( \sin^2(\theta) \) really means \( (\sin(\theta))^2 \) we then have that

\[
\sin(\theta) = \pm \sqrt{0.64} = \pm 0.8
\]

so that the two possible values of \( \sin(\theta) \) are 0.8 and –0.8.

**Answer:** 0.8 and –0.8

Consider the geometry behind Example 3.2.9. In the problem statement we are given an angle \( \theta \) such that \( \cos(\theta) = 0.6 \). Assuming that we don’t know \( \sin(\theta) \), let’s try to imagine how to place the inscription point corresponding to \( \theta \) on the unit circle. Since that inscription point is the point \((\cos(\theta), \sin(\theta))\) we know that it has an \( x \)-coordinate of 0.6. All of the points in the \( xy \)-plane whose \( x \)-coordinate is 0.6 lie on the (vertical) line \( x = 0.6 \) so the inscription point corresponding to \( \theta \) must lie on the intersection of the line \( x = 0.6 \) and the unit circle. Both of these things are drawn in Figure 3.2.10 and we can see that there are two different possible inscription points.

Figure 3.2.10: The two possible inscription points for the angle \( \theta \) in Example 3.2.9.

Figure 3.2.10 explains the two possible values for \( \sin(\theta) \) that we found in Example 3.2.9. Based on that calculation, one of the possible inscription points is \((0.6,0.8)\) and the other is \((0.6,–0.8)\). If all we know about \( \theta \) is that \( \sin(\theta) = 0.6 \) then there is no way to determine
which of those is actually the inscription point. However, a little bit of additional information will allow us to make that decision.

Example. Suppose that $\theta$ is some angle such that $\cos(\theta) = 0.6$ and $\sin(\theta) > 0$. Find $\sin(\theta)$.

**Solution:**
This example is similar to Example 3.2.9 except that we also know that $\sin(\theta) > 0$. Using the calculations in Example 3.2.9 we know that the only two possibilities for $\sin(\theta)$ are $0.8$ and $-0.8$. Since we know that $\sin(\theta) > 0$, it must be the case that $\sin(\theta) = 0.8$.

**Answer:** $\sin(\theta) = 0.8$

In Example 3.2.11 we are given the fact that $\sin(\theta) > 0$ so, geometrically, we know that, with this additional piece of information, the inscription point corresponding to $\theta$ must be $(0.6, 0.8)$. That extra piece of information allowed us to determine the inscription point uniquely. However, there are other ways of determining the value of $\sin(\theta)$ and hence the inscription point.

Example. Suppose that $\theta$ is some angle such that $\cos(\theta) = -0.4$ and $180^\circ < \theta < 360^\circ$. Find $\sin(\theta)$.

**Solution:**
First consider the geometry. The inscription point corresponding to $\theta$ has an $x$-coordinate of $-0.4$ and hence must lie on the line $x = -0.4$. The line $x = -0.4$ and the unit circle are shown below.
The image shows the two possible inscription points. Using **Theorem 3.2.7** we can find the two inscription points. Since \( \cos(\theta) = -0.4 \) we have

\[
\begin{align*}
\cos^2(\theta) + \sin^2(\theta) &= 1 \\
(-0.4)^2 + \sin^2(\theta) &= 1 \\
0.16 + \sin^2(\theta) &= 1 \\
\sin^2(\theta) &= 0.84 \\
\sin(\theta) &= \pm 0.92
\end{align*}
\]

so that either \( \sin(\theta) = 0.92 \) or \( \sin(\theta) = -0.92 \) and the two possible inscription points are \((-0.4, 0.92)\) and \((-0.4, -0.92)\).

The one piece of information that we have not used yet is that \( 180^\circ < \theta < 360^\circ \). This is going to allow us to decide which of the two values of \( \sin(\theta) \) is the correct value.

The previous image shows the two possible inscription points; let’s look at the inscription angles corresponding to those two points.

The two angles above are labeled \( \theta_1 \) and \( \theta_2 \) because we haven’t yet determined which is actually \( \theta \). Now, from the image we see that \( \theta_1 < 180^\circ \) and that \( \theta_2 > 180^\circ \). If you have trouble seeing this, go back and look at **Figure 3.2.4**. Since \( 180^\circ < \theta < 360^\circ \), it must be the case that \( \theta \) is actually \( \theta_2 \). Certainly the inscription point corresponding to \( \theta_2 \) has a negative y-coordinate so \( \sin(\theta) \) must be a negative number and hence \( \sin(\theta) = -0.92 \).

**Answer:** \( \sin(\theta) = -0.92 \)

---

**Practice Exercise:** Suppose that \( \theta \) is some angle such that \( \cos(\theta) = 0.35 \) and \( 0^\circ < \theta < 180^\circ \). Find \( \sin(\theta) \). Round your answer to two decimal places.
So far we have only looked at examples where \( \cos(\theta) \) is given and we’re looking for \( \sin(\theta) \). It is a similar process to fix \( \sin(\theta) \) and find \( \cos(\theta) \).

\( \text{(3.2.13) Example.} \) Suppose that \( \theta \) is some angle such that \( \sin(\theta) = 0.3 \) and \( \theta < 90^\circ \). Find \( \cos(\theta) \).

\[ \begin{align*}
\text{Solution:} \\
\text{We first use Theorem 3.2.7 to find the two possible values of } \cos(\theta). \\
\cos^2(\theta) + \sin^2(\theta) &= 1 \\
\cos^2(\theta) + (0.3)^2 &= 1 \\
\cos^2(\theta) + 0.09 &= 1 \\
\cos^2(\theta) &= 0.91 \\
\cos(\theta) &= \pm 0.95 \\
\end{align*} \]

Hence either \( \cos(\theta) = 0.95 \) or \( \cos(\theta) = -0.95 \).

To decide between the two possible values we consider the geometry. Since the inscription point corresponding to \( \theta \) is \((\cos(\theta), \sin(\theta))\) and since \( \sin(\theta) = 0.3 \) it follows that the inscription point lies on the line \( y = 0.3 \) (since values of \( \sin \) are \( y \)-coordinates). This is shown below.

![Diagram showing points on a circle and a horizontal line at \( y = 0.3 \).]

The \( y \)-coordinates of the inscription points are determined by the two possible values of \( \cos(\theta) \) that we found above. The inscription angles, \( \theta_1 \) and \( \theta_2 \), which correspond to the two possible inscription points are also drawn above. Since \( \theta \) must be one of \( \theta_1 \) and \( \theta_2 \) and \( \theta < 90^\circ \), it must be that \( \theta \) is actually \( \theta_1 \). From the diagram, the inscription point corresponding to \( \theta_1 \) is \((0.95, 0.3)\) and hence \( \cos(\theta) = 0.95 \).

**Answer:** \( \cos(\theta) = 0.95 \)

Be careful when working on problems like Example 3.2.12 and Example 3.2.13; they can be tricky. If you’re drawing a diagram, make sure you draw a horizontal line when you’re given the value of \( \sin \).
and a vertical line when you’re given the value of cos. This is done because sin values are the y-coordinates of inscription points and cos values are the x-coordinates of inscription points.

So far, we have defined the sine and cosine functions and looked at their values, but there has been no mention of their usefulness. Their first application will be to relationships between sides in right triangles. Consider a right triangle whose legs have lengths a, b, and c with an interior angle of θ as oriented in Figure 3.2.14.

![Figure 3.2.14: The orientation of the triangle discussed in Theorem 3.2.16.](image)

Assume that the hypotenuse, c, is larger than 1 (for convenience). Now place this triangle on the xy-plane such that the angle θ is inscribed in the unit circle with the hypotenuse as the terminal leg. This placement is shown in Figure 3.2.15. Then the intersection of the hypotenuse with the unit circle is the inscription point corresponding to θ. We can then draw another right triangle whose hypotenuse is a radius of the unit circle which is parallel with the terminal leg of the inscription point which is also shown in Figure 3.2.15. Note (this is important) that the inscription point corresponding to θ is \((\cos(\theta), \sin(\theta))\) which means that the legs of this new triangle have lengths \(\cos(\theta)\) and \(\sin(\theta)\). Figure 3.2.15 shows this more clearly.

The two triangles drawn in Figure 3.2.15 are similar triangles since they have the same angles. Hence, as a consequence of Lemma 3.1.40, we have that

\[
\frac{\sin(\theta)}{b} = \frac{1}{c} = \frac{\cos(\theta)}{a}.
\]

Then, after rearranging the above equation, we have

\[
\sin(\theta) = \frac{b}{c} \quad \text{and} \quad \cos(\theta) = \frac{a}{c}.
\]

This is exactly the result in Theorem 3.2.16.

**Theorem 3.2.16.** Consider a right triangle which has a hypotenuse of length c and legs of length a and b. Let θ be the
interior angle of the triangle formed between the hypotenuse and the leg of length \( a \). (This is shown in Figure 3.2.14.) Then

\[
\sin(\theta) = \frac{b}{c} \quad \text{and} \quad \cos(\theta) = \frac{a}{c}.
\]

It is straightforward to apply Theorem 3.2.16 directly.

(3.2.17) Example. Find the values of \( a \) and \( b \) in the triangle below.

\[
\begin{align*}
\text{Solution:} & \\
& \text{Notice that the triangle in this example is arranged the same as the triangle in Theorem 3.2.16 where } c = 5 \text{ and } \theta = 40^\circ. \text{ It is important that } a \text{ is the leg of the triangle which}
\end{align*}
\]
is also a leg of the angle \( \theta \) and we see that it is. Using the equation in Theorem 3.2.16 we then have that

\[
\sin(40^\circ) = \frac{b}{5} \quad \text{and} \quad \cos(40^\circ) = \frac{a}{5}.
\]

By rearranging those equations we have that

\[
b = 5 \sin(40^\circ) \quad \text{and} \quad a = 5 \cos(40^\circ).
\]

A calculator tells us that \( \sin(40^\circ) = 0.6428 \) and that \( \cos(40^\circ) = 0.766 \), hence

\[
b = 5 \sin(40^\circ) = 3.21 \quad \text{and} \quad a = 5 \cos(40^\circ) = 3.83.
\]

It follows that \( b = 3.21 \) and \( a = 3.83 \).

[ANSWER: \( a = 3.83 \) and \( b = 3.21 \)]

When we use the Pythagorean Theorem it is not important which leg is \( a \) and which leg is \( b \). However, the arrangement of the triangle in Theorem 3.2.16 is extremely important. It is important that \( c \) is the hypotenuse but it is also important that \( a \) is a leg of \( \theta \). Remember from Definition 3.1.3 that the legs of an angle are the lines that form the angle. You’ll note that the side \( a \) is “touching” the angle \( \theta \) while \( b \) is not. We often call \( a \) the leg that is \textit{adjacent} to \( \theta \) whereas \( b \) is the leg that is \textit{opposite} to \( \theta \). When we attempt to apply Theorem 3.2.16 to a right triangle we must first identify the opposite and adjacent sides. Figure 3.2.18 shows some examples of the sides opposite and adjacent to an angle \( \theta \).

It is important to realize that Theorem 3.2.16 can only be applied to right triangles.
Once we’ve learned how to designate the opposite and adjacent sides to an angle we can reformulate Theorem 3.2.16 as

\[
\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{and} \quad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}},
\]

which will make it easier to apply the theorem. However, we must be careful when designating the opposite and adjacent side because they depend on the angle involved.

Consider the triangle in Figure 3.2.19 and the angle \( \theta \). The leg adjacent to \( \theta \) is \( a \) and the leg opposite to \( \theta \) is \( b \). It follows that

\[
\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c} \quad \text{and} \quad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}.
\]

These two equations show that \( \sin(\theta) = \cos(\phi) \) and \( \sin(\phi) = \cos(\theta) \).

Now consider the angle \( \phi \). Based on Figure 3.2.19 we see that the leg adjacent to \( \phi \) is \( b \) and the leg opposite to \( \phi \) is \( a \). Hence

\[
\sin(\phi) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \quad \text{and} \quad \cos(\phi) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}.
\]

It can be tempting to think that these equations contradict each other but they do not. If you find that difficult to believe, go back and repeat Example 3.2.17 except with the focus on the other angle in the triangle.

Figure 3.2.19: A right triangle with two angles. The side \( a \) is adjacent to \( \theta \) but opposite to \( \phi \) whereas the side \( b \) is adjacent to \( \phi \) but opposite to \( \theta \).

Make sure that you practice using the sine and cosine functions to determine information about right triangles. We end the theory with a couple of examples.

(3.2.20) Example. Find \( x \) in the triangle below.
Solution:
We are going to invoke Theorem 3.2.16 with \( \theta = 21^\circ \). Hence we will use either

\[
\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{or} \quad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}
\]

to solve for \( x \). Observe that the side with length 20 is the opposite side to the 21° angle and that we have no information on the adjacent side. Since we do not know the length of the adjacent side and we are not trying to solve for its length, there is no need to use the equation with the adjacent side in it. Hence we use

\[
\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}
\]

where \( \theta = 21^\circ \), the hypotenuse is \( x \), and the opposite side has length 20. Thus

\[
\sin(21^\circ) = \frac{20}{x}.
\]

After rearranging this equation we have that

\[
x = \frac{20}{\sin(21^\circ)} = 55.81.
\]

Answer: \( x = 55.81 \)

(3.2.21) Example. Find \( x \) in the triangle below.

Solution:
We will examine the 34° angle. Observe that we know the length of the hypotenuse and we want the length of the leg adjacent to the 34°. Hence we will use

\[
\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}
\]
where \( \theta = 34^\circ \), the hypotenuse has length 3, and the adjacent leg is \( x \). This yields

\[
\cos(34^\circ) = \frac{x}{3}
\]

or

\[
x = 3 \cos(34^\circ) = 2.49
\]

**Answer:** \( x = 2.49 \)

Be careful of the difference between \( 3 \cos(34^\circ) \) and \( \frac{3}{\cos(34^\circ)} \). Both of these expressions show up sometimes in different situations and so you should be careful that you’re using the right one. When in doubt, check your triangles. If you’re using \( \cos(\theta) \) then you’re relating the hypotenuse and one of the legs. The hypotenuse should always be longer than each leg.

**Applications**

The applications in this section are straightforward. Right triangles are everywhere in nature and these techniques can be used to find distances. When dealing with trigonometry applications you should always try to draw a picture. In this section we will try to set up a right triangle and solve for the missing information using the information given.

**Example.** A lumberjack is using a rope to help anchor a tree. This rope is tied to the very top of the tree and then attached to the ground. If the rope is 200 ft long and makes an angle of \( 72^\circ \) with the ground, how tall is the tree?

**Solution:**
The first thing to do is draw a picture. Now, it was not stated in the problem, but we will need to assume that the tree makes a right angle with the ground. Though this is not necessarily the case, it is a reasonable assumption as trees usually stick straight out of the ground. This gives us the following picture:

\[ \sin(72^\circ) = \frac{h}{200} \]

so that

\[ h = 200 \sin(72^\circ) = 190.21 \]

and hence the tree is 190.21 ft tall.

\[ \text{Answer: } 190.21 \text{ ft} \]

Example 3.2.23 below examines the example from Section 3.1 involving the trusses on a house.

(3.2.23) Example. A carpenter is working on a house and wants to know the dimensions of the trusses used to hold up the roof. Trusses have the shape of an isosceles triangle whose base is the width of the house. The carpenter uses a tool called a pitch finder to find that the pitch of the roof is at an angle of 32° and the floor plans tell him that the house is 125 ft wide. These dimensions are shown in the figure below.

A. Find the lengths of the two other sides of the truss.

B. Find the height of the truss.
Solution:
Before we address the questions, we first examine the geometry of the truss. The problem statement dictates that the truss is an isosceles triangle so the two unknown sides of the triangle are equal. Let’s call this length \( \ell \). We also want to consider the height whose length is \( h \). These things are shown below:

Note that the height splits the base into two equal parts. This is a general fact about isosceles triangles; if it is not clear, take a moment to think about it. Once we have drawn in this height, the rest of the problem becomes much easier.

a. To find \( \ell \) we can look at one of the smaller triangles. Since \( \ell \) is the hypotenuse of the smaller triangle and the leg adjacent to the 32° angle has length 62.5 we have that
\[
\cos(32^\circ) = \frac{62.5}{\ell}
\]
or
\[
\ell = \frac{62.5}{\cos(32^\circ)} = 73.7.
\]
Hence the length of the other two sides of the truss is 73.7 ft.
[Answer: 73.7 ft]

b. We can now solve
\[
\sin(32^\circ) = \frac{h}{\ell} = \frac{h}{73.7}
\]
so that
\[
h = 73.7 \sin(32^\circ) = 39.05.
\]
Hence the truss is 39.05 ft tall.
If you find it easier, you can solve this part with the Pythagorean Theorem. By looking at one of the smaller triangles we have
\[
h^2 + (62.5)^2 = \ell^2
\]
so that
\[
h = \sqrt{\ell^2 - (62.5)^2} = \sqrt{(73.7)^2 - (62.5)^2} = 39.05.
\]
Either method is perfectly acceptable. 

**Answer:** 39.05 ft

It’s important to be careful with rounding. In the final calculation we computed $73.7 \sin(32^\circ)$. Remember that 73.7 is only an approximation of $\ell$ and what we’re really trying to calculate is $\ell \sin(32^\circ)$. When I put the calculation into my calculator I actually found

$$\left( \frac{62.5}{\cos(32^\circ)} \right) \sin(32^\circ)$$

and then rounded to 39.05. It is always preferable to round at the end of a problem. If you round at each step, those small bits of error can sometimes compound and make the answer very far off at the end.

Finally, we wish to revisit the example with which we opened this section regarding the dimension of a television.

(3.2.24) **Example.** In the technical specifications for a particular television it says that the screen’s diagonal measures 40 in and that the diagonal makes a 30° angle with the bottom of the screen as shown below:

![Diagram of television screen dimensions](image)

Find the dimensions of the television, $\ell$ and $h$.

**Solution:**

We will use the 30° angle for our solution. Again, although it is not explicitly stated in the problem, it is safe to assume that the corners of the television form right angles. This is a fair approximation. Hence

$$\sin(30^\circ) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{40} \quad \text{and}$$

$$\cos(30^\circ) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\ell}{40}$$

so that

$$h = 40 \sin(30^\circ) = 20 \quad \text{and} \quad \ell = 40 \cos(30^\circ) = 34.65.$$
Hence the television is approximately 34.65 ft wide and 20 ft tall.

**Answer:** 34.65 ft wide and 20 ft tall

(3.2.c) **Practice Exercise.** Simon shoots a bottle rocket such that its flight path makes an angle of 80° with the ground. It flies 600 ft in a straight line before it erupts. How far off of the ground is it when it erupts? Round your answer to two decimal places.
3.2.1. Find \( \sin(\theta) \) and \( \cos(\theta) \) in the triangle below. Leave your answer in exact form.

\[
\begin{array}{c}
\triangle \quad \theta \\
15 \\
25
\end{array}
\]

3.2.2. Find \( \sin(\theta) \) and \( \cos(\theta) \) in the triangle below. Leave your answer in exact form.

\[
\begin{array}{c}
2 \\
5
\end{array}
\]

3.2.3. Find the length of the hypotenuse of the right triangle shown below. Round your answer to two decimal places.

\[
\begin{array}{c}
\triangle \quad 40^\circ \\
15
\end{array}
\]

3.2.4. Find \( a \) and \( b \) below. Round to two decimal places.

\[
\begin{array}{c}
\triangle \quad 75^\circ \\
4
\end{array}
\]

3.2.5. Suppose that \( \cos(\theta) = \frac{3}{10} \). What are the two possible values of \( \sin(\theta) \)? Round your answers to two decimal places.

3.2.6. Suppose that \( \sin(\theta) = -\frac{4}{13} \). What are the two possible values of \( \cos(\theta) \)? Round your answers to two decimal places.

3.2.7. Suppose that \( \sin(\theta) = -\frac{3}{5} \) and \( \cos(\theta) > 0 \). Find \( \cos(\theta) \). Round your answer to two decimal places.

3.2.8. Suppose that \( \cos(\theta) = -\frac{2}{3} \) and \( 180^\circ \leq \theta \leq 360^\circ \). Find \( \sin(\theta) \). Round your answer to two decimal places.

3.2.9. Suppose that \( \sin(\theta) = 0.36 \) and \( \theta \leq 90^\circ \). Find \( \cos(\theta) \). Round your answer to two decimal places.

3.2.10. Find \( \ell \) below. Round to two decimal places.

\[
\begin{array}{c}
\triangle \quad \ell \\
2 \\
5
\end{array}
\]

3.2.11. A ladder is 20 ft long and is leaning against the side of a house. The ladder is making an angle of 80° with the ground. How far is the base of the ladder from the wall? Round your answer to two decimal places.

3.2.12. When an airplane is taking off, it leaves the runway at an angle of 6°. That is, after its wheels leave the ground it is flying in a straight line that makes an angle of 6° with the ground. For safety purposes, the FAA requires that, after takeoff, the plane must be flying over the runway until it is at least 100 ft in the air. If a pilot wishes to follow this requirement on takeoff, what is the closest he or she can be to the end of the runway when his or her wheels leave the ground? Round your answer to two decimal places. Note: The FAA has no such requirement to my knowledge; I made it up for the sake of the problem.
3.2.13. Emilio likes to fly kites. His favorite kite has exactly 200 ft of string and on a particularly windy day he can attach the loose end of the string to the ground and watch the kite fly on its own. On one such day I used a protractor to measure the angle that the kite string made with the ground at 41°. How far off the ground was the kite at that particular time? Round your answer to two decimal places.

3.2.14. A ladder is 4 m long and is leaning against a house. If it makes a 75° angle with the ground, how far is the base of the ladder from the house? Round to two decimal places.

3.2.15. A rope is used to anchor a sign post to the ground and it is attached to the post halfway between the ground and the top of the post. Given that 7 ft of rope is needed and the rope makes an angle of 65° with the ground, find the height of the sign post. Round your answer to two decimal places.

3.2.16. A strand of a spider web extends at an angle from the ground to a flag pole. If the strand is 15 ft long and makes an angle of 70° with the ground, how high up on the flag pole does the web reach? Round your answer to two decimal places.

3.2.17. A ladder is leaning against a wall at an angle of 80°. If the base of the ladder is 3 ft from the wall, how high on the wall does the ladder reach? Round your answer to two decimal places.

3.2.18. Two roads diverged in a yellow wood at a right angle, both of which go in perfectly straight lines. Luckily, a straight dirt path has been built between the ends of these two roads for those who cannot make up their mind. The road less traveled is 3 mi long and meets the dirt path at a 42° angle. How long is the dirt path? Round to two decimal places.

3.2.19. A hiking path goes straight from the base of a mountain to the top of the mountain. It is 4000 ft long and makes an angle of 27° with the horizontal. How tall is the mountain?

Challenge Problems

(3.2.c1) Challenge Problem. Find the measures of all of the angles and the lengths of all of the lines in the figure below:

You should draw a picture as a part of your answer.

(3.2.c2) Challenge Problem. In this exercise we will prove the Law of Cosines. Consider the triangle drawn below:
Using only the techniques developed up to Section 3.2, show that

\[ c^2 = a^2 + b^2 - 2ab \cos(\theta). \]

This equation is used in Section 4.2 but it is not proved there. **Hint:**

*Draw in a height.*
3.3 SPECIAL ANGLES

While material in this section is certainly contained in Connally et al.’s text, it does not correspond to any particular section. Up to this point we have assumed that angles must live between 0° and 360°. Any angle that can be measured with a protractor does live in that range, but when you think of angles as rotations it is perfectly reasonable to think of angles which are smaller than 0° and larger than 360°. In addition to these new angles, we also want to explore some convenient angles for which we can compute the values of \( \sin(\theta) \) and \( \cos(\theta) \).

Motivation

The sine and cosine functions are very powerful tools and in Section 3.4 we will explore the graphs of these functions more carefully. Before we can do that, however, we need to do two things:

- First, we need to extend the domain of these two functions. Until now we have assumed that all angles measure between 0° and 360° so that the domain of both sine and cosine is \([0°, 360°]\). By defining angles that are smaller than 0° and larger than 360° we can extend the domain of sine and cosine to the set of all real numbers in a very natural way.

- Second, we would like to be able to calculate exact values for some more outputs of the sine and cosine functions. At this point in the notes, we can calculate exactly the values of \( \cos(\theta) \) and \( \sin(\theta) \) that are shown in Table 3.2.5 but we need a calculator for the rest. The values in Table 3.2.5 are graphed in Figure 3.3.1 and we can see that, without knowing more about the outputs of these functions, it is unclear what the shape of the graphs should look like.

It will be our goal in this section to fill in a few points on the graphs in Figure 3.3.1 in the interval \([0°, 360°]\) and to understand what in means to have an angle outside of that interval.

Theory

Consider an inscribed angle of 45° as shown in Figure 3.3.3 and examine the triangle drawn in the figure. Since both of its acute angles
are 45° it is an isosceles triangle so that its legs are the same length, which is labeled ℓ. Then, using the Pythagorean Theorem, we have

\[
\ell^2 + \ell^2 = 1
\]

\[
2\ell^2 = 1
\]

\[
\ell^2 = \frac{1}{2}
\]

\[
\ell = \frac{1}{\sqrt{2}}.
\]

Note that ℓ is the length of a side of a triangle so that ℓ must be positive. It follows that the inscription point corresponding to 45° is \(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\) and hence

\[
\cos(45°) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin(45°) = \frac{1}{\sqrt{2}}.
\]  

(3.3.2)

Figure 3.3.3: The triangle used to calculate sin(45°) and cos(45°).

Now consider an angle of 30°. An inscribed angle of 30° is drawn in Figure 3.3.5 along with a triangle that we wish to examine. The geometry is more complicated in this situation, but it is shown in the
Remember, an equilateral triangle has three equal interior angles and three equal sides.

Since the triangle is an equilateral triangle we have that \( h = 1 \) (since two of the sides are radii of the unit circle). Then, since the point \((\ell, \frac{h}{2})\) is on the unit circle, we have that

\[
\ell^2 + \left(\frac{h}{2}\right)^2 = 1
\]

\[
\ell^2 + \frac{1}{4} = 1
\]

\[
\ell^2 = 3
\]

\[
\ell = \frac{\sqrt{3}}{2}.
\]

Note that \(\ell\) is the length of a side of a triangle so that \(\ell\) must be positive. It follows that

\[
\cos(30^\circ) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(30^\circ) = \frac{1}{2}. \quad (3.3.4)
\]

Equation 3.3.2 and Equation 3.3.4 give two more angles for which we can calculate the outputs of sine and cosine without a calculator. These values, however, will allow us to calculate many more values using Lemma 3.3.6.
Lemma 3.3.6. A right triangle is uniquely determined by the length of its hypotenuse and by the measure of one of its interior angles. Two important special cases are given below.

The only right triangle with interior angles of $30^\circ$ and $60^\circ$ that has a hypotenuse of length 1 is the following:

![Right Triangle](image)

The only right triangle with interior angles of $45^\circ$ that has a hypotenuse of length 1 is the following:

![Right Triangle](image)

How does Lemma 3.3.6 help us calculate new values of sine and cosine? Consider an angle of $60^\circ$ and observe the triangle shown in Figure 3.3.7. We see that it’s a right triangle whose hypotenuse has length 1 which has interior angles of $60^\circ$ and $30^\circ$. By Lemma 3.3.6 there is only one triangle with those dimensions so the side lengths must be $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$. The longer side is $q$ so $q = \frac{\sqrt{3}}{2}$ and the shorter side is $p$ so $p = \frac{1}{2}$ (since $\frac{1}{2} < \frac{\sqrt{3}}{2}$). Hence

$$\cos(60^\circ) = \frac{1}{2} \quad \text{and} \quad \sin(60^\circ) = \frac{\sqrt{3}}{2}.$$ 

The procedure outlined above can be repeated for any angle which is a multiple of $30^\circ$ or a multiple of $45^\circ$. To do this we use reference triangles. Definition 3.3.8 below may seem confusing, but it is extremely intuitive in practice. If the definition seems overwhelming, read the explanation that follows.
Figure 3.3.7: The reference triangle used to calculate $\sin(60^\circ)$ and $\cos(60^\circ)$.

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$q$

$60^\circ$

$(p, q)$

$p$

(3.3.8) Definition. Let $\theta$ be an angle. The reference triangle corresponding to $\theta$ is the right triangle whose vertices are the points $(0,0)$, $(\cos(\theta), \sin(\theta))$, and $(\cos(\theta), 0)$. The reference angle corresponding to $\theta$ is the interior angle of the reference triangle which is formed at the vertex $(0,0)$.

In Figure 3.3.3 and Figure 3.3.7 we used reference triangles corresponding to angles of $45^\circ$ and $60^\circ$, respectively. Things get a little more complicated with angles that are larger than $90^\circ$. Figure 3.3.9 shows the reference triangle corresponding to an angle of $210^\circ$. In that figure we’ve called the inscription point $(p, q)$ which is, of course, the point $(\cos(210^\circ), \sin(210^\circ))$. Then the third point in the reference triangle should be $(p,0)$ which is on the x-axis.

One other thing that we need to consider in Figure 3.3.9 is the lengths of the legs of the triangle. In Figure 3.3.7 we labeled the inscription point by $(p, q)$ and we were able to label the lengths of the legs by $p$ and $q$. That is because the point $(p, q)$ is in the first quadrant in Figure 3.3.7 and hence $p$ and $q$ are both positive. However, in Figure 3.3.9, the point $(p, q)$ is still the inscription point but because it is in the third quadrant we have that $p$ and $q$ are both negative. In this case, $p$ and $q$ cannot be the lengths of the legs of the triangle because they are negative. However, $|p|$ and $|q|$ are positive numbers and give the length of the triangle.

It is important to realize that the inscription point can have coordinates that are either positive or negative, but that the lengths of the sides of a triangle must always be positive. Soon we will use reference triangles to calculate values of $\sin(\theta)$ and $\cos(\theta)$. In the procedure used to do that we will first find the lengths of the sides of the
reference triangle and then we will have to decide the signs on the coordinates of the inscription point.

The other part of Definition 3.3.8 is the definition of a reference angle. The reference angle is the angle in the reference triangle which is formed at the origin. In Figure 3.3.9 we see that the reference angle is $30^\circ$. How do we know that the angle is actually $30^\circ$? Let’s call the reference angle $\theta$. The angle formed between the positive $x$-axis and the negative $x$-axis is $180^\circ$ and, since angles add we have that

$$\theta + 180^\circ = 210^\circ$$

and hence $\theta = 30^\circ$. If this is calculation is uncomfortable, go back and review Example 3.1.15 where we first encountered these types of calculations.

Note that we don’t always need to form reference triangles to figure out a reference angle. The reference angle is the smallest angle formed between the leg of the inscription angle and the $x$-axis. Reference triangles look a little bit different in each of the four quadrants of the $xy$-plane and Figure 3.3.10 shows how they work in each of those cases. In each of those figures, the angle in question is $\theta$ and the corresponding reference angle is $\alpha$. Each figure also shows how to calculate the reference angle in that particular quadrant. You should practice finding reference triangles and reference angles.

The next three examples explore how to find reference triangles and reference angles. They are all very similar, but take place in different quadrants. Take great care to study and practice examples like
Examples of reference triangles in each quadrant. If $\theta$ is an angle with $0^\circ \leq \theta \leq 360^\circ$, the associated reference triangle is drawn and the reference angle corresponding to $\theta$ is drawn in as $\alpha$. Each figure also tells how to find $\alpha$ in each quadrant.

Figure 3.3.10: Examples of reference triangles in each quadrant. If $\theta$ is an angle with $0^\circ \leq \theta \leq 360^\circ$, the associated reference triangle is drawn and the reference angle corresponding to $\theta$ is drawn in as $\alpha$. Each figure also tells how to find $\alpha$ in each quadrant.

It is important to be able to find reference angles. Make sure that you remember that reference angles must be between $0^\circ$ and $90^\circ$ regardless of the measure of the original angle. It is important that you understand the procedure behind reference angles and reference triangles. The formulas provided in Figure 3.3.10 only work for angles between $0^\circ$ and $360^\circ$ and, in general, there is no formula.

(3.3.11) Example. Consider an angle of $52^\circ$. Draw the inscription angle corresponding to $52^\circ$ and draw its reference triangle. Then calculate the reference angle corresponding $52^\circ$. 

(a) Examples of reference triangles in the first quadrant. In this case, $\alpha = \theta$.

(b) Examples of reference triangles in the second quadrant. In this case, $\alpha = 180^\circ - \theta$.

(c) Examples of reference triangles in the third quadrant. In this case, $\alpha = \theta - 180^\circ$.

(d) Examples of reference triangles in the fourth quadrant. In this case, $\alpha = 360^\circ - \theta$. 
Solution:

We first draw an inscription angle of 52°.

We now draw in the reference triangle by drawing a straight line from the inscription point to the x-axis (making sure that the line makes a right angle with the x-axis).

In the first quadrant it is easy to calculate the reference angle since the reference angle is just the same as the original angle. Answer: The reference angle is 52° and the reference triangle is shown above.

(3.3.12) Example. Consider an angle of 147°. Draw the inscription angle corresponding to 147° and draw its reference triangle. Then calculate the reference angle corresponding 147°.

Solution:

We first draw an inscription angle of 147°.
We now draw in the reference triangle by drawing a straight line from the inscription point to the x-axis.

In the second quadrant the reference angle is given by $180^\circ - 147^\circ = 33^\circ$. You should think about what this last calculation means. Draw an angle of $180^\circ$ in the previous picture and see if you can justify that the reference angle is calculated by $180^\circ - 147^\circ$.

**Answer:** The reference angle is $33^\circ$ and the reference triangle is shown above.

(3.3.13) **Example.** Consider an angle of $319^\circ$. Draw the inscription angle corresponding to $319^\circ$ and draw its reference triangle. Then calculate the reference angle corresponding $319^\circ$.

**Solution:**

We first draw an inscription angle of $319^\circ$. **
We now draw in the reference triangle by drawing a straight line from the inscription point to the x-axis.

In the fourth quadrant the reference angle is given by $360° - 319° = 41°$. Again, you should think about what this last calculation means. Draw an angle of $360°$ in the previous picture and see if you can justify that the reference angle is calculated by $360° - 319°$.

**Answer:** The reference angle is $41°$ and the reference triangle is shown above.

Reference angles will allow us to calculate sine and cosine of many more angles. Go back and look at Figure 3.3.9. The reference triangle has a hypotenuse of length 1 and an interior angle (the reference angle) of $30°$. It follows by Lemma 3.3.6 that the triangle must have legs of length $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$. Since the leg of length $|q|$ is shorter than the leg of length $|p|$ we have that $|q| = \frac{1}{2}$ and $|p| = \frac{\sqrt{3}}{2}$.

Remember that, although we have figured out the lengths of the sides of the reference triangle, we have not yet found the values of $p$ and $q$; we still need to determine their signs. Let’s first decide the sign of $p$. From Figure 3.3.9 we can see that the inscription point is in the third quadrant and points in the third quadrant all have negative values.
x-coordinates. Since we know that $|p| = \frac{\sqrt{3}}{2}$ and $p < 0$ it must be that $p = -\frac{\sqrt{3}}{2}$. Similarly, points in the third quadrant all have negative $y$-coordinates so $q$ must be negative. Since $|q| = \frac{1}{2}$ and $q < 0$ it must be that $q = -\frac{1}{2}$.

We have now found that the inscription point corresponding to 210° is $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ so that

$$\cos(210°) = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(210°) = -\frac{1}{2}.$$  

The process used to find these values can be used in some other special cases. Any time we have an angle $\theta$ and we can determine the lengths of the legs of the reference triangle corresponding to $\theta$, we can determine the values of $\cos(\theta)$ and $\sin(\theta)$.

(3.3.14) Example. Find $\cos(300°)$ and $\sin(300°)$.

Solution:

In order to calculate these, you need to draw the reference triangle. This has been drawn for you below, but in practice you’ll need to do it step-by-step as done in the last three examples.

We have called the inscription point $(p, q)$ so that $p = \cos(300°)$ and $q = \sin(300°)$. Observe that the reference triangle has a hypotenuse of length 1 and an interior angle of 60° so, by Lemma 3.3.6, its legs must have length $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$. Since $|p|$ is smaller than $|q|$, we have that $|p| = \frac{1}{2}$ and that $|q| = \frac{\sqrt{3}}{2}$.

It remains only to find the signs on $p$ and $q$. Observe that the inscription point is in the fourth quadrant. Points in the fourth quadrant have a positive $x$-coordinate and a negative
y-coordinate so that $p > 0$ and $q < 0$. Hence $p = \frac{1}{2}$ and $q = -\frac{\sqrt{3}}{2}$. It follows, finally, that

$$\cos(300^\circ) = \frac{1}{2} \quad \text{and} \quad \sin(300^\circ) = -\frac{\sqrt{3}}{2}. $$

**Answer:** $\cos(300^\circ) = \frac{1}{2}$ and $\sin(300^\circ) = -\frac{\sqrt{3}}{2}$.

---

**Example 3.3.15.** Find $\cos(135^\circ)$ and $\sin(135^\circ)$.

**Solution:**

The reference triangle has been shown below.

![Reference Triangle](image)

We have called the inscription point $(p, q)$ so that $p = \cos(135^\circ)$ and $q = \sin(135^\circ)$. Observe that the reference triangle has a hypotenuse of length 1 and an interior angle of $45^\circ$ so, by Lemma 3.3.6, its legs must both have length $\frac{\sqrt{2}}{2}$ from which it follows that $|p| = \frac{\sqrt{2}}{2}$ and that $|q| = \frac{\sqrt{2}}{2}$.

It remains only to find the signs on $p$ and $q$. Observe that the inscription point is in the second quadrant. Points in the second quadrant have a negative $x$-coordinate and a positive $y$-coordinate so that $p < 0$ and $q > 0$. Hence $p = -\frac{\sqrt{2}}{2}$ and $q = \frac{\sqrt{2}}{2}$. It follows, finally, that

$$\cos(135^\circ) = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(135^\circ) = \frac{\sqrt{2}}{2}. $$

**Answer:** $\cos(135^\circ) = -\frac{\sqrt{2}}{2}$ and $\sin(135^\circ) = \frac{\sqrt{2}}{2}$.

In Example 3.3.11 we make the claim that points in the fourth quadrant have a positive $x$-coordinate and a negative $y$-coordinate. If that is hard
for you to see, imagine a point in the fourth quadrant (perhaps the inscription point in Figure 3.3.10d). This point is to the right of the y-axis so its x-coordinate must be positive. Also, this point is below the x-axis so its y-coordinate must be positive. Using similar arguments, can you justify the statement in Example 3.3.12 that points in the second quadrant have a negative x-coordinate and a positive y-coordinate?

(3.3.A) Practice Exercise. Calculate cos(θ) and sin(θ) for the following angles:

(i) θ = 315°
(ii) θ = 120°
(iii) θ = 240°

We now address our other concern for this section: angles which are larger than 360° or smaller than 0°. In Definition 3.1.3 we defined an angle to be the geometric object formed by two rays emanating from the same point. However, we have since come to use inscribed angles as an intuition for measuring angles. That is, we can think of angles as counterclockwise rotation from the positive x-axis. If we accept this notion of angles as a guide, it is natural to extend the definition to include angles with measure outside of the interval [0°, 360°].

Think about an angle of 400°. We have not yet defined this, but let’s think about what it should be. Of course, 400° = 360° + 40° since angles add. We also understand both a 360° angle and a 40° angle. An angle of 360° corresponds to a complete rotation around the unit circle and it seems that a 400° angle should be 40° past 360°. This situation is depicted in Figure 3.3.16.

Note in Figure 3.3.16 that the only difference between the 40° angle and the 400° angle is that we draw the arc going around an extra time in the 400° angle.
Figure 3.3.16 shows the general procedure for defining angles larger than 360°. Every angle $\theta$ which is larger than 360° can be written as

$$\theta = k \cdot 360^\circ + \alpha$$

for some integer $k$ and some angle $\alpha$ with $0^\circ \leq \alpha \leq 360^\circ$. To draw an inscription angle of $\theta$ we then go counterclockwise around the circle $k$ times and then around by an additional angle of $\alpha$.

In the example of 400° above, we had that $k = 1$ and $\alpha = 40^\circ$. Now consider an angle of 1000°. You can see that

$$1000^\circ = 2 \cdot 360^\circ + 280^\circ.$$  \hspace{1cm} (3.3.17)

It follows that an angle of 1000° will go around the circle twice and then an additional 280°. Similarly we have that

$$750^\circ = 2 \cdot 360^\circ + 30^\circ$$

and

$$1500^\circ = 4 \cdot 360^\circ + 60^\circ.$$  

Inscription angles of 1000°, 750°, and 1500° are shown in Figure 3.3.18.

Figure 3.3.18: Various large angles.

There is something worth mentioning at this point. The decomposition in Equation 3.3.17 was helpful in drawing the 1000° inscription angle. That equation is certainly believable, but how would we find it if it wasn’t given? That is, if we didn’t have Equation 3.3.17, how could we find values of $k$ and $\alpha$ such that

$$1000^\circ = k \cdot 360^\circ + \alpha?$$

Well, to find $k$ we divide 1000 by 360 and round down. Use your calculator to see that

$$\frac{1000}{360} = 2.78$$

so that $k = 2$. Once we know $k$ we simply subtract $k \cdot 360^\circ$ from 1000° to find $\alpha$. That is,

$$\alpha = 1000^\circ - k \cdot 360^\circ = 1000^\circ - 2 \cdot 360^\circ = 280^\circ.$$  

1000/360 = 2.78 \hspace{1cm} tells us that 1000° is more than 2 \cdot 360° but less than 3 \cdot 360°.
(3.3.19) Example. Draw an inscription angle of $1200^\circ$.

Solution:

We first need to write

$$1200^\circ = k \cdot 360^\circ + \alpha$$

for an integer $k$ and an angle $\alpha$. Note that

$$\frac{1200}{360} = 3.33$$

so that $k = 3$. Hence

$$\alpha = 1200^\circ - 3 \cdot 360^\circ = 120^\circ$$

so that $1200^\circ = 3 \cdot 360^\circ + 120^\circ$.

To draw an inscription angle of $1200^\circ$ we go around the unit circle three times and then go an extra $120^\circ$.

\[ \text{Answer: Shown above} \]

Negative angles are straightforward, as well. We should think of a negative angle as a “backwards” rotation. That is, they are simply rotations in the clockwise direction instead of the counterclockwise direction. An angle of $-60^\circ$ should be a rotation of $60^\circ$ but in the clockwise direction, an angle of $-214^\circ$ should be an angle of $214^\circ$ in the clockwise direction, etc. Figure 3.3.20a and Figure 3.3.20b compare and contrast angles of $60^\circ$ and $-60^\circ$ whereas Figure 3.3.20c and Figure 3.3.20d compare and contrast angles of $214^\circ$ and $-214^\circ$.

Of course we can also draw very large negative angles, as well. For example, $-1000^\circ$ should be $1000^\circ$ but in the clockwise direction. We have already found that an angle of $1000^\circ$ is twice around the circle in the counterclockwise direction and then an additional $280^\circ$. Hence an angle of $-1000^\circ$ is twice around the circle in the clockwise direction and then an additional $280^\circ$. An angle of $-1000^\circ$ is shown in Figure 3.3.21.
3.3 SPECIAL ANGLES

(a) A $60^\circ$ angle.
(b) A $-60^\circ$ angle.
(c) A $214^\circ$ angle.
(d) A $-214^\circ$ angle.

Figure 3.3.20: Examples of negative angles.

![Inscription angle of $-1000^\circ$.](image)

Figure 3.3.21: An inscription angle of $-1000^\circ$. Compare this with Figure 3.3.18a.

The last thing that we need to address in this section is how to evaluate the sine and cosine functions outside of the interval $[0^\circ, 360^\circ]$. Remember that we defined $\cos(\theta)$ and $\sin(\theta)$ by referencing the inscription point corresponding to $\theta$. Well, we know how to make inscription angles and hence inscription points for angles outside of the interval $[0^\circ, 360^\circ]$ so the same definition of the sine and cosine functions still apply. That is, if $\theta$ is a real number whose degree measure is any real number then $\cos(\theta)$ is the x-coordinate of the inscription point corresponding to $\theta$ and $\sin(\theta)$ is the y-coordinate of the inscription point.

For most angles outside of $[0^\circ, 360^\circ]$ we still need to use a calculator to approximate values of $\cos(\theta)$ and $\sin(\theta)$ but if $\theta$ is a multiple of $30^\circ$ or $45^\circ$ we can use the same techniques that we used in Example 3.3.14 and Example 3.3.15.
(3.3.22) Example. Find $\cos(660^\circ)$ and $\sin(660^\circ)$.

Solution:
We first draw a reference triangle. To find the inscription angle we observe that $660/360 = 1.83$ and $660^\circ - 1 \cdot 360^\circ = 300^\circ$ so that

$$660^\circ = 1 \cdot 360^\circ + 300^\circ.$$  

Hence the inscription angle, the reference triangle, and the reference angle are drawn below.

![Reference Triangle](image)

We have called the inscription point $(p, q)$ so that $p = \cos(660^\circ)$ and $q = \sin(660^\circ)$. By Lemma 3.3.6, the legs of the triangle must have length $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$. Since $|p|$ is smaller than $|q|$, we have that $|p| = \frac{1}{2}$ and that $|q| = \frac{\sqrt{3}}{2}$.

It remains only to find the signs on $p$ and $q$. Observe that the inscription point is in the fourth quadrant. Points in the fourth quadrant have a positive $x$-coordinate and a negative $y$-coordinate so that $p > 0$ and $q < 0$. Hence $p = \frac{1}{2}$ and $q = -\frac{\sqrt{3}}{2}$. It follows, finally, that

$$\cos(660^\circ) = \frac{1}{2} \quad \text{and} \quad \sin(660^\circ) = -\frac{\sqrt{3}}{2}.$$  

Answer: $\cos(660^\circ) = \frac{1}{2}$ and $\sin(660^\circ) = -\frac{\sqrt{3}}{2}$

(3.3.23) Example. Find $\cos(-675^\circ)$ and $\sin(-675^\circ)$.

Solution:
We first draw a reference triangle. To find the inscription angle we observe that $\frac{675}{360} = 1.875$ and $675^\circ - 1 \cdot 360^\circ = 315^\circ$ so that

$$675^\circ = 1 \cdot 360^\circ + 315^\circ.$$

Hence the inscription angle is obtained by one rotation and an extra $315^\circ$ around the circle in the clockwise direction. The inscription angle, the reference triangle, and the reference angle are drawn below.

We have called the inscription point $(p, q)$ so $p = \cos(-675^\circ)$ and $q = \sin(-675^\circ)$. By Lemma 3.3.6, the legs of the triangle must both have length $\frac{\sqrt{2}}{2}$. Hence $|p| = \frac{\sqrt{2}}{2}$ and that $|q| = \frac{\sqrt{2}}{2}$.

It remains only to find the signs on $p$ and $q$. Observe that the inscription point is in the first quadrant. Points in the fourth quadrant have a positive $x$-coordinate and a positive $y$-coordinate so that $p > 0$ and $q > 0$. Hence $p = \frac{\sqrt{2}}{2}$ and $q = \frac{\sqrt{2}}{2}$. It follows, finally, that

$$\cos(-675^\circ) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(-675^\circ) = \frac{\sqrt{2}}{2}.$$  

**Answer:** $\cos(-675^\circ) = \frac{\sqrt{2}}{2}$ and $\sin(-675^\circ) = \frac{\sqrt{2}}{2}$

Note that it is *always* possible to write an angle as

$$\theta = n \cdot 360^\circ + \alpha.$$

This can be done even when $\theta$ is negative. Some students find this approach to negative angles easier than the one presented above.
(3.3.b) Practice Exercise. Calculate \( \cos(\theta) \) and \( \sin(\theta) \) for the following angles:

(i) \( \theta = -135^\circ \)

(ii) \( \theta = 930^\circ \)

Compare Example 3.3.14 and Example 3.3.22. You will note that

\[
\cos(300^\circ) = \frac{1}{2} = \cos(660^\circ) \quad \text{and} \quad \sin(300^\circ) = -\frac{\sqrt{3}}{2} = \sin(660^\circ).
\]

This follows from the simple fact that \( 300^\circ \) and \( 660^\circ \) correspond to the same inscription point. This demonstrates a subtle point, which is that the sine and cosine function represent a measure of the net positional change incurred by a rotation. It is clear that \( 300^\circ \) and \( 660^\circ \) are not the same angle. It is even clear that if you take an object and rotate it by an angle of \( 660^\circ \) then it undergoes a lot more movement than if you rotate it by an angle of \( 300^\circ \). However, after the rotation has occurred, the object would be in the same position under rotation by those two angles. The value of \( \cos(\theta) \) indicates the position change of something in the \( x \)-direction (whatever that means in the context of the situation) under a rotation of \( \theta \) and the value of \( \sin(\theta) \) indicates the position change of something in the \( y \)-direction (whatever that means in the context of the situation) under a rotation of \( \theta \).

It seems odd at first to consider angles which cannot be measured with a protractor. These angles are not, however, a matter of mathematical curiosity but rather an important physical distinction. Consider the steering wheel in your car. If you are going straight and then turn the steering wheel by an angle of \( 410^\circ \) that will certainly result in a much sharper turn than if you turned it by an angle of \( 50^\circ \). However, in each of those cases the wheel would look like it was in the same position.
3.3.1. Calculate \( \cos(\theta) \) and \( \sin(\theta) \) for the following angles. Leave your answers in exact form.
   
   a.) \( \theta = 150^\circ \)
   b.) \( \theta = 225^\circ \)
   c.) \( \theta = 330^\circ \)

3.3.2. Find the reference angles corresponding to the following angles.

   a.) \( \theta = 214^\circ \)
   b.) \( \theta = 100^\circ \)
   c.) \( \theta = 651^\circ \)
   d.) \( \theta = -75^\circ \)
   e.) \( \theta = -414^\circ \)

3.3.3. Calculate \( \cos(\theta) \) and \( \sin(\theta) \) for the following angles. You should first draw the inscription angle. However, for this problem you do not need to use a reference triangle to find the coordinates of the inscription point.

   a.) \( \theta = 450^\circ \)

b.) \( \theta = 1080^\circ \)

c.) \( \theta = -990^\circ \)

d.) \( \theta = -180^\circ \)

3.3.4. Calculate \( \cos(\theta) \) and \( \sin(\theta) \) for the following angles. Leave your answers in exact form.

   a.) \( \theta = 1395^\circ \)
   b.) \( \theta = 1560^\circ \)
   c.) \( \theta = -480^\circ \)
   d.) \( \theta = -1575^\circ \)

3.3.5. You are given the following two values:

   \[
   \cos(15^\circ) = \frac{1 + \sqrt{3}}{2\sqrt{2}}
   \]

   \[
   \sin(15^\circ) = \frac{\sqrt{3} - 1}{2\sqrt{2}}
   \]

Find \( \cos(285^\circ) \) and \( \sin(285^\circ) \). Leave your answers in exact form. Hint: Draw the reference triangles corresponding to 15° and 285°.
3.4 **GRAPHS OF SINE AND COSINE**

*This section corresponds to section 7.3 in Functions Modeling Change [4]*

The sine and cosine functions were defined in Section 3.2. Here we will learn the complete shape of the graphs of these functions and will learn to apply them to some rudimentary application.

**Motivation**

The sine and cosine functions provide a convenient “shape” that allows us to model all sorts of interesting applications. Let’s consider the application introduced in Section 2.4. Someone is sitting on a Ferris wheel with a radius of 80 ft, whose center is 100 ft off of the ground, and which rotates at a rate of 1° every second. When the ride starts, this person is sitting level with the center of the wheel and is moving upward. A rudimentary image of this Ferris wheel is shown in Figure 2.4.1. We define \( h \) such that the rider is \( h(t) \) feet off of the ground after riding for \( t \) seconds.

Now imagine that this ride has gone on for less than 90 seconds. Since the Ferris wheel is moving at a rate of 1° every second, the rider’s position looks like the one shown in Figure 3.4.1.

![Figure 3.4.1: Our Ferris wheel rider’s position after t seconds.](image)

Look at the triangle formed in Figure 3.4.1. Using the techniques in Section 3.2 we have that

\[
\sin(t) = \frac{\ell}{80}
\]

since the radius of the Ferris wheel is 80 ft. Hence

\[
\ell = 80 \sin(t).
\]

Now, \( \ell \) describes the vertical distance between the center of the Ferris wheel and the rider. Since the center is 100 ft from the ground, we have that the rider is \( \ell + 100 \) ft from the ground, so

\[
h(t) = 80 \sin(t) + 100. \quad (3.4.2)
\]
Of course, we performed the previous calculation assuming that \( t \) was less than 90. As we will see, however, the function in Equation 3.4.2 works for all values of \( t \). Translations of the sine and cosine functions will provide periodic models for a wide range of applications.

**Theory**

In Section 3.3 we found several new and exciting outputs of the sine and cosine functions. Several of these values are shown in Figure 3.4.3. Now that we have these values we can see a shape of these functions start to emerge. The entire graph of the sine and cosine functions are also shown in Figure 3.4.3.

![Graph of sine and cosine functions](image)

(a) The graph of the function \( y = \cos(x) \).

(b) The graph of the function \( y = \sin(x) \).

Figure 3.4.3: The graph of the sine and cosine functions, including the points found in Section 3.3.

We would now like to investigate several of the geometric features of the graphs of the sine and cosine function.

**Domain:** As discussed in Section 3.3, the domain of both the sine and cosine function is the set of all real numbers, \( \mathbb{R} = (-\infty, \infty) \). That is to say that we can make sense of \( \cos(\theta) \) and \( \sin(\theta) \) for angles \( \theta \) whose degree measure is any real number.

**Range:** Let’s consider the outputs of the cosine function for a moment. To find cosine of an angle we look at its inscription point and then take the x-coordinate. Since inscription points are on the unit circle and the unit circle has radius 1, the biggest possible x-value of an inscription point is 1 and the smallest is -1. If you’re having trouble seeing that, take a look at the unit circle. Similarly, the values of the sine function are y-coordinates on the unit circle which must lie between -1 and 1. Hence both the sine and cosine function have a range of \([-1, 1]\).
roots: For what angles \( \theta \) is \( \sin(\theta) = 0 \)? Well, that happens any time the inscription point corresponding to \( \theta \) falls on the x-axis, i.e. has a y-coordinate of 0. We see that this happens for angles like \(-180^\circ, 0^\circ, 180^\circ, 360^\circ, \) etc. This is made more clear in Figure 3.4.3b. Hence the roots of the sine function are points of the form \((n \cdot 180^\circ, 0)\) for integers, \(n\). Similarly, \(\cos(\theta) = 0\) whenever the inscription point corresponding to \( \theta \) falls on the y-axis, which happens for angles like \(-90^\circ, 90^\circ, 270^\circ, \) etc. This can be seen in Figure 3.4.3a. It follows that the roots of the cosine function are points of the form \((90^\circ + n \cdot 180^\circ, 0)\) for integers, \(n\).

y-intercept: We already know that \(\cos(0^\circ) = 1\) and \(\sin(0^\circ) = 0\) so that the y-intercept of \(y = \cos(x)\) is \((0, 1)\) and the y-intercept of \(y = \sin(x)\) is \((0, 0)\).

symmetry: Look back at Figure 3.3.20 and try to imagine an arbitrary angle \( \theta \) with positive degree measure and then the angle \(-\theta\). Suppose that the inscription point corresponding to \( \theta \) is \((p, q)\). You can see that the inscription point corresponding to \(-\theta\) will just be a reflection of \((p, q)\) across the x-axis, which is the point \((p, -q)\). It follows that

\[
\cos(\theta) = p \quad \text{and} \quad \cos(-\theta) = p
\]

while

\[
\sin(\theta) = q \quad \text{and} \quad \sin(-\theta) = -q.
\]

From this we can conclude that

\[
\cos(-\theta) = \cos(\theta) \quad \text{and} \quad \sin(-\theta) = -\sin(\theta)
\]

for any angle \( \theta \). It follows that \( \cos \) is an even function and \( \sin \) is an odd function.

period: The sine and cosine functions are both periodic with period \(360^\circ\). We can justify this using the fact that the sine and cosine functions reflect the position of the inscription point corresponding to an angle. And, based on the discussion in Section 3.3, rotation by \(360^\circ\) does not change the position of an inscription point. That is, angles of \( \theta \) and \( \theta + 360^\circ \) have the same inscription point. Since the sine and cosine functions record the coordinates of the inscription point, we have

\[
\cos(\theta) = \cos(\theta + 360^\circ) \quad \text{and} \quad \sin(\theta) = \sin(\theta + 360^\circ)
\]

and so these two functions are periodic. The graphs in Figure 3.4.3 may also help convince you of this fact.

midline: For each of the sine and cosine functions, the maximum value is 1 and the minimum value is \(-1\) (see the discussion of the range). Hence the midline of each of these functions is \(y = 0\).
AMPLITUDE: Again, for both functions the maximum value is 1 and the minimum value is $-1$ so their amplitudes are 1.

These results are summarized in Table 3.4.4.

<table>
<thead>
<tr>
<th>FEATURE</th>
<th>GRAPH OF $y = \cos(x)$</th>
<th>GRAPH OF $y = \sin(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>$\mathbb{R} = (-\infty, \infty)$</td>
<td>$\mathbb{R} = (-\infty, \infty)$</td>
</tr>
<tr>
<td>Range</td>
<td>$[-1, 1]$</td>
<td>$[-1, 1]$</td>
</tr>
<tr>
<td>Roots</td>
<td>$(90^\circ + n \cdot 180^\circ, 0)$</td>
<td>$(n \cdot 180^\circ, 0)$</td>
</tr>
<tr>
<td>$y$-intercept</td>
<td>$(0, 1)$</td>
<td>$(0, 0)$</td>
</tr>
<tr>
<td>Symmetry</td>
<td>Even</td>
<td>Odd</td>
</tr>
<tr>
<td>Period</td>
<td>$360^\circ$</td>
<td>$360^\circ$</td>
</tr>
<tr>
<td>Midline</td>
<td>$y = 0$</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>Amplitude</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3.4.4: The geometric features of the sine and cosine functions.

We would also like to examine what happens to these features when we shift their graphs. That is, what changes about the geometry when we look at functions of the form

$$f_1(x) = A \cos(x) + k \quad \text{and} \quad f_2(x) = A \sin(x) + k$$

for values of $A$ and $k$. Remember that both of these transformations correspond to vertical changes in the graphs. That is, multiplication by $A$ stretches the graph vertically by a factor of $A$ and addition of $k$ shifts the graph vertically by $k$ units.

Let’s first examine the graphs of $y = A \cos(x)$ and $y = A \sin(x)$. Again, multiplication by $A$ stretches the graph vertically. However, if
A is negative then the graph is also reflected about the x-axis. Figure 3.4.5 shows examples when $A = 0.5$ and when $A = -2$. Observe that the maximum and minimum values are multiplied by $|A|$ and hence the maximum value becomes $|A|$ and the minimum values becomes $-|A|$. It follows that the midline is still $y = 0$ but the amplitude becomes $|A|$. We see that when the graphs of sine and cosine are stretched vertically by a factor of $A$, the midline does not change and the amplitude becomes $|A|$.

Now let’s examine the effect of $k$. This will change the graph by a vertical shift of $k$ units. Examples are shown in Figure 3.4.6 when $k = -1$ and $k = 2$. We see that the maximum and minimum values of these graphs get shifted upward by $k$ to $k + 1$ and $k - 1$, respectively. It follows that the midline becomes $y = k$ and the amplitude is still 1. Under a vertical shift by $k$ units, the midline gets shifted up by $k$ but the amplitude remains unchanged.

To review the geometric differences between graphs of the form $y = \cos(x)$ and $y = \sin(x)$ and graphs of the form $y = A \cos(x) + k$ and $y = A \sin(x) + k$, Table 3.4.8 summarizes the geometric features of these transformed graphs.

The first and most accessible use of these types of functions is to model the position of things that live on a circle. The way that we defined the sine and cosine functions was so that $p = \cos(\theta)$ and $q = \sin(\theta)$ describe the position of a point on a circle centered at...
Figure 3.4.7: Some examples of graphs of the form \( y = A \cos(x) + k \) and \( y = A \sin(x) + k \).

<table>
<thead>
<tr>
<th>Feature</th>
<th>( y = A \cos(x) + k )</th>
<th>( y = A \sin(x) + k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>( \mathbb{R} = (-\infty, \infty) )</td>
<td>( \mathbb{R} = (-\infty, \infty) )</td>
</tr>
<tr>
<td>Range</td>
<td>([k -</td>
<td>A</td>
</tr>
<tr>
<td>( y )-intercept</td>
<td>((0, k +</td>
<td>A</td>
</tr>
<tr>
<td>Symmetry</td>
<td>Even</td>
<td>None</td>
</tr>
<tr>
<td>Period</td>
<td>360°</td>
<td>360°</td>
</tr>
<tr>
<td>Midline</td>
<td>( y = k )</td>
<td>( y = k )</td>
</tr>
<tr>
<td>Amplitude</td>
<td>(</td>
<td>A</td>
</tr>
</tbody>
</table>

Table 3.4.8: The geometric features of transformations of the sine and cosine functions.

\((0,0)\) with a radius of 1. As Theorem 3.4.10 explains, if \( A > 0 \) then \( p = A \cos(\theta) + h \) and \( q = A \sin(\theta) + k \) describe the position of a point on a circle centered at \((h, k)\) with a radius of \( A \).

\((3.4.10)\) Theorem. Let \( A \) be positive and consider a circle of radius \( A \) centered at the point \((h, k)\). Let \((p, q)\) be the point on this circle which makes an angle of \( \theta \) with the horizontal. That is, \((p, q)\) is the point on the circle which is an angle of \( \theta \) counterclockwise from the point \((h + A, k)\). This situation is shown in Figure 3.4.9. Then

\[
p = A \cos(\theta) + h \quad \text{and} \quad q = A \sin(\theta) + k \quad (3.4.11)
\]

Theorem 3.4.10 is straightforward to apply. After this section there will not be a lot of occasion to apply Theorem 3.4.10 directly to a problem. Later in the section we will discuss how to use Theorem 3.4.10 to extend apply geometric intuition to angles which don’t lend them-
selves to geometry, but for now we examine the most rudimentary examples.

(3.4.12) **Example.** The circle below is centered at the point $(3, -2)$ and has a radius of 5 units. Find the point $(p, q)$.

**Solution:**

A direct application of Theorem 3.4.10 with $h = 3$, $k = -2$, $A = 5$, and $\theta = 300^\circ$ yields

\[
p = A \cos(\theta) + h = 5 \cos(300^\circ) + 3 = \frac{11}{2}
\]

\[
q = A \sin(\theta) + k = 5 \sin(300^\circ) - 2 = -\frac{5\sqrt{3} + 4}{2}.
\]

**Answer:** $p = \frac{11}{2}$ and $q = -\frac{5\sqrt{3} + 4}{2}$.
Example 3.4.13. The circle below is centered at the point $(-250, -250)$ and has a radius of 250 units. Find the point $(p, q)$.

Solution:

We will use another application of Theorem 3.4.10, this time with $h = -250$, $k = -250$, and $A = 250$. However, while it is tempting to use $\theta = 30^\circ$ in Equation 3.4.11, that will not yield the correct answer. If you read Theorem 3.4.10 it requires that $\theta$ be measured from the point $(h + A, k)$ which, in this case, is $(0, -250)$. Hence we need to use $\theta = 180^\circ + 30^\circ = 210^\circ$ in Equation 3.4.11.

\[
\begin{align*}
p &= A \cos(\theta) + h = 250 \cos(210^\circ) - 250 \\
&= 250 \left( -\frac{\sqrt{3}}{2} \right) - 250 = -466.51 \\
q &= A \sin(\theta) + k = 250 \sin(210^\circ) - 250 \\
&= 250 \left( -\frac{1}{2} \right) - 250 = -375.0
\end{align*}
\]

**Answer:** $p = -466.51$ and $q = -375.0$

Example 3.4.13 illuminates an issue that gives a lot of students problems. When applying Theorem 3.4.10, it is important that you choose the right value of $\theta$. In this text, equations will always measure angles from the right-most point on the circle. This should not be a big leap as it is the same way we measure inscription angles on the unit circle.

There is another thing worth mentioning about Example 3.4.12 and Example 3.4.13. These things can be calculated as we did using Theorem 3.4.10, but they can also be calculated using the same techniques that we learned in Section 3.3 to find inscription points. I now wish to outline an alternative way to solve Example 3.4.13. Consider, again, the figure in the example. Now, the point $(p, q)$ looks a little like an
inscription point and the $30^\circ$ angle looks like a reference angle. The circle is not a unit circle so these definitions do not apply, but we can use similar techniques. A portion of this circle is shown in Figure 3.4.14, along with a triangle similar to a reference triangle.

By looking at the triangle, we see that

$$a = 250 \cos(30^\circ) \quad \text{and} \quad b = 250 \sin(30^\circ).$$

Now, since we know $a$ and $b$ and we know that the center of the circle is $(-250, -250)$, we can find $(p, q)$. You can see from the figure that $(p, q)$ is $a$ units to the left of $(-250, -250)$ so that

$$p = -250 - a = -250 - 250 \cos(30^\circ) = -250 - 250 \left(\frac{\sqrt{3}}{2}\right) = -466.51.$$  

Similarly, the figure shows that $(p, q)$ is $b$ units below $(-250, -250)$ so that

$$q = -250 - b = -250 - 250 \sin(30^\circ) = -250 - 250 \left(\frac{1}{2}\right) = -375.$$  

Compare these results with the results shown above in the solution to Example 3.4.13.

(3.4.A) **Practice Exercise.** Find the coordinates of the point $(p, q)$ shown below. Round to two decimal places.
When finding a point on a circle you may use Theorem 3.4.10 if you’d like, but you may also use the geometry. Using the geometry is often a more intuitive approach and it is important that you understand it. We end the theory in this section with an example which uses Theorem 3.4.10 to model the position of a point on a circle as a function of its angle.

(3.4.15) Example. The circle below is centered at $(6,5)$ and has a radius of 4. Find functions $w$ and $v$ which describe the coordinates of a point on this circle as a function of the angle it makes counterclockwise from the right-most point on the circle (the point $(10,5)$). That is, when the point makes an angle of $\theta$ with the horizontal (in the usual way), its coordinates should be $(w(\theta), v(\theta))$. 

![Graph of sine and cosine functions](image)
Solution:
This example is merely a restatement of Theorem 3.4.10. The setup is the same in the case where $h = 6$, $k = 5$, $A = 4$, and $\theta$ is a variable. We then have that
\[ w(\theta) = 4 \cos(\theta) + 6 \quad \text{and} \quad v(\theta) = 4 \sin(\theta) + 5. \]
Answer: $w(\theta) = 4 \cos(\theta) + 6$ and $v(\theta) = 4 \sin(\theta) + 5$

Example 3.4.15 seems to be a bit trivial, but it is helpful in the applications.

Applications

Let’s return to the example introduced at the beginning of the section. A Ferris wheel rider is going around a Ferris wheel with a radius of 80 ft whose center is 100 ft off the ground. They start at the right-most point on the Ferris wheel when the ride starts and the wheel rotates at $1^\circ$ per second so that $t$ seconds after the ride starts, the rider’s carriage makes an angle of $t$ degrees with the horizontal (in the usual way). This is shown in Figure 3.4.1.

The last time we addressed this problem we assumed that $t < 90$ but now let $t$ be any value. We are trying to find a function $h$ such that the rider’s height after $t$ seconds on the Ferris wheel is $h(t)$ feet. Let’s apply Theorem 3.4.10. In order to use the theorem we need to have an $xy$-coordinate system in place. We get to choose where to place the coordinates, so we should make a choice that will work nicely for us.

It might seem natural to put the origin at the center of the circle, but that’s not the best choice. If the origin were at the center of the circle then applying Theorem 3.4.10 would give us the rider’s coordinates relative to the center; that is, we would have the rider’s horizontal and vertical distance from the center. We don’t care about horizontal
distance, but we do care about vertical distance. However, we care about vertical distance from the ground, not from the center.

![Ferris wheel diagram](image)

Figure 3.4.16: The Ferris wheel in Figure 3.4.1 with an imposed $xy$-coordinate system.

Let’s instead place the origin on the ground in the center of the base of the Ferris wheel as shown in Figure 3.4.16. Then we can apply Theorem 3.4.10 with $h = 0$, $k = 100$, $A = 80$, and $\theta = t$. It follows that the rider’s position is

$$(80 \cos(t), 100 + 80 \sin(t)).$$

The function $h$ should record the rider’s height which is precisely the $y$-coordinate of the rider’s position in our $xy$-coordinate system. Hence

$$h(t) = 100 + 80 \sin(t).$$

The above argument is very theoretical in nature. That is, it requires us to appeal to a technical Theorem and impose a coordinate system. This is a very effective technique but not a very intuitive one. However, recall the argument used to develop Equation 3.4.2. It was very geometric in nature and feels more natural than the one just provided. Unfortunately, it only worked in the case when $t > 90$.

You may use Theorem 3.4.10 to handle problems like this if you’d like; it is a perfectly acceptable technique. However, in practice it is not always the easiest way. The value of Theorem 3.4.10 is that it allows us to extend the geometric intuition to the rest of the circle. That is to say, when approaching problems that are similar to our Ferris wheel problem, you are allowed to approach the problem geometrically under the assumption that the angle is small and then apply the derived equations to larger angles, as well. In general, the “small” angles that you should examine are the ones small enough to be put into a triangle. The argument in the beginning of the section worked because we were able to make a triangle with $t$ as one of the angles. This, in general, is the strategy that you should employ.
Example. A compact disc (CD) is 120 mm in diameter and the hole in the center is 15 mm in diameter. An $xy$-coordinate system is implemented whose origin is at the center of the CD as shown below.

Find a function $H$ which describes the height of a point on the outer edge when it makes an angle of $\theta$ with the positive $x$-axis (in the usual way). That is, if a point is an angle of $\theta$ counterclockwise from the right-most point on the unit circle, $H(\theta)$ should be its distance from the $x$-axis in millimeters.

Solution:
Consider an angle $\theta$ which is between $0^\circ$ and $90^\circ$ and consider the geometry drawn below.

The height of the triangle drawn should be the value of $H(\theta)$. Observe that the hypotenuse of the triangle is 60 mm since the diameter of the CD is 120 mm. Since it is a right triangle, we have

$$\sin(\theta) = \frac{H(\theta)}{60}$$
and hence \( H(\theta) = 60 \sin(\theta) \). Then Theorem 3.4.10 allows us to use this equation for any value of \( \theta \).

**Answer:** \( H(\theta) = 60 \sin(\theta) \)

(3.4.18) **Example.** The center of a wind turbine is attached to the top of a 60 m tower and it has four spinning blades that are 40 m long. Find a function \( h \) such that when one of the blades makes an angle of \( \theta \) with the horizontal (in the usual way), the tip of the blade is at a height of \( h(\theta) \) feet off of the ground. This is shown in the figure below.

\[ \begin{align*}
\text{Solution:} \\
\text{A right triangle is drawn into the diagram below. Note that we put one of the vertices at the origin so that } \theta \text{ is an interior angle in the triangle.} \\
\text{We see, based on this figure, that } h(\theta) = \ell + 60 \text{ where } \ell \text{ depends on the value of } \theta. \text{ Then, using the triangle, we have} \\
\sin(\theta) &= \frac{\ell}{40} \\
\text{so that } \ell &= 40 \sin(\theta). \text{ Hence } h(\theta) = 40 \sin(\theta) + 60.
\end{align*} \]

**Answer:** \( h(\theta) = 40 \sin(\theta) + 60 \)
As mentioned earlier, you are welcome to use Theorem 3.4.10 to solve problems like this in the same manner as we solved Example 3.4.15. However, students often find the geometric approach easier to understand.

(3.4.8) Practice Exercise. Britney is standing in place and twirling a baton in her hand (assume that her hand is in the center of the baton) so that its ends are spinning in a circle. The baton is 28 in long and her hand is at a height of 40 in. Find a function $f$ such that when one end of the baton is making an angle of $\theta$ with the horizontal (in the usual way), the tip of that end of the baton is at a height of $f(\theta)$ inches.
3.4.1. Let $f(\theta) = 2 \sin(\theta) - 1$ and let $g(\theta) = \frac{1}{2} \cos(\theta) + \frac{3}{2}$.

A.) What is the midline and amplitude of $f$?

B.) What is the midline and amplitude of $g$?

C.) Sketch the graphs of both $f$ and $g$ on the same plot. (Be sure to identify which is which.)

3.4.2. Sketch a graph of the following functions; assume that $x$ is in degrees and be sure to label your axes. Additionally, give their midline and amplitude.

A.) $f(x) = 3 \cos(x) - 3$

B.) $g(x) = \frac{1}{3} \sin(x) + 2$

C.) $h(x) = 5 \cos(x) - 7$

D.) $p(x) = \frac{3}{4} \sin(x) + \frac{1}{2}$

3.4.3. Consider the circle shown below. Round all answers to two decimal places.

A.) Find the coordinates of $a$.

B.) Find the coordinates of $b$.

C.) Find the coordinates of $c$.

D.) Find the coordinates of $d$.

3.4.4. Consider a water wheel with a radius of 15 ft which uses the flowing water of a river to turn a shaft. The center of the water wheel is 12 ft off of the surface of the water (so that the lowest point of the wheel is 3 ft underwater). If a beetle lands on the very edge of the wheel, find a function $f$ so that $f(\theta)$ is the height of the beetle (in feet) when the beetle makes an angle of $\theta$ with the horizontal (in the usual way). Note: Look up the wikipedia article called water wheel if you’re not sure what it is.

3.4.5. A windmill has blades that are 20 ft in length and reach a maximum height of 50 ft. A small light is placed at the very end of one of the blades. Find a function $f$ such that when the blade on which the light is placed makes an angle of $\theta$ with the horizontal, $f(\theta)$ gives the height of the light in feet.

3.4.6. Consider the propeller of an airplane. The radius of a single blade of the propeller is 6 ft and the center of the propeller is 20 ft off the ground. Find a function $f$ such that when a particular blade makes an angle of $\theta$ with the horizontal, the end of the blade is a height of $f(\theta)$ feet off the ground.

3.4.7. Elmer is on a Ferris wheel which has a diameter of 140 ft and whose center is 80 ft off the ground. Find a function $h$ such that $h(\theta)$ is the height (in feet) of Elmer’s carriage when it makes an angle of $\theta$ with the horizontal (measured in the usual way).

3.4.8. The tire of a particular car is 620 mm in diameter and the valve stem on the wheel is at a radius of 200 mm. The diagram below demonstrates the geometry; the dashed line represents the ground, the large circle represents the outside diameter of the tire, and the point $v$ represents the position of the valve stem. Find a function $f$ such that when the valve stem is at an angle of $\theta$ with the horizontal (in the usual manner) then $f(\theta)$ describes the height of the valve stem off the ground in
millimeters. Note: The valve stem is the valve into which one pumps air when filling their tire. It is not important to understand a valve stem to do this problem.

3.4.9. A box fan sits in a window. The center of the fan is 4 ft off of the ground and each fan blade is 1 ft long. Find a function $f$ such that the tip of a fan blade is at a height of $f(\theta)$ when the blade makes an angle of $\theta$ with the horizontal (measured counter-clockwise, as usual).

3.4.10. Andy is riding a Ferris wheel which has a radius of 200 ft and its center is 250 ft off of the ground. The function $A$ describes Andy's angle on the Ferris wheel during his ride. After he has been riding the Ferris wheel for $t$ minutes, the line from the center of the Ferris wheel to Andy's seat makes an angle of $A(t)$ degrees with the horizontal (in the usual way) where

$$A(t) = 6t - 30.$$

A.) Find a function $h$ which describes Andy’s height from the ground in terms of his angle from the horizontal. That is, if Andy’s seat makes an angle of $\theta$ with the horizontal then it should be $h(\theta)$ feet from the ground.

b.) Find a function $M$ such that when Andy’s seat is an angle of $\theta$ from the horizontal, $M(\theta)$ should be the amount of time (in minutes) for which Andy has been riding the Ferris wheel.

c.) Find a function $\ell$ which describes Andy’s height as a function of time. That is, when Andy has been riding the Ferris wheel for $t$ minutes, $\ell(t)$ should be his height (in feet) from the ground.

3.4.11. The graph of $y = f(\theta)$ is drawn below.

A.) Find the period, midline, and amplitude of $f$.

b.) The function $f$ describes the height of the rider on a Ferris wheel. That is, if a rider is at an angle of $\theta$ degrees from the horizontal then his or her height is $f(\theta)$ feet from the ground. What is the radius of the Ferris wheel? How far from the ground is the center of the Ferris wheel?

c.) Suppose you are told that $f(\theta) = A \sin(\theta) + k$ for some values of $A$ and $k$. Find $A$ and $k$. 

**Challenge Problems**

(3.4.c1) **Challenge Problem.** In each of the problems below, assume that trigonometric functions always evaluate numbers as if they were angles in degree measure. For example, \( \sin(x) \) should mean sine of the angle whose degree measure is \( x \). *Hint: Use techniques from Section 2.4.*

A. The graph of \( y = f(x) \) is shown below. Find a formula for \( f(x) \).
   
   *Hint: you will need the sine function.*

   ![Graph of y = f(x)](image1)

B. The graph of \( y = g(x) \) is shown below. Find a formula for \( g(x) \).
   
   *Hint: you will need the sine function.*

   ![Graph of y = g(x)](image2)

C. The graph of \( y = p(x) \) is shown below. Find a formula for \( p(x) \).
   
   *Hint: you will need the cosine function.*

   ![Graph of y = p(x)](image3)
D. The graph of $y = q(x)$ is shown below. Find a formula for $q(x)$.

*Hint: you will need the cosine function.*

(3.4.c2) **Challenge Problem.** The center of a Ferris wheel is 300 ft high and is 400 ft (along the ground) from the park entrance. The Ferris wheel has a radius of 200 ft as shown in the diagram below.
We are going to be interested in a function \( d \). If a rider on this Ferris wheel makes an angle of \( \theta \) with the horizontal as shown in the diagram, then \( d(\theta) \) is the distance (in feet) between the park entrance and the rider.

A. Find and simplify \( d(\theta) \). *Hint: Use the Pythagorean identity to simplify.*

B. What are the maximum and minimum values of \( d \). That is, what is the closest and the farthest that the rider ever gets from the park entrance?

c. Sketch the graph of \( y = d(x) \).

d. The graph that you found in the previous part should have a familiar form. Using the things that you’ve discovered so far about \( d \), try to guess a “nicer” formula for \( d(\theta) \).
3.5 THE TANGENT FUNCTION

This section corresponds to section 7.4 in Functions Modeling Change [4].

The sine and cosine functions are called TRIGONOMETRIC FUNCTIONS for obvious reasons. There are four other (common) trigonometric functions, one of which is the tangent function. This section examines the definition and usefulness of this new function.

Motivation

Consider the triangle drawn in Figure 3.5.2. This triangle is similar to many triangles that we’ve seen before in Section 3.2. Suppose that we want to find the length of \( \ell \). Now, the side adjacent to the \( 34^\circ \) angle has length 4 and the side opposite has length \( \ell \) so we know that

\[
\cos(34^\circ) = \frac{4}{h} \quad \text{and} \quad \sin(34^\circ) = \frac{\ell}{h}.
\]

The only equation that has an \( \ell \) in it also contains an \( h \) and, unfortunately, we don’t know \( h \). We can still find the value of \( \ell \), we just have to combine those two equation. Using the equation for \( \cos(34^\circ) \) we know that

\[
h = \frac{4}{\cos(34^\circ)}
\]

and using the equation for \( \sin(34^\circ) \) we know that

\[
\ell = h \sin(34^\circ) = \left( \frac{4}{\cos 34^\circ} \right) \sin(34^\circ) = 4 \left( \frac{\sin(34^\circ)}{\cos(34^\circ)} \right) = 2.70 \quad (3.5.1)
\]

Figure 3.5.2: A right triangle which needs the tangent function.

The triangle in Figure 3.5.2 gives us the side adjacent to the \( 34^\circ \) angle and asks us to find the side opposite to the \( 34^\circ \) angle. The cosine function relates the adjacent side to the hypotenuse and the sine function relates the opposite side to the hypotenuse but, to this point, there is no function which relates the adjacent side and the opposite side. We were able to get the job done by combining the sine and cosine functions but it would have been more convenient if there were a function that related the sides involved. This is the purpose of the tangent function.
We begin with Definition 3.5.3. Recall from Equation 3.5.1 (and some manipulation) that 
\[
\frac{\ell}{4} = \frac{\sin(34^\circ)}{\cos(34^\circ)}
\]
with respect to the triangle in Figure 3.5.2. We had that \( \ell \) was the length of the opposite side and that 4 was the length of the adjacent side so, in this case, \( \frac{\sin(34^\circ)}{\cos(34^\circ)} \) related the opposite and adjacent sides. Hopefully that explains the definition of the tangent function.

**(3.5.3) Definition.** Define the **TANGENT** function, \( \tan \), such that
\[
\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}
\]
for any angle \( \theta \) where \( \cos(\theta) \neq 0 \).

Before explaining the tangent function and its usefulness, let’s look at some calculations.

**(3.5.4) Example.** Compute the following:

A. \( \tan(0^\circ) \)
B. \( \tan(90^\circ) \)
C. \( \tan(60^\circ) \)
D. \( \tan(495^\circ) \)
E. \( \tan(-71^\circ) \)

**Solution:**

A. Using the definition,
\[
\tan(0^\circ) = \frac{\sin(0^\circ)}{\cos(0^\circ)} = \frac{0}{1} = 0.
\]
\[\text{Answer:} \ \tan(0^\circ) = 0\]

B. Using the definition,
\[
\tan(90^\circ) = \frac{\sin(90^\circ)}{\cos(90^\circ)} = \frac{1}{0}.
\]
It follows that \( \tan(90^\circ) \) is not defined. However, this is to be expected. Note that \( \cos(90^\circ) = 0 \) and Definition 3.5.3 stipulates that \( \tan(\theta) \) is only defined when \( \cos(\theta) \neq 0 \).

**Answer:** \( \tan(90^\circ) \) is undefined

**c.** Using the definition,

\[
\tan(60^\circ) = \frac{\sin(60^\circ)}{\cos(60^\circ)} = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{2} \cdot 2 = \sqrt{3}.
\]

**Answer:** \( \tan(60^\circ) = \sqrt{3} \)

**d.** Using the definition,

\[
\tan(495^\circ) = \frac{\sin(495^\circ)}{\cos(495^\circ)} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = \frac{\sqrt{2}}{2} \cdot -\frac{2}{\sqrt{2}} = -1.
\]

**Answer:** \( \tan(495^\circ) = -1 \)

**e.** Using the definition,

\[
\tan(-71^\circ) = \frac{\sin(-71^\circ)}{\cos(-71^\circ)} = -2.90.
\]

Note that \(-71^\circ\) is not one of the special angles for which we can find exact values of the sine and cosine function so we are forced to approximate. It is better to put \(\frac{\sin(-71^\circ)}{\cos(-71^\circ)}\) into the calculator all at once than to approximate the numerator and the denominator separately before dividing. However, you will also find that your calculator has a tangent button and if you use that button directly you can also find

\[
\tan(-71^\circ) = -2.90
\]

without using the fraction in Definition 3.5.3.

**Answer:** \( \tan(-71^\circ) = -2.90 \)

One thing which is worth noticing about Example 3.5.4 is that \( \tan(-71^\circ) = -2.90 \). We know that sine and cosine always have outputs between \(-1\) and \(1\) but the tangent function can have any real number as an output. Also mentioned in Example 3.5.4 is the fact that your calculator has a tangent button which works analogously to the sine and cosine button.
(3.5.A) Practice Exercise. Compute the following:

(i) \( \tan(45^\circ) \)
(ii) \( \tan(120^\circ) \)
(iii) \( \tan(900^\circ) \)
(iv) \( \tan(-570^\circ) \)
(v) \( \tan(342^\circ) \)

You should be able to find exact values for all of these except for the last one.

In Definition 3.2.2 we defined the sine and cosine functions using the coordinates of the inscription point corresponding to an angle. Since then, inscription points have given us a nice way of visualizing the sine and cosine function. We now investigate what the tangent function describes about an inscription point.

![Figure 3.5.5: An arbitrary inscription angle, \( \theta \).](image)

Consider the inscription angle, \( \theta \), shown in Figure 3.5.5. We already know that \( p = \cos(\theta) \) and \( q = \sin(\theta) \). The question is, where does \( \tan(\theta) \) show up in Figure 3.5.5? The answer is that \( \tan(\theta) \) is the slope of the leg of the inscription angle; that is, \( \tan(\theta) \) is the slope of the line between \((0,0)\) and \((p,q)\). This easy to verify. If \( m \) is the slope then we have

\[
m = \frac{q - 0}{p - 0} = \frac{q}{p} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta).
\]

In fact, the tangent function can be used to find the slope of any line. If the line \( y = mx + b \) intersects the x-axis at an angle of \( \theta \) (measured in the usual way), then \( m = \tan(\theta) \). This idea can be used directly, as shown in Example 3.5.6 below.
(3.5.6) Example. Find the equation of the line drawn below:

Solution:

We know that the line is of the form \( y = mx + b \) for some values of \( m \) and \( b \). Since we know that the angle the line makes with the \( x \)-axis is \( 42^\circ \), we have

\[
m = \tan(42^\circ) = 0.9.
\]

Then \( y = 0.9x + b \). To find \( b \) we use the fact that the line goes through the point \((-2, 0)\). Hence

\[
0 = 0.9(-2) + b \quad \text{so that} \quad b = 1.8.
\]

Finally, \( y = 0.9x + 1.8 \).

Answer: \( y = 0.9x + 1.8 \)

We would now like to use the tangent function to relate the legs of a right triangle. Consider the same triangle that we examined when we defined the sine and cosine functions, as shown in Figure 3.2.14. We know that

\[
\sin(\theta) = \frac{b}{c} \quad \text{and} \quad \cos(\theta) = \frac{a}{c}.
\]

so we have that

\[
\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{b/c}{a/c} = \frac{b \cdot c}{a \cdot b} = \frac{b}{c}.
\]

This is the result in Lemma 3.5.7. In the triangle above, we called \( a \) the side adjacent to \( \theta \) and we called \( b \) the side opposite to \( \theta \). According to these conventions we can write

\[
\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}.
\]
(3.5.7) Lemma. Consider a right triangle with a hypotenuse of length $c$ and legs of length $a$ and $b$. Let $\theta$ be the interior angle of the triangle formed between the hypotenuse and the leg of length $a$. (This is shown in Figure 3.2.14.) Then
\[ \tan(\theta) = \frac{b}{a}. \]

Lemma 3.5.7 gets applied in the same manner as Theorem 3.2.16. We investigate some examples below.

(3.5.8) Example. Find $\ell$ below.

\[ \begin{array}{c}
25 \\
53^\circ \\
\ell
\end{array} \]

Solution:
Since the triangle in question is a right triangle we can use Lemma 3.5.7. The side opposite to the $53^\circ$ angle has length 25 and the side adjacent to the $53^\circ$ has length $\ell$ so we have
\[ \tan(53^\circ) = \frac{25}{\ell} \]
and hence
\[ \ell = \frac{25}{\tan(53^\circ)} = 18.84. \]

Answer: $\ell = 18.84$

(3.5.9) Example. Find $\ell$ below.

\[ \begin{array}{c}
19^\circ \\
12 \\
\ell
\end{array} \]
Solution:
Since the triangle in question is a right triangle we can use Lemma 3.5.7. The side opposite to the \(19^\circ\) angle has length \(\ell\) and the side adjacent to the \(19^\circ\) has length 12 so we have
\[
\tan(19^\circ) = \frac{\ell}{12}
\]
and hence
\[
\ell = 12 \tan(19^\circ) = 4.13
\]
**Answer:** \(\ell = 4.13\)

(3.5.b) Practice Exercise. Find \(x\) below:

![Triangle Diagram]

Figure 3.5.10: A visualization of some values of \(\tan(\theta)\).

We should look at the graph of the tangent function. We will not be as thorough in our investigation of the graph of the tangent function as we were with the sine and cosine functions, but we will give a rudimentary justification of its shape. It turns out that the tangent function is periodic with period \(180^\circ\). Consider the tangent function on the interval \([-90^\circ, 90^\circ]\). Remember, \(\tan(\theta)\) represents the slope of the leg of the inscription angle corresponding to \(\theta\). Look at Figure 3.5.10. As \(\theta\) gets very close to \(0^\circ\) we see that the leg of the inscription angle corresponding to \(\theta\) gets very flat and hence has a small slope. When \(\theta\) is just above \(0^\circ\) we see that \(\tan(\theta)\) is very small and positive whereas when \(\theta\) is just below \(0^\circ\) we see that \(\tan(\theta)\) is very small but negative. When the inscription point is very far away from the \(x\)-axis, however,
the magnitude of \( \tan(\theta) \) gets very large. When \( \theta \) is just below 90° we have that \( \tan(\theta) \) is very large and when \( \theta \) is just above −90° we have that \( \tan(\theta) \) is very large but negative. Since \( \tan(\theta) \) is undefined whenever \( \cos(\theta) \neq 0 \), the graph of \( y = \tan(x) \) has a vertical asymptote at \( x = \theta \) whenever \( \cos(\theta) = 0 \). The graph of \( y = \tan(x) \) is shown in Figure 3.5.11.

![Graph of \( y = \tan(x) \) with vertical asymptotes at \( x = \theta \) whenever \( \cos(\theta) = 0 \).](image)

Figure 3.5.11: The graph of \( y = \tan(x) \).

For reference, the geometric features of the tangent function are given in Table 3.5.12. Note that, although the tangent function is periodic, it has no amplitude and midline because it has no maximum and minimum.

<table>
<thead>
<tr>
<th>FEATURE</th>
<th>GRAPH OF ( y = \tan(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>All ( x \in \mathbb{R} ) except ( x = 90° + n \cdot 180° )</td>
</tr>
<tr>
<td>Range</td>
<td>( \mathbb{R} = (-\infty, \infty) )</td>
</tr>
<tr>
<td>Roots</td>
<td>( (n \cdot 180°, 0) )</td>
</tr>
<tr>
<td>y-intercept</td>
<td>( (0, 0) )</td>
</tr>
<tr>
<td>Symmetry</td>
<td>Odd</td>
</tr>
<tr>
<td>Period</td>
<td>( 180° )</td>
</tr>
</tbody>
</table>

Table 3.5.12: The geometric features of the tangent function.

Finally, we can also use the tangent function to do examples similar to those in Section 3.2. Compare Example 3.5.13 below with Example 3.2.12.

(3.5.13) Example. Suppose that \( \theta \) is some angle such that \( \cos(\theta) = -0.4 \) and \( \tan(\theta) < 0 \). Find \( \sin(\theta) \).
Solution:

First consider the geometry. The inscription point corresponding to $\theta$ has an $x$-coordinate of $-0.4$ and hence must lie on the line $x = -0.4$. The line $x = -0.4$ and the unit circle are shown below.

The image shows the two possible inscription points corresponding to inscription angles of $\theta_1$ and $\theta_2$. Using Theorem 3.2.7 we can find the two inscription points. Since $\cos(\theta) = -0.4$ we have

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$(-0.4)^2 + \sin^2(\theta) = 1$$

$$0.16 + \sin^2(\theta) = 1$$

$$\sin^2(\theta) = 0.84$$

$$\sin(\theta) = \pm 0.92$$

so that either $\sin(\theta) = 0.92$ or $\sin(\theta) = -0.92$ and the two possible inscription points are $(-0.4, 0.92)$ and $(-0.4, -0.92)$. It remains only to find the sign of $\sin(\theta)$. The one piece of information that we have not used yet is that $\tan(\theta) < 0$. There are two ways to use this information to find the desired sign; either way is acceptable but both are presented for convenience.

On one hand, we can think of $\tan(\theta)$ as the slope of the leg of the inscription angle. We know that $\tan(\theta) < 0$ so that the slope of the leg of the inscription angle is negative. In the figure above, the leg of $\theta_1$ has a negative slope and the leg of $\theta_2$ has a positive slope so it must be that $\theta = \theta_1$. The inscription point corresponding to $\theta_1$ is in the second quadrant and hence has a positive $y$-coordinate. Thus $\sin(\theta) = 0.92$.

On the other hand, we can write

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$ 

We know that $\cos(\theta) = -0.4 < 0$. If $\sin(\theta)$ were a negative number then $\tan(\theta)$ would have to be positive as the quotient...
of two negative numbers is positive, whereas if \( \sin(\theta) \) were a positive number then \( \tan(\theta) \) would be negative as a positive number divided by a negative number is negative. Since we know that \( \tan(\theta) < 0 \) it follows that \( \sin(\theta) \) must be positive and hence \( \sin(\theta) = 0.92 \).

Obviously either of those two arguments produces the same result (as we would expect); either one is perfectly valid.

**Answer:** \( \sin(\theta) = 0.92 \)

---

### Applications

The applications in this section will be very similar to those in Section 3.2.

(3.5.14) **Example.** A tree is casting a shadow in the early afternoon. The tree’s shadow is 30 ft long and the line between the top of the tree and the end of the shadow makes an angle of 55° with the ground. Find the height of the tree.

**Solution:**

As always, we start by drawing a picture:

We then see a triangle develop. Again, it is not stated that the triangle should be a right triangle, but it is a fair assumption that the tree comes out of the ground at a right angle. Hence we have a right triangle with a 55° interior angle whose adjacent side has length 30 and whose opposite side is the length we want; let’s call it \( h \). Then

\[
\tan(55°) = \frac{h}{30}
\]
so that

\[ h = 30 \tan(55^\circ) = 42.84 \]

and hence the tree is 42.84 ft tall.

**Answer:** 42.84 ft

The tangent function also allows us to address one of the problems discussed in Section 3.1 involving the height of a building. The figure shown in the example below is the same as Figure 3.1.46.

(3.5.15) **Example.** A surveyor is standing 60 ft away from a building and uses a theodolite to determine that his line of sight with the top of the building makes an angle 65° with the ground as shown below. How tall is the building?

![Diagram of a building and theodolite]

**Solution:**

In the image we can clearly see a triangle. Of course, it is reasonable to assume that the building comes out of the ground at a right angle so that the triangle is a right triangle.

Then, if the building’s height is \( h \), we have

\[ \tan(65^\circ) = \frac{h}{60} \]

and hence

\[ h = 60 \tan(65^\circ) = 128.67 \]

so that the building’s height is about 128.67 ft.

**Answer:** 128.67 ft

Of course, we can also tackle more complicated problems such as Example 3.5.16.
Example. An earthquake has damaged a small section of a bridge. This bridge is shown below and the missing section is the gray section of length $d$. The bridge is fragile and, unfortunately, it is not safe for someone to walk onto the bridge and measure the length of the gap. However, it is possible to take certain measurements. The schematics from the bridge dictate that it is at a constant height of 75 ft from the ground. Measurements taken from the base of the bridge find an angle of elevation of $43^\circ$ to the closer edge of the gap and an angle of elevation of $34^\circ$ to the farther edge of the gap. Using this information, find the length, $d$, of the gap.

Solution:
There are two triangles that we will use in this figure. They are drawn in below along with a new length, $\ell$.

We want to analyze those two triangles separately so, to make more sense of them, they have been removed from the rest of the figure below.
Things now start to become a lot more clear. Using the triangle on the left we have that

\[ \tan(43^\circ) = \frac{75}{\ell} \]

so that

\[ \ell = \frac{75}{\tan(43^\circ)}. \]

Using the triangle on the right we have that

\[ \tan(34^\circ) = \frac{75}{\ell + d} \]

and hence

\[ \ell + d = \frac{75}{\tan(34^\circ)}. \]

Combining these two equations yields

\[ \frac{75}{\tan(43^\circ)} + d = \frac{75}{\tan(34^\circ)} \]

so that, finally,

\[ d = \frac{75}{\tan(34^\circ)} - \frac{75}{\tan(43^\circ)} = 30.67. \]

Hence the gap is 30.76 ft across.

**Answer:** 30.76 ft
3.5.1. Compute the following and leave your answers in exact form:
   A.) \( \tan(30^\circ) \)
   B.) \( \tan(45^\circ) \)
   C.) \( \tan(60^\circ) \)
   D.) \( \tan(135^\circ) \)
   E.) \( \tan(225^\circ) \)
   F.) \( \tan(-60^\circ) \)

3.5.2. Find \( \tan(\theta) \) in the triangle below and leave your answer in exact form.

3.5.3. Find \( \tan(\theta) \) in the triangle below and leave your answer in exact form.

3.5.4. Find the value of \( a \) below. Round your answer to two decimal places.

3.5.5. Find \( \ell \) below and round to two decimal places.

3.5.6. Find \( \ell \) below. Round to two decimal places.

3.5.7. Find \( y \) in the following diagram. Round your answer to two decimal places.

3.5.8. Find an equation for the line graphed below and leave your answer in exact form. (Assume that the line goes through the point \((1, 0)\) as it appears.)

3.5.9. The function \( f(x) \) is a linear function. It has an x-intercept of \((-3, 0)\) and it intersects
the x-axis at an angle of 60°. Find the y-intercept of f and leave your answer in exact form.

3.5.10. The function f is linear, f(1) = −4, and f intersects the x-axis at an angle of 150° (measured counterclockwise from the x-axis, as usual). Find a formula for f and leave your answer in exact form.

3.5.11. The top of a 200 ft vertical tower is to be anchored by cables that make an angle of 60° with the ground. Round your answers to two decimal places.
   a.) How long must the cables be?
   b.) How far from the base of the tower should anchors be placed?

3.5.12. The front door to the student union is 20 ft above the ground and it is reached by a flight of steps. The school wants to build a wheelchair ramp with an incline of 15° from the ground to the door. How much horizontal distance is needed for the ramp? Round to two decimal places.

3.5.13. The area of the outermost triangle shown below is 5 in². Find the perimeter of the large triangle and round your answer to two decimal places. Remember: The perimeter of a shape is the distance around the outside of it.

3.5.14. A ramp is to be built that has a 12° angle of incline which climbs a total of 4 ft. How long does the base of the ramp need to be? Round your answer to two decimal places.

3.5.15. An airplane takes off at an angle of 8° from a runway. If there are 2000 ft of runway remaining when the airplane lifts off, find the airplane’s height above the ground when it passes the end of the runway. Round to two decimal places.

**Challenge Problems**

(3.5.c1) Challenge Problem. Fix an angle θ. Draw the inscrip-
tion point corresponding to θ. There is exactly one line which passes through that point and passes through no other point on the circle. (By “line” we mean the graph of a linear function.) This line is called a **tangent line** to the circle. Let x and y be functions such that x(θ) is the x-coordinate of the x-intercept of the tangent line and y(θ) is the y-coordinate of the y-intercept of the tangent line. This is demonstrated in the diagram below for one choice of θ.
A. First, find the equation of the tangent line (in terms of \( \theta \)), find \( x(\theta) \), and find \( y(\theta) \). Justify and simplify your answer.

B. Now consider a second circle with radius one centered at \((4, 0)\). There are four lines which are tangent to both circles as shown in the diagram below. Find all of them. *Hint: Two of them are easy. Use the previous part of this problem to find the other two.*
3.6 INVERSE TRIG FUNCTIONS

This section corresponds to section 7.5 in Functions Modeling Change [4]

In previous sections we’ve used the angles in a right triangle to find the lengths of its sides. In this section we will use the lengths of the sides of a right triangle to find its angles.

Motivation

To this point in our study of right triangle trigonometry, if given the measure of an acute angle of a right triangle and the length of any of its sides, we are able to determine all of the rest of the information about the triangle. Using any of the sine, cosine, or tangent functions we can find the length of a second side, and the Pythagorean Theorem allows us to find the third side.

Consider the figure in Figure 3.6.1. We know that

\[ \tan(\theta) = \frac{b}{c}. \]  

(3.6.2)

As we explored in Section 3.5, if we know \( \theta \) and either \( a \) or \( b \) then we can find the length of the other side. It stands to reason that if we know both of \( a \) and \( b \) then we should be able to uniquely determine \( \theta \). You can explore this for yourself if you’d like. Pick any two positive numbers and try to make a triangle whose legs have those numbers as lengths. You will quickly realize that there is only one triangle that you can make. Hence if we pick \( b \) and \( c \) we should be able to find \( \theta \).

Remember back to Section 2.3. In that section we discussed how it meant to be able to solve equations like those in Equation 3.6.2. We would like to use an inverse function and say that

\[ \theta = \tan^{-1}\left( \frac{b}{c} \right). \]  

(3.6.3)

However, go back and examine Figure 3.5.11. It is clear from that graph and from Theorem 2.3.8 that the tangent function is not invertible which means that Equation 3.6.3 makes no sense.

There is still hope, though. In Section 2.3 we also discussed how it is possible to force a function which is not invertible to act like an
invertible function by restricting its domain. It turns out that all three trigonometric functions are invertible when we only consider angles which can live in a right triangle and in this section we endeavor to use inverse trigonometric functions to find angles in right triangles.

**Theory**

Go back and review the graphs of the cosine, sine, and tangent functions. These can be found in Figure 3.4.3a, Figure 3.4.3b, and Figure 3.5.11, respectively. It is clear from these graphs that none of these three functions are invertible. However, Figure 3.6.4 shows the graphs of these three functions on a domain of $(0^\circ, 90^\circ)$. These functions do not look as exciting on this smaller interval as they do on the entire domain of $\mathbb{R}$, but they are certainly invertible on this interval.

In fact, no periodic functions are invertible. Can you see why?

![Graphs of trigonometric functions restricted to $(0^\circ, 90^\circ)$](image)

Figure 3.6.4: The cosine, sine, and tangent functions restricted to a domain of $(0^\circ, 90^\circ)$.

As long as we restrict the domain of cosine, sine, and tangent to the interval $(0^\circ, 90^\circ)$ we can invert them. Remember, though, that in Section 3.3, Section 3.4, and Section 3.5 we made quite a fuss over the fact that the domain of trigonometric functions is the entire real line (or most of the real line in the case of the tangent function). Hence it would seem that writing $\cos^{-1}, \sin^{-1}, \tan^{-1}$ to represent inverses on this restricted domain is an abuse of notation. Unfortunately, we do it anyway. Some people write these functions as arccos, arcsin, and arctan, but we will not call them that in this text.

We already know how to make an inverse function, so it may seem that Definition 3.6.5 is superfluous. However, since we need to restrict the domain to make sense of inverses in this case, it is important to be precise.

(3.6.5) **Definition**.

- If $x$ is in the interval $(0, 1)$ then $\cos^{-1}(x)$ is the angle between $0^\circ$ and $90^\circ$ whose cosine is $x$. That is, $\cos^{-1}(x) = \theta$ if and only if $\cos(\theta) = x$ and $0^\circ < \theta < 90^\circ$. 
• If \( x \) is in the interval \((0,1)\) then \( \sin^{-1}(x) \) is the angle between 0° and 90° whose sine is \( x \). That is, \( \sin^{-1}(x) = \theta \) if and only if \( \sin(\theta) = x \) and \( 0^\circ < \theta < 90^\circ \).

• If \( x \) is in the interval \((0,\infty)\) then \( \tan^{-1}(x) \) is the angle between 0° and 90° whose tangent is \( x \). That is, \( \tan^{-1}(x) = \theta \) if and only if \( \tan(\theta) = x \) and \( 0^\circ < \theta < 90^\circ \).

There is a bit to discuss about Definition 3.6.5. The input of a standard trigonometric function is an angle and its output is a ratio. That is, if \( \theta \) is an angle then \( \cos(\theta) \) is the ratio of the side adjacent to \( \theta \) and the hypotenuse in a triangle. Conversely, the input of an inverse trigonometric function is a ratio and its output is an angle. It is important to remember that we use a standard trigonometric function when we have an angle and we use an inverse trigonometric function when we want to find an angle.

Also, note that \( \cos^{-1}(x) \) and \( \sin^{-1}(x) \) are defined for values of \( x \) on the interval \((0,1)\) whereas \( \tan^{-1}(x) \) is defined for values of \( x \) on the interval \((0,\infty)\). This is because, for angles of \( \theta \) in \((0^\circ,90^\circ)\), \( \cos(\theta) \) and \( \sin(\theta) \) output numbers in \((0,1)\) whereas \( \tan(\theta) \) output numbers that are arbitrarily large. It makes sense, then, that we should be able to solve \( \tan(\theta) = x \) for arbitrarily large values of \( x \) but only be able to solve \( \cos(\theta) = x \) and \( \sin(\theta) = x \) for values of \( x \) in \((0,1)\).

Before applying inverse trigonometric functions to right triangles, Example 3.6.6 performs some naive calculations.

<table>
<thead>
<tr>
<th>(3.6.6) Example</th>
<th>Compute the following:</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. ( \cos^{-1} \left( \frac{1}{2} \right) )</td>
<td></td>
</tr>
<tr>
<td>B. ( \sin^{-1} \left( \frac{\sqrt{2}}{2} \right) )</td>
<td></td>
</tr>
<tr>
<td>C. ( \tan^{-1} (1.5) )</td>
<td></td>
</tr>
</tbody>
</table>

Solution:

A. Remember that the outputs of \( \cos^{-1} \) are angles so that \( \cos^{-1} \left( \frac{1}{2} \right) \) should be an angle. If \( \cos^{-1} \left( \frac{1}{2} \right) = \theta \) then \( \theta \) should be an angle such that \( \cos(\theta) = \frac{1}{2} \). We know plenty of angles whose cosine is \( \frac{1}{2} \) but we need \( 0^\circ < \theta < 90^\circ \) and there is only such angle with a cosine of \( \frac{1}{2} \), which is \( \theta = 60^\circ \). Hence \( \cos^{-1} \left( \frac{1}{2} \right) = 60^\circ \).

**Answer:** \( \cos^{-1} \left( \frac{1}{2} \right) = 60^\circ \)
b. Similarly, to find $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ we need to find an angle $\theta$ with $0^\circ < \theta < 90^\circ$ such that $\sin(\theta) = \frac{\sqrt{2}}{2}$. If you remember from Section 3.3, the only such angle is $\theta = 45^\circ$ so $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$.

**Answer:** $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$

c. To find $\tan^{-1}(1.5)$ we must find an angle $\theta$ such that $\tan(\theta) = 1.5$. Unfortunately, this is not one of the angles for which trigonometric functions have exact outputs. Fortunately, though, your calculator should have a $\tan^{-1}$ button. Using this $\tan^{-1}$ button, we can calculate $\tan^{-1}(1.5) = 56.31^\circ$. You certainly can check this by calculating $\tan(56.31^\circ) = 1.5$.

**Answer:** $\tan^{-1}(1.5) = 56.31^\circ$

You calculator should have buttons for each of the three inverse trigonometric functions. All calculators are different, of course, but any calculator which can evaluate trigonometric functions should be able to evaluate inverse trigonometric functions, as well. However, you should also be able to recognize when inverse trigonometric functions will output the special angles discussed in Section 3.3. Now, the astute among us might have realized that the choice to restrict these trigonometric functions to the interval $(0^\circ, 90^\circ)$ in order to invert them was somewhat arbitrary. There are other choices of restrictions that would yield invertible functions. The value of restriction to $(0^\circ, 90^\circ)$ is that those are precisely the angles which show up in a right triangle. This will allow us to solve for angles in right triangles as discussed in the beginning of the section.

The next few examples explore how to use inverse trigonometric functions to solve for angles in a right triangle. These problems are all set up similarly to those in Section 3.2 and Section 3.5. However, instead of solving for one of the sides in the equation we will be using an inverse function to solve for one of the angles.

(3.6.7) **Example.** Find the value of $\theta$ in the triangle below.
Solution:

We still want to think about how trigonometric functions relate sides to angles. That is, the quantities involved in this triangle are the angle $\theta$, the side adjacent to $\theta$, and the side opposite to $\theta$. We know that

$$\tan(\theta) = \frac{30}{60}$$

or

$$\tan(\theta) = \frac{1}{2}. \quad (3.6.8)$$

Now, according to Definition 3.6.5, $\tan^{-1} \left( \frac{1}{2} \right)$ should be the angle between $0^\circ$ and $90^\circ$ whose tangent is $\frac{1}{2}$ which is precisely what $\theta$ should be. Hence

$$\theta = \tan^{-1} \left( \frac{1}{2} \right) = 26.57^\circ.$$  

**Answer:** $\theta = 26.57^\circ$

Note that we were really careful in the last part of Example 3.6.7. It is tempting to think that Equation 3.6.8 immediately implies that $\theta = \tan^{-1} \left( \frac{1}{2} \right)$, however Definition 3.6.5 requires a little bit more. If $f$ is any generic invertible function and $f(x) = y$ then we know that $f^{-1}(y) = x$ but the tangent function isn’t actually invertible except on a restricted domain and the $\tan^{-1}$ function does not always work as an inverse of tan.

The moral of the story here is that $\tan(\theta) = \frac{1}{2}$ implies that $\theta = \tan^{-1} \left( \frac{1}{2} \right)$ only if we already know that $0^\circ < \theta < 90^\circ$. In Example 3.6.7 it is obvious that $0^\circ < \theta < 90^\circ$ since $\theta$ is an angle in a right triangle. In this section we will only be dealing with angles in right triangles so it is not important to perform this check, but in future sections things will get more complicated and it is important to realize that it is not always possible to use inverse trigonometric functions as we expect inverse functions to work.

(3.6.9) Example. Find the value of $\theta$ in the triangle below.
Solution:
Observe that we are given the lengths of the side opposite to $\theta$ and the length of the hypotenuse of the triangle and we are asked to find $\theta$. The trigonometric function which relates these three things is the cosine function. Hence

$$\cos(\theta) = \frac{8}{10} = \frac{4}{5}.$$  

Since $\theta$ is an angle in a right triangle we have that

$$\theta = \cos^{-1}\left(\frac{4}{5}\right) = 36.87^\circ.$$  

**Answer:** $\theta = 36.87^\circ$

---

(3.6.10) Example. Find the value of $\theta$ in the triangle below.

\[ \begin{array}{c}
16 \\
\theta \\
20 \\
\end{array} \]

Solution:
From the diagram we see that

$$\sin(\theta) = \frac{16}{20} = \frac{4}{5}$$

and, since $\theta$ is an angle in a right triangle, we have

$$\theta = \sin^{-1}\left(\frac{4}{5}\right) = 53.13^\circ.$$  

**Answer:** $\theta = 53.13^\circ$

---

(3.6.a) Practice Exercise. Find the value of $\theta$ in the triangle below.
Applications

The applications in this section are straightforward. In each we try to set up a triangle and then use an inverse trigonometric function to solve for the desired angle. Like the examples earlier in the section, the setup of the applications will look a lot like those in Section 3.2 and Section 3.5. The difference in this section will be that that we use the sides of the triangle to find an angle instead of using an angle to find the lengths of the sides.

Example. A ladder leans against the side of a house. The ladder is 12 ft long and it reaches a height of 11.5 ft high on the wall. Find the angle the ladder makes with the ground.

Solution:
The geometry of the situation is shown below.

We know the length of the ladder, we know the distance along the wall between the ground and the ladder, and we can assume that the wall makes a right angle with the ground (as we have done before). Hence we have that

\[ \sin(\theta) = \frac{11.5}{12} \]

and, since \( \theta \) is an angle in a right triangle,

\[ \theta = \sin^{-1}\left(\frac{11.5}{12}\right) = 73.4^\circ. \]
(3.6.12) **Example.** The Oregon Ducks’ quarterback is standing on the field at the 20 yard line, which is 30 yd from the nearest goal post. The quarterback wants to throw a football and hit the very top of the goal post, which is 10 yd off of the ground. Assuming that he can throw the ball in a perfectly straight line, at what angle with the ground must he throw the ball? *Note: Goal posts form a right angle with the ground.*

**Solution:**

As always, we draw a picture.

The angle $\theta$ in the diagram is the one for which we are looking. From the diagram we see that

$$\tan(\theta) = \frac{10}{30} = \frac{1}{3}$$

and, since $\theta$ is an angle in a right triangle,

$$\theta = \tan^{-1}\left(\frac{1}{3}\right) = 18.43^\circ.$$

**Answer:** $\theta = 18.43^\circ$

Make sure that you are careful in the setup of your picture and the triangle that you use. Remember that if you find yourself trying to evaluate $\cos^{-1}(x)$ or $\sin^{-1}(x)$ for a value of $x$ which is bigger than 1 then you’ve made a mistake somewhere. It is, however, perfectly acceptable to evaluate $\tan^{-1}(x)$ when $x$ is bigger than 1. And, as always, check to see if the answer that you got makes sense in the context of the problem.
(3.6.b) Practice Exercise. John is sledding down the side of a hill. His sled starts at an elevation of 50 m above the base of the hill. If his sled travels 95 m before reaching the base of the hill (that is, 95 m measured parallel to the slope of the hill), find the (acute) angle that the hill makes with the horizontal.
3.6.1. Compute the following and leave your answers in exact form.

A.) \(\sin^{-1}\left(\frac{1}{2}\right)\)
B.) \(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\)
C.) \(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\)
D.) \(\tan^{-1}(\sqrt{3})\)
E.) \(\tan^{-1}(1)\)

3.6.2. Compute the following and leave your answers in exact form.

A.) \(\cos^{-1}(\sin(60^\circ))\)
B.) \(\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)\)
C.) \(\sin(\sin^{-1}(0.34))\)
D.) \(\sin^{-1}(\sin(120^\circ))\)
E.) \(\cos^{-1}(\cos(-63^\circ))\)

3.6.3. Find \(\theta\) below. Round your answer to two decimal places.

3.6.4. Find \(\theta\) below. Round your answer to two decimal places.

3.6.5. Find \(\theta\) below. Round your answer to two decimal places.

3.6.6. Find \(\theta\) below. Round to two decimal places.

3.6.7. The basepaths on a baseball diamond are 90 ft. On a particular pitch, a baserunner tries to steal second base. The runner is 40 ft away from first base when the catcher catches the pitch. At this moment what angle is the line between the catcher and the baserunner making with the first base line? Assume that the catcher is exactly on the intersection of the first and third base lines and assume that the baserunner is exactly on the line between first and second base. Round your answer to two decimal places. *Hint: It is less than 45°.*

3.6.8. The grade of a road is calculated from its vertical rise per 100 ft. For example, if a road is sloped such that it climbs 5 ft for every 100 ft of horizontal distance traveled then its grade is calculated to be \(\frac{5}{100} = 0.05\). If a particular road has a grade of 0.09, what is its angle of incline? Round your answer to two decimal places.

3.6.9. Christina is launching a model rocket. For safety reasons she clears a circle of radius 50 m around the launch point. When she launches the rocket she launches it at a slight angle so that when the rocket passes the edge of the circle it is 250 m in the air. At what angle was the rocket launched? Make sure that your angle is measured from the ground (that is, your angle should be between 0° and 90°). Round to two decimal places.
3.6.10. A rope helps anchor a flag pole. The rope is tied to the top of the flagpole at one end and is anchored into the ground at the other end. The flag pole is 100 ft tall and the rope is 125 ft long. What angle does the rope make with the flag pole? Round to two decimal places.

3.6.11. Stevie’s kite has a string which is 100 m long and, at a particular moment, it is 70 m off of the ground. What angle does the kite string make with the ground? Round to two decimal places.

3.6.12. Two birds - Dave and Jonas - fly out of their nest in a tree. Dave flies 4 mi south and Jonas flies 7 mi east. Later, Dave wants to fly to Jonas.

(a) How far will Dave need to fly? Round your answer to two decimal places.

(b) Dave needs to fly roughly northeast to get to Jonas. At what angle north of east does he need to fly? Round your answer to two decimal places. (This angle is shown as θ below.)

3.6.13. Find the (acute) angle that the function \( f(x) = 2x + 3 \) makes with the horizontal. Round to two decimal places.

Challenge Problems

(3.6.c1) Challenge Problem. Assume that \( 0 \leq x \leq 1 \). Write an expression for each of the following that does not involve any trigonometric functions or inverse trigonometric functions.

A. \( \sin(\cos^{-1}(x)) \)
B. \( \tan(\cos^{-1}(x)) \)
C. \( \sin(\tan^{-1}(x)) \)
D. \( \cos(\tan^{-1}(x)) \)
E. \( \cos(\sin^{-1}(x)) \)
F. \( \tan(\sin^{-1}(x)) \)
Chapter 3 developed trigonometry in triangles. We used right triangles in the unit circle to develop trigonometric functions and we used those trigonometric functions to study triangles. In this chapter we take a closer look at those trigonometric functions. This chapter covers the content of Chapter 8 in *College Algebra and Trigonometry for the University of Oregon*, by Connally et al. [4]
4.1 RADIANS AND ARC LENGTH

This section corresponds to section 8.1 in Functions Modeling Change [4]

This section covers an alternate unit used to measure angles. These units are called radians. Though degrees are more often used in practice, radians are much more convenient in mathematics. Unfortunately, it is not possible at this level to extoll all of the virtues of radians. The first place where the benefit is blindingly obvious is in calculus, but for now try to take our word for it.

**Motivation**

For now, there is one place where we can see the benefit of radians. **Lemma 4.1.1** reminds us how to find the circumference and area of a circle.

(4.1.1) **Lemma.** The circumference of a circle of radius \( r \) is \( 2\pi r \) and its area if \( \pi r^2 \).

Consider a circular walking track with a radius of 80 m. One lap around this track will be a distance of \( 2\pi \cdot 80 = 503 \) meters. What if we only walk 40 \% of the way around the track? Obviously that would be 40 \% of the total distance around the track, or \( 0.4(2\pi \cdot 80) = 201 \) meters. Now imagine that we go 120° around the circle. How far would that be? This distance is shown as \( S \) in **Figure 4.1.2**.

![Figure 4.1.2: An angle of 120° traversed in a circle of radius 80 m.](image)

Well, a complete lap around the track would be 360°, so 120° around the track would be \( \frac{120}{360} \) of the way around the circle and hence

\[
S = \frac{120}{360}(2\pi \cdot 80) = \frac{120\pi}{180}(80).
\]

Similarly, if we travel \( \theta^\circ \) around the track then we have gone a distance of

\[
S = \frac{\theta^\circ \pi}{180}(80).
\]
This is a reasonably simple calculation. However, we will define radian measure of angles so that $120^\circ$ is the same as $\frac{120\pi}{180}$ radians. According to this convention, if we travel an angle of $\theta$ around the track then we have gone a distance of

$$S = r\theta,$$

which is a more convenient equation. Of course, this only works if $\theta$ is measured in radians.

**Theory**

We first define radian measure. We now have to be very careful with our angles. If $\theta$ is an angle then we now have two different ways of measuring it. Until this point we’ve been identifying angles with their degree measure. From this point forward, though, we will have two different units that we can use.

**Definition**. Let $\theta$ be an angle. If its degree measure is $D$ then we define its *radian* measure by

$$D \frac{\pi}{180}.$$

Conversely, if its radian measure is $R$ then its degree measure can be found by

$$R \frac{180}{\pi}.$$

Conversion is relatively straightforward. An angle of $20^\circ$ is

$$20 \frac{\pi}{180} = 0.35$$

radians. An angle of 3 radians is

$$3 \frac{180}{\pi} = 171.89$$

degrees. No units are used when describing radian measure. That is, to say that an angle $\theta$ has a measure of 1 degree we write $\theta = 1^\circ$, but to say that an angle of $\theta$ has a measure of 1 radian we write $\theta = 1$. From this point forward, any time the degree symbol is left off of an angle it should be assumed that the angle is being measured in radians. Example 4.1.4 is meant to help you practice converting from degrees to radians and from radians to degrees.
(4.1.4) **Example.**

A. Convert $60^\circ$ from degrees to radians.

B. Convert $100^\circ$ from degrees to radians.

C. Convert $135^\circ$ from degrees to radians.

D. Convert $5$ from radians to degrees.

E. Convert $\frac{7\pi}{9}$ from radians to degrees.

**Solution:**

These are straightforward using **Definition 4.1.3**.

A. An angle of $60^\circ$ is

$$60 \frac{\pi}{180} = \frac{\pi}{3} = 1.05$$

radians.

**Answer:** $\frac{\pi}{3}$

B. An angle of $100^\circ$ is

$$100 \frac{\pi}{180} = \frac{5\pi}{9} = 1.75$$

radians.

**Answer:** $\frac{5\pi}{9}$

C. An angle of $135^\circ$ is

$$135 \frac{\pi}{180} = \frac{3\pi}{4} = 2.36$$

radians.

**Answer:** $\frac{3\pi}{4}$

D. An angle of $5$ radians is

$$5 \frac{180}{\pi} = \frac{900}{\pi} = 286.48$$

degrees.

**Answer:** $286.48^\circ$

E. An angle of $\frac{7\pi}{9}$ radians is

$$\frac{7\pi}{9} \cdot \frac{180}{\pi} = 140$$

degrees.

**Answer:** $140^\circ$
For reference, the degree and radian measure of some important angles are shown in Table 4.1.5.

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>π/6 ≈ 0.52</td>
</tr>
<tr>
<td>45°</td>
<td>π/4 ≈ 0.79</td>
</tr>
<tr>
<td>60°</td>
<td>π/3 ≈ 1.05</td>
</tr>
<tr>
<td>90°</td>
<td>π/2 ≈ 1.57</td>
</tr>
<tr>
<td>120°</td>
<td>2π/3 ≈ 2.09</td>
</tr>
<tr>
<td>135°</td>
<td>3π/4 ≈ 2.36</td>
</tr>
<tr>
<td>150°</td>
<td>5π/6 ≈ 2.62</td>
</tr>
<tr>
<td>180°</td>
<td>π ≈ 3.14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>210°</td>
<td>7π/6 ≈ 3.67</td>
</tr>
<tr>
<td>225°</td>
<td>5π/4 ≈ 3.93</td>
</tr>
<tr>
<td>240°</td>
<td>4π/3 ≈ 4.19</td>
</tr>
<tr>
<td>270°</td>
<td>3π/2 ≈ 4.71</td>
</tr>
<tr>
<td>300°</td>
<td>5π/3 ≈ 5.24</td>
</tr>
<tr>
<td>315°</td>
<td>7π/4 ≈ 5.50</td>
</tr>
<tr>
<td>330°</td>
<td>11π/6 ≈ 5.76</td>
</tr>
<tr>
<td>360°</td>
<td>2π ≈ 6.28</td>
</tr>
</tbody>
</table>

Table 4.1.5: Degree and radian measure of some important angles.

For the rest of this section we will focus on examples from other sections but with radians instead of degrees. Example 4.1.6 demonstrates how angles still add in radians as we would expect.

(4.1.6) Example. Find the angle θ below.

```
\[ \frac{\pi}{6} + \frac{2\pi}{3} + \theta = \pi \]
```

Solution:
We know that the interior angles of a triangle sum to 180°. In radians, 180° is an angle of \( \pi \). Hence, in radians, the interior angles of a triangle sum to \( \pi \). Hence

\[ \frac{\pi}{6} + \frac{2\pi}{3} + \theta = \pi \]

and

\[ \theta = \pi - \frac{\pi}{6} - \frac{2\pi}{3} = \frac{6\pi}{6} - \frac{\pi}{6} - \frac{4\pi}{6} = \frac{\pi}{6} \]

Answer: \( \theta = \frac{\pi}{6} \)
Trigonometric functions work as expected. That is, \( \cos(\theta) \) should be the \( x \)-coordinate of the inscription angle corresponding to \( \theta \) regardless of whether \( \theta \) is being measured in degrees or radians. That is, \( \cos(60^\circ) \) should be the same as \( \cos(\pi/3) \) since \( 60^\circ \) and \( \pi/3 \) are the same angle.

(4.1.7) Example. Find the following values exactly:

A. \( \cos(\pi/3) \)
B. \( \sin(3\pi/4) \)
C. \( \tan(3\pi/2) \)
D. \( \cos(-\pi/6) \)
E. \( \sin(35\pi/6) \)

Solution:
These can be calculated by drawing reference triangles as done in Section 3.3 if you’d like. However, we will calculate them by looking at the same angle in degrees.

A. \( \cos(\pi/3) = \cos(60^\circ) = \frac{1}{2} \)

Answer: \( \frac{1}{2} \)

B. \( \sin(3\pi/4) = \sin(135^\circ) = \frac{\sqrt{2}}{2} \)

Answer: \( \frac{\sqrt{2}}{2} \)

C. \( \tan(3\pi/2) = \tan(270^\circ) \) so \( \tan(3\pi/2) \) is undefined.

Answer: undefined

D. \( \cos(-\pi/6) = \cos(-30^\circ) = \frac{\sqrt{3}}{2} \)

Answer: \( \frac{\sqrt{3}}{2} \)

E. \( \sin(35\pi/6) = \sin(1050^\circ) = -\frac{1}{2} \)

Answer: \( -\frac{1}{2} \)

Calculators can also evaluate trigonometric functions using radians. Obviously you could simply convert angles to degrees before using your calculator, but your calculator should be able to handle radians. Earlier in Section 3.2 we discussed the importance of putting calculators in degree mode. It is now time to put them in radian mode. If you type the expression “\( \cos(30) \)” into your calculator, it will interpret the expression differently depending on which mode it’s using. If the calculator is in degree mode then it will assume that the \( 30 \) in
cos(30) should be 30 degrees whereas if it is in radian mode then it will assume that the 30 should be 30 radians.

\[ \text{(4.1.8) Example. Use your calculator to compute the following:} \]
\[ \text{A. } \sin(0.4) \]
\[ \text{B. } \cos(8) \]
\[ \text{C. } \tan(3.5) \]

**Solution:**

\[ \text{A. From a calculator, } \sin(0.4) = 0.39. \]
\[ \boxed{\text{Answer: } \sin(0.4) = 0.39} \]

\[ \text{B. From a calculator, } \cos(8) = -0.15. \]
\[ \boxed{\text{Answer: } \cos(8) = -0.15} \]

\[ \text{C. From a calculator, } \tan(3.5) = 0.37. \]
\[ \boxed{\text{Answer: } \tan(3.5) = 0.37} \]

You should keep your calculator in radian mode. From this point forward, all angles will be measured in radians. Every example and exercise that asks for an angle will want an answer in radians unless otherwise specified.

Of course, trigonometric functions still define the same relationships in right triangles.

\[ \text{(4.1.9) Example. Find } h \text{ below.} \]

\[ \text{Solution:} \]

\[ \text{Since the triangle is a right triangle, we have} \]
\[ \sin(4\pi/15) = \frac{h}{5} \]
\[ \text{so that} \]
\[ h = 5 \sin(4\pi/15) = 3.72. \]
The graphs of trigonometric functions change slightly when we use radians for their inputs. The shape remains the same, but the scale on the x-axis changes. Compare the graphs in Figure 4.1.10 with those in Figure 3.4.3. Since the shape of the graph doesn’t change when we change to radians, all the work that we did in Section 3.4 works the same when we change our units from degrees to radians.

(a) The graph of the function $y = \cos(x)$ where $x$ is in radians.

(b) The graph of the function $y = \sin(x)$ where $x$ is in radians.

Figure 4.1.10: The graphs of the sine and cosine functions with inputs in radians.

Inverse trigonometric functions will work just as we expect them to work. Regardless of whether we’re measuring angles in degrees or radians, $\sin^{-1}(0.5)$ should be the acute angle $\theta$ such that $\sin(\theta) = 0.5$. Writing both $\sin^{-1}(0.5) = 30^\circ$ and $\sin^{-1}(0.5) = \frac{\pi}{6}$ are perfectly valid since $30^\circ$ and $\frac{\pi}{6}$ are the same angle.

(4.1.A) Practice Exercise. Find $\theta$ and $x$ in the figure below.

The last theoretical topic we should mention is the idea of arc length with which we opened the section.
(4.1.11) Lemma. Consider a circle of radius $r$ and an angle $\theta$. An arc which traverses an angle of $\theta$ around the circumference of the circle has a length of $S$ where

$$S = r\theta,$$

assuming that the radian measure of $\theta$ is used in the above equation. This is shown in the diagram below.

Usage of Lemma 4.1.11 is relatively straightforward except that we need to ensure that the equation $S = r\theta$ uses the radian value of $\theta$. When an angle appears in a trigonometric function, it is irrelevant whether we use degrees or radians. However, whenever an angle appears in an equation outside of a trigonometric function it make a big difference. In mathematics it should always be assumed that the radian value of an angle is used in equations unless otherwise specified. The equation for arc length is no exception.

(4.1.12) Example. Find $S$ below.

Solution:
An angle of $200^\circ$ has a radian measure of

$$\frac{200 \pi}{180} = \frac{10\pi}{9}.$$
Hence
\[ S = 35 \frac{10\pi}{9} = 122.17 \]

from Lemma 4.1.11.

Answer: \( S = 122.17 \)

Certainly the usage of the arc length formula can be more involved, as shown in Example 4.1.13.

\begin{enumerate}
  \item \textbf{Example.} Round all answer to two decimal places.
  \item Find \( \theta \) in the figure below.
  \item Find \( x \) in the figure below.
  \item Find \( y \) in the figure below.
  \item Find \( w \) in the figure below.
\end{enumerate}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure}
\caption{Diagram for Example 4.1.13}
\end{figure}

\textit{Solution:}

\begin{enumerate}
  \item The radius of this circle is 10 so, from Lemma 4.1.11, we have
  \[ 20 = 10\theta \]
  \[ \theta = 2. \]

  Answer: \( \theta = 2 \)
  \item The point \( p \) is the point \((10 \cos(\theta), 10 \sin(\theta))\) and \( x \) is the absolute value of the first coordinate of \( p \), so
  \[ x = |10 \cos(\theta)| = |10 \cos(2)| = |-4.16| = 4.16. \]

  Answer: \( x = 4.16 \)
\end{enumerate}
c. Similarly, \( y \) is the absolute value of the second coordinate of \( p \), so
\[
y = |10 \sin(\theta)| = |10 \sin(2)| = |9.09| = 9.09.
\]

**Answer:** \( y = 9.09 \)

d. Using the Pythagorean Theorem, we have
\[
w^2 = y^2 + (x + 10)^2
\]
\[
w = \sqrt{9.09^2 + 14.16^2}
\]
\[
w = 16.83
\]

**Answer:** \( w = 16.83 \)

---

**Applications**

We first look at some applications using arc length.

(4.1.14) Example. A farmer has a circular fence which encloses an area of 30,000 \( \text{ft}^2 \). A portion of this fence which traverses an angle of 125° has been damaged in a storm. What length of fence will he need to use to replace the damaged section?

**Solution:**

The fence is circular and the area of the circle that it encloses is 30,000 \( \text{ft}^2 \). If the radius of this circle is \( r \) then

\[
30000 = \pi r^2
\]

so that

\[
r = \sqrt{\frac{30000}{\pi}} = 97.72
\]

and the radius of the circle is 97.72 ft. The damaged area traverses an angle of 125° around the circle. An angle of 125° is

\[
125 \frac{\pi}{180} = 2.18
\]

radians so that if the length of fence is \( \ell \) then

\[
\ell = (97.72)(2.18) = 213.18.
\]

Thus the farmer needs 213.18 ft of fence.

**Answer:** 213.18 ft
Example. Consider the circular walking track of radius 80 m from the beginning of the section.

A. Jerry walks 750 m around the track. What angle did he traverse during his walk?

B. Donna also goes on a walk during which she traverses an angle of 600°. How much farther was Donna’s walk than Jerry’s walk?

Solution:

A. If $\theta$ is the angle in question then

$$750 = 80\theta$$

so that

$$\theta = \frac{750}{80} = 8.375$$

Answer: 8.375

B. We need to find out how far Donna walked. We want to use the equation $S = r\theta$. An angle of 600° is

$$600 \cdot \frac{\pi}{180} = 10.47$$

radians. It follows that Donna walked $S$ meters where

$$S = (80)(10.47) = 837.76.$$ 

Since Jerry walked 750 m we have that Donna walked approximately

$$837.76 - 750 = 87.76$$

feet further than Jerry.

Answer: 87.76 ft

Practice Exercise. Two lighthouses are 100 mi apart. If you could draw a straight line from each lighthouse to the center of the earth, the two lines would meet at an angle of 1.4333°. Find the radius of the earth.

We end the section by practicing some applications from earlier sections with radians instead of degrees.
(4.1.16) Example. Find the equation of the line shown below. Round to two decimal places.

\[ y = 0.48x - 0.48 \]

Solution: 

The desired line has the form \( y = mx + b \). The slope of this line is given by \( m = \tan(\pi/7) = 0.48 \).

Since the line goes through the point \((1, 0)\) we have that

\[ 0 = 0.48 + b \]

so that \( b = -0.48 \). Hence \( y = 0.48x - 0.48 \).

Answer: \( y = 0.48x - 0.48 \)

(4.1.17) Example. Alexis flies a kite with a string of length 100 m. If the kite string makes an angle of 1.2 (radians) with the ground, find the altitude of the kite.

Solution: 

If the altitude of the kite is \( A \) then we have the following triangle:

We can see that \( A = 100 \sin(1.2) = 93.23 \) so that the kite is at an altitude of 93.2 m.

Answer: 93.2 m
Homework Assignment

4.1.1. Convert the following angles from degrees to radians. Give both an exact answer and an answer rounded to two decimal places.

A.) \(225^\circ\)
B.) \(300^\circ\)
C.) \(810^\circ\)
D.) \(10^\circ\)
E.) \(60^\circ\)
F.) \(231^\circ\)
G.) \(4370^\circ\)

4.1.2. Convert the following from radians to degrees:

A.) \(\frac{7\pi}{6}\)
B.) \(\frac{7\pi}{4}\)
C.) 5 (Round to two decimal places.)
D.) \(\pi\)
E.) \(\frac{5\pi}{6}\)
F.) 2 (Round to two decimal places.)
G.) \(\frac{17\pi}{5}\) (Round to two decimal places.)

4.1.3. Compute the following and leave your answers in exact form.

A.) \(\sin\left(\frac{3\pi}{4}\right)\)
B.) \(\cos\left(-\frac{\pi}{4}\right)\)
C.) \(\tan\left(\frac{5\pi}{6}\right)\)

4.1.4. Consider the following circle of radius 6 cm.

A.) Find the arc length traversed by \(\phi\) if \(\phi = \frac{3\pi}{5}\). Round your answer to two decimal places.
B.) Find the arc length traversed by \(\phi\) if \(\phi = 145^\circ\). Round your answer to two decimal places.

4.1.5. A perfectly circular lake has a diameter of 1000 m. If I walk 400° around the lake, how far have I walked? Round your answer to two decimal places.

4.1.6. A lake is perfectly circular with a radius of 200 m.

A.) On Monday I walked around the lake for 500 m. What angle did I traverse during that walk?
B.) During a walk around the lake on Tuesday I traversed an angle of 450°. How far did I walk on Tuesday?

4.1.7. If I drive 10.5 laps around a circular race track and during that drive I go 37 mi, what is the radius of the track? Round to two decimal places.

4.1.8. Elmer is on a Ferris wheel which has a diameter of 140 ft and whose center is 80 ft off the ground. If the angle that Elmer traverses over the course of the ride is 1500°, what total distance did he travel around the edge of the Ferris wheel?

4.1.9. Find \(\theta\) below.

4.1.10. The line drawn below goes through the points \((-1, -3)\) and \((2, 3)\). Find the angle \(\theta\) and round to two decimal places.
4.1.11. Find the perimeter of the shape shown below (assume that the arc is a section of a circle). Round your answer to two decimal places. Hint: The triangle is not a right triangle but it is an isosceles triangle. Try drawing a height.

### Challenge Problems

(4.1.c1) **Challenge Problem.** (This problem is not nearly as scary as it looks. Most of the work is just simple computation.) Radians are convenient in calculus. Derivatives will not be the subject of this exercise, but when we use radians as inputs for trigonometric functions we get

$$\frac{d}{dx}(\sin(x)) = \cos(x) \quad \text{and} \quad \frac{d}{dx}(\cos(x)) = -\sin(x).$$

If we assume that $x$ is in degrees, those equations become unpleasant. Radians are also convenient for calculators. Calculators do not naturally understand functions like sin, cos, log, and ln. When a calculator is programmed to calculate the outputs of such functions it must do so using only the operations of addition, subtraction, multiplication, and division. Hence we are forced to be clever when asking a calculator to evaluate complicated functions. The solution comes from calculus and it is called a Taylor Expansion. The idea is that a “smooth” function (whatever that means) can be approximated by a polynomial. Furthermore, this approximation can be as accurate as you want, provided that you are willing to use a large polynomial. In this exercise we will explore how to approximate the sine function with polynomials.

Some advanced notation will now be introduced. First, we define what is called a **factorial.** If $n$ is some integer then we define $n!$ (read “$n$ factorial”) to be the product of all positive integers which are no larger than $n$. For example, $3! = 3 \cdot 2 \cdot 1 = 6$ and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. As a matter of convenience we also define $0! = 1$. Now, assume that $h$ is any function. It is sometimes convenient to add values of $h$ over consecutive integers. For example, we might...
want to compute the sum of $h(k)$ where $k$ ranges over the integers from 0 to 4. This is a natural thing to do which has a complicated notation. Mathematicians would write
\[
\sum_{k=0}^{k=4} h(k) = h(0) + h(1) + h(2) + h(3) + h(4).
\]
The $\sum$ symbol means that things are going to be added. Whatever comes after the $\sum$ symbol is the thing that we will be adding (in this case $h(k)$). The $k = 0$ at the bottom and the $k = 4$ at the top means start with $h(k)$ when $k = 0$ and keep adding until you get to $h(k)$ when $k = 4$. This notation is not terribly helpful in this case, but when we are adding a lot of terms it is nice. For example, if we are considering the sum
\[
\sum_{k=17}^{k=148} h(k) = h(17) + h(18) + h(19) + \ldots + h(147) + h(148)
\]
than it is nice to be able to use the compact form of the new notation. It is also nice when we want to be able to vary how many of these things we want to add, and it is very useful when defining function.

For each integer $n$ with $n \geq 0$ define a polynomial $p_n$ such that
\[
p_n(x) = \sum_{k=0}^{k=n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.
\]
In this case, $h(k) = (-1)^k \frac{x^{2k+1}}{(2k+1)!}$. This is confusing. To help demonstrate, the polynomials $p_0$, $p_1$, $p_2$, and $p_3$ are shown below.

\[
p_0(x) = x
\]
\[
p_1(x) = x - \frac{x^3}{3!}
\]
\[
p_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}
\]
\[
p_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}
\]
Hopefully you can see the pattern. You can see that $p_{n+1}(x)$ is the same as $p_n(x)$ except that there is one extra term. The idea here is if we could imagine something like $p_\infty(x)$ where we add up all of the terms, then this crazy infinite polynomial would actually equal $\sin(x)$.

For the entirety of this problem, assume that $x$ is in radians when computing $\sin(x)$. Note that most scientific (or higher) calculators and mathematics-oriented computer software can compute factorials for you. The TI-83 has a function for factorials even though it does not have a factorial button.
A. First, make sure you understand how these polynomials work. (Do not use the \( \sum \) notation in these parts.)
   (i) Write down the polynomial \( p_4(x) \).
   (ii) Write down the polynomial \( p_5(x) \).
   (iii) Write down the polynomial \( p_6(x) \).

Feel free to leave factorials in your answer if you want. I definitely don’t want you to bother computing 13!.

B. The first of these polynomials - \( p_0(x) = x \) - looks nothing like \( \sin(x) \). However, it is not long before you can start to see how the polynomials look more and more like \( \sin(x) \) as \( n \) gets larger. We now want to convince ourselves of that fact.

   (i) Plot \( y = p_4(x) \) and \( y = \sin(x) \) on the same graph.
   (ii) Plot \( y = p_5(x) \) and \( y = \sin(x) \) on the same graph.
   (iii) Plot \( y = p_6(x) \) and \( y = \sin(x) \) on the same graph.

This stage of the exercise is not meant to be difficult. *I do not expect you to make these graphs by hand.* It is probably best to use a computer to make these graphs for you and print them out. WolframAlpha is a great resource, but there are other ways. Feel free to get help with this if you need it.

C. Fill out the following table. Round to four decimal places.

<table>
<thead>
<tr>
<th>x</th>
<th>( p_0(x) )</th>
<th>( p_1(x) )</th>
<th>( p_2(x) )</th>
<th>( p_3(x) )</th>
<th>( p_4(x) )</th>
<th>( \sin(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
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<tr>
<td>3</td>
<td></td>
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</tr>
<tr>
<td>4</td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*It is not advisable to do these computations by hand.* That is, it is probably not the best approach to use your calculator to compute each box one at a time. Find a systematic approach. The “table” feature of TI-8x calculators, Microsoft Excel, and the Spreadsheet functionality of Google Docs all work nicely for this. There are, of course, many other ways.

D. Imagine that you are in charge of programming a calculator.
   You know that you need to use one of these polynomials in place of the sine function. Use the table that you made to answer the following questions:
(i) What is the smallest polynomial which accurately calculates \( \sin(1) \) to three decimal places? That is, what is the smallest value of \( n \) such that \( p_n(1) \) is equal to \( \sin(1) \) when both are rounded to three decimal places?

(ii) What is the smallest polynomial which accurately calculates \( \sin(2) \) to three decimal places?

(iii) What is the smallest polynomial which accurately calculates \( \sin(3) \) to two decimal places? Hint: this one might not be on the table.

E. Let’s assume that you are trying to program your calculator so that it is accurate to four decimal places. After doing some work you find for values of \( x \) in \([-\pi, \pi]\) that the value of \( p_6(x) \) is equal to \( \sin(x) \) when rounded to four decimal places. However, for large values of \( x \) you find that \( p_6(x) \) is not even close to \( \sin(x) \). As an example, \( \sin(4\pi) = 0 \) but \( p_6(4\pi) = 14388.1256 \).

After playing with these functions for a while you find that any time you pick an interval and find a polynomial that is accurate on that interval, the polynomial fails miserably on some larger interval. This is a problem because you need to be able to compute \( \sin(\theta) \) for any value of \( \theta \). Suggest a strategy for using \( p_6 \) to calculate \( \sin(\theta) \) accurate to four decimal places for any value of \( \theta \). Hint: Use the fact that \( p_6 \) is accurate to four decimal places on \([-\pi, \pi]\) along with the geometry of the graph of the sine function.
4.2 NON-RIGHT TRIANGLES

This section corresponds to section 7.6 in Functions Modeling Change [4].

To this point we’ve focused on right triangles. In this section we look at other triangles. The two tools that we have for working with non-right triangles are called the Law of Cosines and the Law of Sines. Proofs of these things are not provided in this section, but the proofs rely heavily on the techniques that we have developed for right triangles. The reason that right triangles are such an important special case is because they are easier to analyze and because the techniques developed for right triangles can be generalized to other triangles.

Motivation

The motivation to study triangles which are not right triangles should be clear. We’ve developed a mastery of right triangles. If a right triangle provides enough information to determine it uniquely we can find the rest of its information. We would like to be able to do the same with other types of triangles.

It is important to understand what information is involved in a right triangle. Consider the right triangle shown in Figure 4.2.1. We know how to find the lengths of $a$ and $c$ using a combination of the Pythagorean Theorem and a trigonometric function.

![Figure 4.2.1: An arbitrary right triangle.](image)

Now, when we first look at the triangle in Figure 4.2.1, it looks as though we’re given two pieces of information: the angle of $\frac{\pi}{9}$ and the leg of length 2. In reality, though, we’re also given the fact that the triangle is a right triangle which tells us the measure of a second angle. Hence there are three known pieces of information about the triangle in Figure 4.2.1. In this section we develop techniques so that if we know three pieces of information about a non-right triangle then we can determine the rest (subject to some restrictions).

Theory

The results in this section are reasonably straightforward in their statements. However, they can be difficult to apply at times and it
can be difficult to know when to use each of them. We first introduce
the Law of Cosines in Theorem 4.2.2.

(4.2.2) Theorem (The Law of Cosines). Consider an arbitrary
triangle whose sides have length $a$, $b$, and $c$. Let $C$ be the
angle such that the side opposite to $C$ has length $c$. This is
shown in Figure 4.2.4. Then

$$c^2 = a^2 + b^2 - 2ab \cos(C).$$  \hspace{1cm} (4.2.3)

Figure 4.2.4: The figure accompanying Theorem 4.2.2.

There are two ways that Theorem 4.2.2 can be applied to determine
information about a triangle. The Law of Cosines relates the three
sides of a triangle and one of its angles, so if we know three of those
things we can find the fourth. Hence we can apply it in the following
two situations:

- If we know the lengths of all three of the sides of a triangle then
  we can use the law of cosines to find the measure of any of its
  angles.

- If we know the lengths of two of the sides of a triangle and the
  measure of one of its angles then we can use the law of cosines
  to find the length of the third side.

Examples of the Law of Cosines are shown in Example 4.2.5 through
Example 4.2.7
Solution:

Any time we use the Law of Cosines we need to involve all three sides of the triangle and one of the triangle’s interior angles. We first assign the quantities in our triangle to the variables involved in Theorem 4.2.2. Note that in Figure 4.2.4 the side $c$ is opposite to the angle $C$. When we assign our variables it is important that we observe this convention. Since our problem involves the angle $\theta$ we are forced to let $C = \theta$. Since $c$ needs to be opposite to $C$ we are forced to let $c = 4$. Once we’ve assigned $c$ and $C$, the other two sides of the triangle can be assigned to $a$ and $b$ arbitrarily. We let $a = 6$ and $b = 7$.

Now that we’ve assigned values to $a$, $b$, $c$, and $C$ we can relate them with Equation 4.2.3. Hence

$$c^2 = a^2 + b^2 - 2ab \cos(A).$$

After plugging in the variables and simplifying we have

$$4^2 = 6^2 + 7^2 - 2(6)(7) \cos(\theta)$$
$$16 = 36 + 49 - 84 \cos(\theta)$$
$$-69 = -84 \cos(\theta)$$
$$\frac{23}{28} = \cos(\theta).$$

Remember from Section 3.6 that in order to find $\theta$ using the inverse cosine function we need to know that $0^\circ < \theta < 90^\circ$. Since the problem dictates that $\theta$ is an acute angle, it follows that

$$\theta = \cos^{-1}\left(\frac{23}{28}\right) = 0.61.$$

**Answer:** $\theta = 0.61$

It was mentioned in Example 4.2.5 that the choice of $a$ and $b$ was arbitrary. If you have trouble seeing why either choice is correct, try reversing the choice and repeating this example. Note, also, in this example we required that $\theta$ was an acute angle so that we could use $\cos^{-1}\left(\frac{23}{28}\right)$. However, it was not necessary to stipulate in the problem that $\theta$ was acute; we could have deduced that information even if it were not stated explicitly. Since $\theta$ was opposite to the shortest side, Lemma 3.1.38 tells us that $\theta$ must be the smallest interior angle of the triangle. We also know that no triangle can have more than one obtuse angle, so $\theta$ must be an acute angle.

The tricky part of using the Law of Cosines is making sure that the right quantities go into the variables in Equation 4.2.3. The Law of
Cosines always involves all of the sides of the triangle and one of its angles. As mentioned in Example 4.2.5, it is important that the side we choose to be $c$ is opposite to the angle $C$. Once those two things are chosen, the sides which are chosen to be $b$ and $c$ are irrelevant.

In Example 4.2.5 we carefully labeled the sides of the triangle so that we could use the equation in Theorem 4.2.2. In future examples, though, we won’t use those labels and will simply use the quantities in the given triangle directly.

\[(4.2.6) \text{ Example. } \text{ Find } \ell \text{ below.}\]

\[
\begin{tikzpicture}
  \draw[thick] (0,0) -- (10,0) -- (14,0) -- cycle;
  \draw[fill=yellow!20] (0,0) -- (10,0) -- (14,0) -- cycle;
  \node at (5,0) {$\ell$};
  \node at (0,3) {10};
  \node at (14,3) {14};
  \node at (7,0) {$\frac{7\pi}{18}$};
\end{tikzpicture}
\]

\textbf{Solution:}

Notice that we are given the length of two sides, the measure of one angle, and we are asked to find the length of the third side. The Law of Cosines is applicable in this situation because we are relating the three sides of the triangle to one of its angles. The angle involved is $\frac{7\pi}{18}$ and $\ell$ is the side opposite to the angle of $\frac{7\pi}{18}$. (Thus, in this case, $c = \ell$ and $C = \frac{7\pi}{18}$.) Hence, after applying Theorem 4.2.2 and simplifying, we see

\[
\ell^2 = 10^2 + 14^2 - 2(10)(14) \cos\left(\frac{7\pi}{18}\right)
\]
\[
\ell^2 = 296 - 280 \cos\left(\frac{7\pi}{18}\right)
\]
\[
\ell^2 = 200.23
\]
\[
\ell = \pm \sqrt{200.23}
\]
\[
\ell = \pm 14.15.
\]

Of course, $\ell$ is the length of a side of a triangle so $\ell = 14.15$.

\textbf{Answer: } $\ell = 14.15$

\[(4.2.a) \text{ Practice Exercise. } \text{ Find } \ell \text{ in the triangle below.}\]
In Example 4.2.5 we were given the lengths of the sides of a triangle and were asked to find one of its angles. Using the Law of Cosines we were able to find the desired angle. That was because the side lengths of a triangle uniquely determine the triangle. In Example 4.2.6 we were given the length of two of the sides and the measure of the angle between them and were able to find the length of the third side. Again, the length of two of the sides of a triangle and the angle between them will uniquely determine everything about the triangle. However, as we will see in Example 4.2.7, two sides of a triangle and the measure of an angle which is not between them does not necessarily define a triangle uniquely.

(4.2.7) Example. Find x below. Assume that the side whose length is x is the longest side of the triangle.

\[ 6^2 = 5^2 + x^2 - 2(5)x \cos(\frac{2\pi}{9}) \]  
\[ 36 = 25 + x^2 - 10 \cos(\frac{2\pi}{9})x \]  
\[ 0 = x^2 - (10 \cos(\frac{2\pi}{9}))x + 11. \]
We see that Equation 4.2.9 asks for the roots of a second degree polynomial which we can find with the quadratic formula. Hence

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-(-12 \cos(2\pi/9)) \pm \sqrt{(-12 \cos(2\pi/9))^2 - 4(1)(11)}}{2(1)} \]

\[ x = 1.414 \quad \text{or} \quad x = 7.778. \]

Unfortunately, the equation in Theorem 4.2.2 yields two possible values for \( x \). However, we know that the side of the triangle whose length is \( x \) is the longest side so that \( x \) must be 7.778 as 1.414 < 5.

**Answer:** \( x = 7.778 \)

In Example 4.2.7 we found two possible values for \( x \). The problem statement told us that we needed the biggest value, but we should explore what both of those values represent. Consider the figure presented in the problem statement of Example 4.2.7. It gives us two side lengths and the length of an angle, but it tells us nothing about the angle between the side of length 6 and length 5. So, there is no reason to expect that the triangle looks anything like the one shown in the figure, or that the triangle is an obtuse triangle. In fact, there are two triangles with side lengths of 6 and 5 and an angle of \( \frac{2\pi}{9} \) adjacent to the side of length 6; one of which corresponds to \( x = 1.414 \) and the other of which corresponds to \( x = 7.778 \).

![Figure 4.2.10: The two possible triangles in Example 4.2.7.](image)

You can see in Example 4.2.7 that the angle at the top of the triangle was not specified so there is no reason that it needs to be an obtuse angle. The two values of \( x \) were obtained by solving Equation 4.2.9. Remember that an equation of the form \( ax^2 + bx + c = 0 \) sometimes has two solutions, sometimes it has one solution, and sometimes it has no solution at all. Getting two solutions means that there are two possible triangles like the two in Figure 4.2.10. If, in a different
example, you find that the Law of Cosines yields no solution, then there is simply no triangle possible with the dimensions stated. Any negative solutions can be ignored, as usual, since a negative number cannot be the length of a side of a triangle.

The other tool that we have in this section is the Law of Sines in Theorem 4.2.11.

**Theorem** (The Law of Sines). Consider a triangle with sides of length $a$ and $b$ and with angles of $A$ and $B$ such that the side of length $a$ is opposite to the angle $A$ and the side of length $b$ is opposite to the angle $B$. This is shown in Figure 4.2.13. Then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \quad (4.2.12)$$

![Figure 4.2.13: The figure accompanying Theorem 4.2.11.](image)

Observe that the Law of Sines involves two of the three sides of the triangle and two of the three angles in the triangle. It follows that the Law of Sines can be used in two situations:

- If we know the lengths of two of the sides and the measure of one of the angles in a triangle, then the Law of Sines can be used to find the measure of a second angle.

- If we know the length of one of the sides and the measure of two of the angles in a triangle, then the Law of Sines can be used to find the length of a second side.

These two situations are different from those where we use the Law of Cosines. We explore the Law of Sines in Example 4.2.14 through Example 4.2.17.
Solution:
We are given the lengths of two of the sides of this triangle and the measure of one of its angles. Since we’re looking for a second angle, the Law of Sines is appropriate. We will start by assigning the variables in Theorem 4.2.11 to our triangle. In Figure 4.2.13 we have that the side \(a\) is opposite to the angle \(A\) and that the side \(b\) is opposite to the angle \(B\). It is, again, important that we observe this convention whenever we apply the Law of Sines. We will let \(a = 4\) and \(b = 2\) which forces \(A = \frac{16\pi}{45}\) and \(B = \theta\) (compare the triangle in this example with Figure 4.2.13). Note that it is perfectly acceptable to swap \(a\) and \(b\) as long as you also swap \(A\) and \(B\) so that the appropriate sides are opposite to the appropriate angles. Then, using Theorem 4.2.11,

\[
\frac{\sin(A)}{a} = \frac{\sin(B)}{b}.
\]

After plugging in the variables and simplifying we have

\[
\frac{\sin(\frac{16\pi}{45})}{4} = \frac{\sin(\theta)}{2}
\]

\[
\sin(\theta) = \frac{1}{2} \sin(64^\circ)
\]

\[
\sin(\theta) = 0.45.
\]

Since \(\sin(\theta) = 0.45\) we want to say that \(\theta = \sin^{-1}(0.45)\). In order to do so, though, we need to know that \(\theta\) is an acute angle. Only the largest angle in a triangle can be obtuse and the largest angle must be opposite to the longest side. Since the side of length 2 cannot be the largest side, \(\theta\) cannot be the largest angle and hence \(\theta\) must be acute. It follows that

\[
\theta = \sin^{-1}(0.45) = 0.47.
\]

**Answer:** \(\theta = 0.47\)

One must be very careful with examples like Example 4.2.14. Compare it with Example 4.2.5. Notice that in both of these examples we were given the length of two sides and the measure of one angle. However, in Example 4.2.5 we were asked to find the length of the third side which required the Law of Cosines while in Example 4.2.14 we were asked to find the measure of a second angle which required the Law of Sines. When deciding between these two laws it is not sufficient to consider only the information which is given, but instead we need to consider all of the information involved in the problem. When we include the thing for which we are trying to solve, Example 4.2.5
involves three sides and an angle while Example 4.2.14 involves two sides and two angles.

Just like with the Law of Cosines, the equation involved in Theorem 4.2.11 is very picky about the assignment of the variables. In Example 4.2.14 we were precise about using the same variables used in Equation 4.2.12 but we will not do that in the future. When using the Law of Sines we will simply have two fractions of the form \( \frac{\sin(\theta)}{\ell} \) which are equal and it is important that the angle in the numerator is opposite to the side in the denominator.

\[
\begin{align*}
(4.2.15) \text{ Example.} & \quad \text{Find } \ell \text{ in the triangle below.} \\
\end{align*}
\]

\[
\begin{align*}
\text{Solution:} & \\
& \text{Since the problem involves two angles and two sides, the Law of Sines is the correct choice. Observe that the angle of 0.54 is opposite to the side of length } \ell \text{ and the angle of 2 is opposite to the side of length 7. Hence from Theorem 4.2.11 we have} \\
& \frac{\sin(0.54)}{\ell} = \frac{\sin(2)}{7} \\
& \ell = \frac{7 \sin(0.54)}{\sin(2)} \\
& \ell = 3.98
\end{align*}
\]

\[\text{Answer: } \ell = 3.98\]

There is one tricky thing that can happen when using the Law of Sines. We can be given two angles and two sides, but it could be that they are not appropriately opposite one another. We explore this possibility in the next three examples.

\[
\begin{align*}
(4.2.16) \text{ Example.} & \quad \text{Find } \ell \text{ in the triangle below.} \\
\end{align*}
\]
Solution:
This problem involves two angles and two sides, which means that we will be using the Law of Sines. Note that the angle of \( \frac{3\pi}{20} \) is across from the side of length \( \ell \) but the angle of \( \frac{5\pi}{18} \) is not across from the side of length 14. Hence it is not the case that

\[
\frac{\sin\left(\frac{3\pi}{20}\right)}{\ell} = \frac{\sin\left(\frac{5\pi}{18}\right)}{14}.
\]

The trick is that we know the third angle in the triangle, as well. The top angle has measure \( \frac{103\pi}{180} \) and that angle is opposite to the side of length 14 so

\[
\frac{\sin\left(\frac{3\pi}{20}\right)}{\ell} = \frac{\sin\left(\frac{103\pi}{180}\right)}{14}
\]

by Theorem 4.2.11, and hence

\[
\ell = \frac{14 \sin\left(\frac{3\pi}{20}\right)}{\sin\left(\frac{103\pi}{180}\right)} = 6.52.
\]

**Answer:** \( \ell = 6.52 \)

---

(4.2.17) Example. Find \( \theta \) in the triangle below. Assume that it is an acute triangle.

![Triangle with sides 9, 10, and \( \theta \)]

Solution:
The problem involves two sides and two angles, so the Law of Sines is the appropriate tool. We see that the angle of 0.92 is opposite to the side of length 9 but the side of length 10 is not opposite to \( \theta \). However, the angle at the top of the triangle is across from the side of length 10. Call the angle at the top of the triangle \( \alpha \). Using Theorem 4.2.11 we have

\[
\frac{\sin(\alpha)}{10} = \frac{\sin(0.92)}{9}.
\]
Of course, the question doesn’t ask us for $\alpha$ but it is the one angle that we can find and it will help us find $\theta$. It follows that

$$\sin(\alpha) = \frac{10\sin(0.92)}{9} = 0.88$$

Since the triangle is acute, $\alpha$ must be acute and hence

$$\alpha = \sin^{-1}(0.88) = 1.08.$$

Again, we wanted to find the measure of $\theta$. Since the sum of the interior angles in a triangle add to $\pi$, we have

$$\theta = \pi - 0.92 - \alpha = \pi - 0.92 - 1.08 = 1.14.$$  

**Answer:** $\theta = 1.14$

(4.2.18) **Example.** Find the angle $\theta$ below.

![Diagram](image)

**Solution:**

Since the problem involves two angles and two sides we will be using the Law of Sines. However, this seems to be the toughest case of the Law of Sines. Notice that the angle we know is not opposite to either of the sides we know. Let $\phi$ be the third angle in the triangle and let $\ell$ be the length of the third leg. By choosing different pairs of angles, we can get three different equations from Theorem 4.2.11:

$$\frac{\sin(\phi)}{9} = \frac{\sin(\theta)}{8},$$

$$\frac{\sin(\theta)}{8} = \frac{0.66}{\ell},$$

and

$$\frac{\sin(0.66)}{\ell} = \frac{\sin(\phi)}{9}. \quad (4.2.19)$$

However, each of those three equations have two unknown quantities in them which means we cannot use any of the equations to solve for any of the variables at this point. However, there is one other thing that we can do; using the Law of Cosines, we can find $\ell$. The problem statement did not ask us
to find $\ell$ but we are unable to find any of the angles with the information given and if we knew $\ell$ then the second line in Equation 4.2.19 will give us $\theta$. Using Theorem 4.2.2 we have

$$\ell^2 = 9^2 + 8^2 - 2(9)(8) \cos(0.66)$$
$$\ell^2 = 31.24$$
$$\ell = 5.59.$$ 

Then, using the third equation in Equation 4.2.19 we have

$$\sin(\theta) = \frac{8 \sin(0.66)}{\ell} = \frac{9 \sin(0.66)}{5.59} = 0.99.$$ 

Since the side of length 8 is not the shortest side, we know that $\theta$ is an acute angle and hence

$$\theta = \sin^{-1}(0.99) = 1.41.$$ 

**Answer:** $\theta = 1.41$

---

(4.2.b) Practice Exercise. Find $\theta$ in the triangle below.

Recall the triangle that we examined in Example 4.2.7. However, suppose that we want to find the right-most angle in the triangle instead of the side, $x$. This is shown as $\theta$ in Figure 4.2.20. This time, using the Law of Sines, we have

$$\frac{\sin(\theta)}{6} = \frac{\sin(2\pi/9)}{5}$$

so that

$$\sin(\theta) = \frac{6 \sin(2\pi/9)}{5} = 0.77.$$ 

We would now like to use the inverse sine function to find $\theta$. However, in order to do that, we would need to know that $\theta$ is an acute angle but there is no reason to expect this to be the case. In fact, the discussion following Example 4.2.7 shows that there are two possible triangles corresponding to two different values of $\theta$. These two triangles are shown in Figure 4.2.10. One of the values of $\theta$ corresponds
to the acute angle \( \theta = \sin^{-1}(0.77) = 50.47^\circ \) but we have not yet developed the techniques to find the other value of \( \theta \). We will explore this situation more in future sections.

![Figure 4.2.20: The triangle from Example 4.2.7.](image)

The final example that we wish to explore is one where all three of the sides of a triangle are known and we wish to find all of the angles. In these examples, always find the angles in order of increasing measure. That is, find the smallest angles first and find the largest angle last. The last angle will be the easiest to find, and we’ve seen already in this section that there can be problems when the Law of Sines and the Law of Cosines involves angles that are possibly obtuse. The only angle which can possibly be obtuse is the largest angle, which is why we find it last.

\[(4.2.21) \text{ Example.} \text{ Find all of the angles in the triangle below.}\]

![Diagram](image)

\textit{Solution:}
We would like to find the smallest angle first. Recall that the smallest angle is opposite to the shortest side, hence we would like to find \( \beta \) first. Since we want to use three sides to find an angle, we need to use the Law of Cosines. Hence

\[
12^2 = 19^2 + 23^2 - 2(19)(23) \cos(\beta)
\]

\[
\cos(\beta) = \frac{12^2 - 19^2 - 23^2}{-2(19)(23)}
\]

\[
\cos(\beta) = 0.85
\]

Since we carefully chose \( \beta \) so that it is an acute angle, we know that

\[
\beta = \cos^{-1}(0.85) = 0.55.
\]

We next try to find the second-smallest angle, which is \( \alpha \). We know all of the lengths of the sides and the measure of one
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of the angles. Hence we have a choice to use either the Law of Sines or the Law of Cosines to find $\alpha$. You may choose either one of them, but here we will use the Law of Sines. Hence

$$\frac{\sin(\beta)}{12} = \frac{\sin(\alpha)}{19} \quad (4.2.22)$$

so that

$$\sin(\alpha) = \frac{19 \sin(\beta)}{12} = \frac{19 \sin(0.55)}{12} = 0.82.$$ 

Again, $\alpha$ was carefully chosen so that it is necessarily acute and hence

$$\alpha = \sin^{-1}(0.82) = 0.97^\circ.$$ 

When we used the Law of Sines to find Equation 4.2.22 we could also have written

$$\frac{\sin(\alpha)}{19} = \frac{\sin(\gamma)}{23}. \quad (4.2.23)$$

This is certainly valid mathematics and accurately relates the variables. However, in Equation 4.2.23 there are two unknowns - namely $\alpha$ and $\gamma$ - so it cannot be used to solve for anything. There is only one unknown in Equation 4.2.22, though, which allowed us to solve for $\alpha$, thus making it more valuable than Equation 4.2.23 in this case.

Now that we know two of the angles in the triangle, we can easily find the third since the interior angles of a triangle sum to $\pi$. Finally,

$$\gamma = \pi - \alpha - \beta = \pi - 0.97 - 0.55 = 1.62.$$ 

Note that $\gamma$ was, in fact, an obtuse angle, which is why it was good to save it until last.

**Answer:** $\alpha = 0.97$, $\beta = 0.55$, and $\gamma = 1.62$. 

---

**Applications**

This whole chapter has dealt with applications which use triangles to find distances and angles. In Section 3.2 and Section 3.5 we learned how to use trigonometric functions to find the lengths of the sides of a right triangle. In Section 3.6 we learned how to use inverse trigonometric functions to find the angles in right triangles. In each of those sections we looked at several applications. The limitation, though, was that these applications required us to use right triangles. We can
now handle all of those same applications in the cases which do not involve right triangles. We will now look at three such examples; the first of which is the example at the end of Section 3.1.

(4.2.24) Example. Two pilots are planning flights to two different cities from the same airport. They use a navigational plotter to determine that their flight paths are an angle of $\pi/3$ apart. One of the planes flies 140 mi before landing in the first city and the other flies 100 mi before landing in the second city as shown below. How far apart are the cities after they’ve both landed?

![Diagram](image)

Solution:
This application is easily handled with the Law of Cosines. The line between the two cities forms a triangle; call the length of this line $\ell$. Then, using Theorem 4.2.2, we have

$$\ell^2 = 100^2 + 140^2 - 2(100)(140)\cos(\pi/3)$$

$$\ell^2 = 15600$$

$$\ell = 124.90$$

and hence the cities are 124.90 mi apart.

[Answer:] 124.90 mi

(4.2.25) Example. A light fixture is hanging from the ceiling. It is held up by two cables which are bolted to the ceiling at one end and tied to the light fixture on the other. The shorter cable is 6 ft long and makes an angle of 0.58 with the ceiling. The longer cable is 8 ft long. Find the angle that the longer cable makes with the ceiling.
**Solution:**
As always, we start by drawing the picture. The angle of interest is $\theta$ below.

Now that we’ve drawn the image, we see immediately that the Law of Sines applies. Then, from Theorem 4.2.11 we have

$$\frac{\sin(\theta)}{6} = \frac{\sin(0.58)}{8}$$

so that

$$\sin(\theta) = \frac{6 \sin(0.58)}{8} = 0.41.$$  

It seems clear from the picture that $\theta$ is an acute angle. However, just to be sure, the side opposite to $\theta$ cannot be the longest side, so $\theta$ must be acute. Hence

$$\theta = \sin^{-1}(0.41) = 0.42.$$  

**Answer:** 24.11°

*(4.2.26) Example. Tim is practicing the shot put. He throws the first shot a distance of 18.1 m. He throws his second shot a distance of 21.3 m at an angle of $\pi/9$ from the first. How far apart are the two shots?*

**Solution:**
The geometry is shown in the triangle below where $d$ is the distance between the two shots.
It then becomes clear that the Law of Cosines will help us find $d$. Hence

$$d^2 = 18.1^2 + 21.3^2 - 2(18.1)(21.3)\cos(\pi/9)$$

$$d = \sqrt{18.1^2 + 21.3^2 - 2(18.1)(21.3)\cos(\pi/9)}$$

$$d = 7.53$$

so that the two shots are 7.53 ft from each other.

**Answer:** 7.53 ft

---

(4.2.27) **Example.** A surveyor is trying to find the height of a mountain. From where he is initially standing, his line of sight with the top of the mountain makes an angle of $\theta = 0.414$ with the ground. He then steps back 1000 ft from the mountain and measures that his line of sight with the top of the mountain makes an angle of $\phi = 0.385$ with the ground. How tall is the mountain?

**Solution:**

Let’s start by drawing a picture (which is not drawn to scale). The desired height is labeled $h$.

![Diagram](image)

We only know one length in the entire picture. Let’s start with the triangle which contains that length, as shown below.

![Diagram](image)

We know that $\alpha = \pi - \theta = 2.728$ and $\beta = \pi - \phi - \alpha = 0.029$. Certainly we can find either of the two missing sides using the Law of Sines; each will allow us to determine the
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height of the mountain. We will focus on finding the side labeled \( \ell \). Using Theorem 4.2.11 we have

\[
\frac{\sin(0.385)}{\ell} = \frac{\sin(0.029)}{1000}
\]

and hence

\[
\ell = \frac{1000 \sin(0.385)}{\sin(0.029)} = 12952.13.
\]

It is more important than usual in this example to be careful with rounding. Since \( \sin(0.029) \) is so small (it is approximately 0.029), it is important that we do not round until the very end of the problem. Approximations will be shown at each step, but in this solution we do not perform any rounding until the final answer.

Now that we know \( \ell \), we can look at the small triangle which contains \( \ell \) and \( h \), as shown below.

![Diagram of a right triangle with labels \( \ell \) and \( h \)]

This triangle is drawn as a right triangle because the height of the mountain should be measured perpendicular to the ground. We know two of the angles and two of the sides of this triangle so we could use the Law of Sines to find \( h \) if we wanted. However, since we have a right triangle in this situation we have

\[
\sin(\theta) = \frac{h}{\ell}
\]

so that

\[
h = \ell \sin(\theta) = 12952.13 \sin(0.414) = 5210.31.
\]

Finally, the mountain is 5210.31 ft tall.

[**Answer:** 5210.31 ft]  

(4.2.c) **Practice Exercise.** Kevin leaves his house and walks 3500 m in a straight line. He then stops, rotates to the left by an angle of \( \frac{7\pi}{18} \), and then walks another 2250 m in a straight line. At this point, how far is he from his house?
4.2.1. Find \( x \) in the triangle below. Round your answer to two decimal places.

4.2.2. Find \( \ell \) below. Round your answer to two decimal places. *Note: This image is not drawn to scale.*

4.2.3. Find \( y \) in the triangle below. Round your answer to two decimal places.

4.2.4. Find \( z \) in the triangle below assuming that all of the internal angles of the triangle are acute. Round your answer to two decimal places.

4.2.5. Find \( \theta \) in the triangle below. Round your answer to two decimal places.

4.2.6. Find \( \ell \) below given that \( \theta = 1.91 \) and \( \phi = 0.51 \). Round your answer to two decimal places.

4.2.7. Find all three angles in the triangle below. Round your answers to two decimal places.

4.2.8. Find all of the angles in the triangle below. Round your answers to two decimal places.

4.2.9. Find all of the angles in the triangle below. Round your answer to two decimal places.
4.2.10. Verify that the triangle below is a right triangle by finding the missing side and the two missing angles.

\[ \begin{align*}
\frac{\pi}{2} & \quad 6 \\
3 & \quad \ell \\
\theta & \quad \phi
\end{align*} \]

4.2.11. A ship leaves the harbor, travels 5 mi north and then turns and travels 2 mi north east. How far is the ship from the harbor at that point (in a straight line)? Round your answer to two decimal places. Hint: What is the angle between north and north east?

4.2.12. Cindy and Katie both leave school at the same time on their bikes and they travel in straight lines which meet at an angle of \( \frac{7\pi}{9} \). After ten minutes Cindy is 3 mi from school and Katie is 2.2 mi from school. How far apart are they after ten minutes? Round your answer to two decimal places.

4.2.13. I set sail from my dock and head in a straight line for 7 mi. The wind then picks up momentarily and I am forced to change direction by \( \frac{5\pi}{16} \). I then sail in a straight line for another 4 mi. At this point, how far am I from the dock? Round to two decimal places.

4.2.14. A particular clock has an hour hand with a length of 5 in and a minute hand with a length of 8 in. If the two hands are currently meeting at \( \frac{23\pi}{16} \), find the distance between their tips. Round to two decimal places.

4.2.15. A rope is helping to anchor a tall, thin sign post. The rope is 20 ft long, it is tied to the very top of the post, and it makes an angle of \( \frac{3\pi}{16} \) with the ground. An engineer wants to use another rope on the opposite side of the post to add more support. If the second rope is 25 ft long and is also to be tied to the top of the post, find the angle that the rope will make with the ground. Round to two decimal places.

4.2.16. A telephone pole is tilted at an angle of \( \frac{\pi}{15} \) toward the sun. At a particular moment, the post casts a shadow whose length is 30 m. The line between the tip of the shadow and the tip of the pole makes a \( \frac{2\pi}{3} \) angle with the ground. Find the length of the pole. Round to two decimal places.

4.2.17. Consider the diagram below.

\[ \begin{align*}
\frac{2\pi}{9} & \quad 13\pi \\
2 & \quad \ell \\
\theta & \quad d
\end{align*} \]

A.) Find \( \theta \). Leave your answer in exact form.

B.) Find \( d \). Round your answer to two decimal places.

C.) Find \( \ell \). Round your answer to two decimal places.

4.2.18. Consider the following diagram:

\[ \begin{align*}
2.75 & \quad \phi \\
\theta & \quad m \\
2 & \quad \ell \\
3 & \quad 5
\end{align*} \]

A.) Find \( \ell \) and round to two decimal places.

B.) Find \( \theta \) and round to two decimal places.
c.) Find \( m \) and round to two decimal places.

d.) Find \( \phi \) and round to two decimal places.

---

**Challenge Problems**

(4.2.c1) **Challenge Problem**. The Law of Sines and the Law of Cosines are generalizations of the rules that we’ve established for right triangles. In this exercise we’re going to explore those relationships. Consider the triangle shown below:

![Triangle Diagram]

For the sake of this exercise, pretend that you don’t know anything about the Pythagorean Theorem and the equations developed in Section 3.2 and Section 3.5. Use only the Law of Cosines and the Law of Sines to show the following relationships:

\[
\begin{align*}
  c^2 &= a^2 + b^2 \\
  \sin(\theta) &= \frac{b}{c} \quad \cos(\theta) = \frac{a}{c} \quad \tan(\theta) = \frac{b}{a}
\end{align*}
\]

Carefully justify your work.
4.3 SINUSOIDAL FUNCTIONS

This section corresponds to section 8.2 in Functions Modeling Change [4]

Imagine the height of a bouncing ball, the length of a spring going back and forth, or the height of water sloshing back and forth in a glass. Imagine the height of a rider on a Ferris wheel, the average temperature of a city over time, or the height of the tide. The commonality in these things is that they all oscillate evenly between two values. A convenient way to model situations like this is with a transformation of the sine function.

Motivation

Let’s go back, yet again, to the example that we explored in the beginning of Section 2.4 and Section 3.4. Someone is sitting on a Ferris wheel with a radius of 80 ft, whose center is 100 ft off of the ground. This is shown again in Figure 4.3.1.

We learned in Section 3.4 that when the rider’s carriage makes an angle of θ with the horizontal, his height is h(θ) feet where

\[ h(θ) = 80 \sin(θ) + 100. \]

This is a useful equation. However, in practice, it is more interesting to know the rider’s position as a function of time. That is, considering the Ferris wheel’s speed and the rider’s starting position, we would like to make a function H such that t seconds into the ride, the rider’s height is H(t).

Suppose that this Ferris wheel takes one minute to complete a revolution and suppose that the rider is sitting in the starting position shown in Figure 4.3.1. As long as we’re given specifics like that, this section explores how to find the function H. Of course, the Ferris wheel is rotating at a constant rate which means that the rider’s an-
gle will be a linear function of time. As we will show later in the section,

\[ H(t) = 80 \sin \left( \frac{\pi}{30} (t - 5) \right) + 100. \]

This equation is one of a general class of functions of the form

\[ f(x) = A \sin(B(x - h)) + k \]

which we will come to call sinusoidal functions.

**Theory**

Roughly speaking, a sinusoidal function is a function which basically looks like the sine or cosine functions. That is, a sinusoidal function is a transformation of the sine or cosine function.

### Definition

A function \( f \) is sinusoidal if it has the form

\[ f(x) = A \sin(B(x - h)) + k \]

for real numbers \( A, B, h, \) and \( k. \)

We wanted sinusoidal functions to look like either sine functions or cosine functions and we defined sinusoidal functions to be a transformation of the sine function. If you look at the graphs of the sine and cosine functions you’ll notice that they are the same graphs up to a horizontal translation. Hence it’s enough to consider only transformations of one of the sine or cosine functions. Here we use the sine function, but it is really an arbitrary choice.

Let’s start by looking at some examples. Figure 4.3.4 shows graphs of the following functions:

\[
\begin{align*}
f(x) &= 2 \sin \left( \frac{2\pi}{3} (x - 3) \right) + 1 \\
g(x) &= 4 \sin \left( \frac{2\pi}{5} (x + 2) \right) - 1 \\
p(x) &= \sin \left( 8 \left( x - \frac{1}{4} \right) \right) - 3 \\
q(x) &= 3 \sin \left( \frac{1}{2} (x - 7) \right) + 2
\end{align*}
\]

Remember, of course, that these \( x \)-values represent radian measure of an angle. You can see how the general shape of these functions resemble the shape of the sine function.

Sinusoidal functions have four parameters. In the sinusoidal function

\[ f(x) = A \sin(B(x - h)) + k \]
the four parameters are \( A, B, h, \) and \( k. \) We already understand two of them. If \( B = 1 \) and \( h = 0 \) then we have \( f(x) = A \sin(x) + k. \) In that case, we already know that the amplitude of \( f \) is \( A \) and the midline is \( y = k. \) Since \( B \) and \( h \) are horizontal shifts and since the amplitude and midline are vertical properties, even for arbitrary sinusoidal functions the amplitude is \( A \) and the midline is \( y = k. \)

The other main feature of a periodic function was the period. We would expect a transformation of a periodic function still to be periodic and, in fact, sinusoidal functions are periodic. A horizontal shift (the \( h \) value) shouldn’t affect the period of a function but a horizontal stretch (the \( B \) value) certainly will affect the period. Recall that the value of \( B \) stretches the function horizontally by a factor of \( \frac{1}{B}. \) The period of the sine function is \( 2\pi \) and stretching the sine function by a factor of \( \frac{1}{B} \) results in a function whose period is \( \frac{2\pi}{B}. \) That is, the period \( p \) of a sinusoidal function is given by

\[
p = \frac{2\pi}{B}.
\]

The value of \( h \) is the horizontal shift. That is, the graph of \( y = A \sin(B(x-h)) + k \) is the same as the graph of \( y = A \sin(Bx) + k \) except shifted to the right by \( h \) units (or to the left by \( |h| \) units if \( h \) is negative). Figure 4.3.5 shows the graph of \( y = \sin(x) \) and the graph of \( y = \sin(x-1). \)

We must be very careful about the value of \( h. \) Consider the function

\[
f(x) = 2 \sin(2x - 3) + 1. \tag{4.3.6}
\]

This function is certainly sinusoidal. In order to study the geometry of this function we want to write \( f \) in the form

\[
f(x) = A \sin(B(x-h)) + k. \tag{4.3.7}
\]
It is clear that $A = 2$, $B = 2$, and $k = 1$. It is tempting to think that $h = -3$, but this is not the case. Notice that the inside of the sine function in Equation 4.3.7 is factored. We want to rewrite Equation 4.3.6 as

$$f(x) = 2 \sin(2(x - 1.5)) + 1.$$ 

Notice the similarities with Equation 4.3.7. This is the form in which we need the function to be when we’re looking for $h$. Still, it is tempting to say that $h = -1.5$. However, because of the form of Equation 4.3.7, we actually have that $h = 1.5$ in this case. The sign on $h$ is important as $h = 1.5$ corresponds to a horizontal shift of 1.5 to the right and $h = -1.5$ corresponds to a horizontal shift of 1.5 to the left. Just to reaffirm, Figure 4.3.8 shows the horizontal shift between the graphs of $y = 2 \sin(2x) + 1$ and $y = 2 \sin(2x - 3) + 1$. 

<table>
<thead>
<tr>
<th>Feature</th>
<th>Graph of $y = A \sin(B(x - h)) + k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude</td>
<td>$A$</td>
</tr>
<tr>
<td>Midline</td>
<td>$y = k$</td>
</tr>
<tr>
<td>Period</td>
<td>$\frac{2\pi}{B}$</td>
</tr>
<tr>
<td>Horizontal Shift</td>
<td>$h$</td>
</tr>
</tbody>
</table>

Table 4.3.9: The geometric features of a sinusoidal function.
The geometric features of sinusoidal functions are shown in Table 4.3.9. An understanding of these geometric features can allow us to graph sinusoidal functions without a calculator. This is shown in Example 4.3.10. Graphing these function is review from MATH 111 so we won’t spend a lot of time on it here. However, sinusoidal functions are easier to graph than most other transformations; they just take some practice.

**Example 4.3.10** Sketch the graph of

\[ f(x) = 3 \sin \left( \frac{\pi}{4} (x - 3) \right) - 1. \]

**Solution:**
First, sketch the midline, maximum, and minimum on your graph. The midline of \( f \) is \( y = -1 \) (since \( k = -1 \)) and the amplitude is \( A = 3 \). Hence the maximum of \( f \) will be \(-1 + 3 = 2\) and the minimum will be \(-1 - 3 = -4\). Below are the graphs of the lines \( y = 2, \ y = -1, \) and \( y = -4. \)

Next, we sketch the graph of \( y = 3 \sin(x) - 1 \). This graph will respect all of the vertical features of \( f \), but none of the horizontal ones.

Remember that the sine function “starts” at the midline. That is, when \( x = 0, \ 3 \sin(0) - 1 = -1. \) Once you know where the function starts you can fill in the shape of the graph to the right of that point. Note that the last graph doesn’t include any units on the \( x \)-axis. This was intentional. We want our eventual graph to have the period of \( f \). If the period of \( f \) is \( p \) then we know that \( \frac{\pi}{4} = \frac{2\pi}{p} \) (since \( B = \frac{\pi}{4} \)) so that \( p = \frac{2\pi}{\pi/4} = 8. \) Now that we know the period of \( f \) should be 8, we can put units on the \( x \)-axis of our graph.
After we put units on the graph, the function that we graphed before is no longer \( y = 3 \sin(x) - 1 \), but is now \( y = 3 \sin(\pi x) - 1 \). Since \( f(x) = 3 \sin(\pi (x - 3)) - 1 \), the graph of \( y = f(x) \) is just a horizontal shift of the graph that we already have to the right by 3 units. Shown below is the previous graph shifted to the right by 3 units, which is the graph of \( y = f(x) \).

**Answer:** Shown on the previous graph.

The real focus of this section will be to use sinusoidal functions for modeling. That is, if we have some features of the graph of a function, we want to be able to come up with a sinusoidal function which reflects those features. For example, suppose that \( f \) is a sinusoidal function with satisfies the following:

- The amplitude of \( f \) is 3.
- The midline of \( f \) is \( y = 2 \).
- The period of \( f \) is 4.
- \( f(2) = 5 \).

We want to find an equation for \( f \). Of course, since \( f \) is a sinusoidal function, we know that

\[
f(x) = A \sin(B(x - h)) + k.
\]  
(4.3.11)

Since \( A \) is the amplitude of \( f \) we know that \( A = 3 \). Since the amplitude of \( f \) is \( y = k \) we know that \( k = 2 \). To find the value of \( B \) we have
\[ B = \frac{2\pi}{p} \text{ where } p \text{ is the period of } f \text{ so that } B = \frac{2\pi}{4} = \frac{\pi}{2}. \] So far we have that
\[ f(x) = 3 \sin\left(\frac{\pi}{2}(x - h)\right) + 2. \]

Generally, finding the value of \( h \) is the hardest part. It is probably not the case that \( h = 0 \), but let’s look at the graph of \( y = 3 \sin\left(\frac{\pi}{2}x\right) + 2 \) in Figure 4.3.12. We know that the graph of \( y = f(x) \) should be a horizontal shift away from this graph. We also know that \( f(2) = 5 \). We can see in Figure 4.3.12 that if we shift the graph of \( y = 3 \sin\left(\frac{\pi}{2}x\right) + 2 \) to the right by 1 unit then we have a graph which contains the point \((2,5)\). This graph satisfies all of the requirements and a horizontal shift to the right by 1 corresponds to \( h = 1 \), so we have
\[ f(x) = 3 \sin\left(\frac{\pi}{2}(x - 1)\right) + 2. \]

Figure 4.3.12: The graph of the function in Equation 4.3.11 and the graph of the same function before the horizontal shift.

After we found the values of \( A, B, \) and \( k \), there was exactly one feature left to satisfy: the fact that \( f(2) = 5 \). When we’re looking for the horizontal shift, the strategy is to graph the function with no horizontal shift and then try to figure out what kind of a shift will satisfy the final feature. In this case, we needed the graph of \( y = f(x) \) to contain the point \((2,5)\). Before the shift we saw that the graph contained the point \((1,5)\) and a shift to the right of 1 unit moved that point to \((2,5)\).

As we mentioned before, the name of the game in this section will be finding the value of \( h \). Unfortunately, there is no sure-fire procedure for doing this; it is just a matter of practicing until you get it down. We explore this more in the following examples.
(4.3.13) Example. The graph of \( y = f(x) \) is shown below.

Assuming that \( f \) is a sinusoidal function, find an equation for \( f \).

Solution:

Since \( f \) is sinusoidal we have

\[
f(x) = A \sin(B(x - h)) + k.
\]

Using the techniques in Section 2.4 we know that the period of \( f \) is 1, the midline of \( f \) is \( y = -\frac{3}{2} \), and the amplitude of \( f \) is \( \frac{5}{2} \). It follows that \( A = \frac{5}{2}, k = -\frac{3}{2}, \) and \( B = 2\pi \). Hence we have

\[
f(x) = \frac{5}{2} \sin(2\pi(x - h)) - \frac{3}{2}.
\]

We are left with the task of finding the value of \( h \). We first graph the function with \( h = 0 \). That is, we graph \( y = \frac{5}{2} \sin(2\pi x) - \frac{3}{2} \). It is shown below along with the graph of \( y = f(x) \).

The point \((0.25, 1)\) is identified on the graph above of \( y = \frac{5}{2} \sin(2\pi x) - \frac{3}{2} \). This is because there is a maximum of a sinusoidal function with no horizontal shift at \( x = \frac{1}{4}p \) where \( p \) is the period and there is a minimum at \( x = \frac{3}{4}p \). In this case, \( p = 1 \).

To find the value of \( h \) we need to figure out how to shift the graph of \( y = \frac{5}{2} \sin(2\pi x) - \frac{3}{2} \) in order to line up with the
graph of \( y = f(x) \). We see that if we shift it to the right by 
\[ 0.65 - 0.25 = 0.4 \] then they will line up; that is, the point 
\( (0.25, 1) \) will be shifted to the point \( (0.65, 1) \). A shift to the 
right by 0.4 corresponds to 
\[ h = 0.4 = \frac{2}{5} \] and hence 
\[ f(x) = \frac{5}{2} \sin\left(2\pi\left(x - \frac{2}{5}\right)\right) - \frac{3}{2}. \]

\[ \text{Answer: } f(x) = \frac{5}{2} \sin\left(2\pi\left(x - \frac{2}{5}\right)\right) - \frac{3}{2} \]

There is something about Example 4.3.13 that deserves mention. We 
decided to shift the graph of 
\( y = \frac{5}{2} \sin(2\pi x) - \frac{3}{2} \) to the right by 0.4 units in order get it to line up with the graph of 
\( y = f(x) \). However, the interested observer might notice that there are a lot of other choices of horizontal shift. Indeed, it seems as if we shift the graph 
of \( y = \frac{5}{2} \sin(2\pi x) - \frac{3}{2} \) to the left by 0.6 units then they will also line up. We can also shift to the left by 1.6 units, or by a whole bunch of other choices. Why, then, can’t it be that 
\( h = -0.6 \) or \( h = -1.6 \)? Actually, those are perfectly reasonable values for \( h \). This is because any sinusoidal function can be shifted in either direction by its period without changing. Once you find one acceptable value of \( h \) (in this case, \( h = 0.4 \)), you can add or subtract the period (in this case 1) from it as many times as you’d like and the function doesn’t change.

The point of the previous digression is that there are many acceptable answers to Example 4.3.13.

\[ \text{(4.3.14) Example. } \] Find a sinusoidal function \( f \) such that:

- The maximum of \( f \) is 14.
- The minimum of \( f \) is 2.
- The period of \( f \) is 2.
- \( f(-0.5) = 14 \).

\[ \text{Solution: } \]

We know that 
\[ f(x) = A \sin(B(x - h)) + k. \]

The amplitude is \( \frac{14 - 2}{2} = 6 \) so that \( A = 6 \). The midline is 
\( y = \frac{14 + 2}{2} = 8 \) so that \( k = 8 \). Since the period is 2, we have 
\( B = \frac{2\pi}{2} = \pi \). It follows that 
\[ f(x) = 6 \sin(\pi(x - h)) + 8. \]
It is almost always straightforward to find $A$, $B$, and $k$. Finding $h$ is the difficult part. We start by looking at what would happen if $h = 0$ and graph $y = 6 \sin(\pi x) + 8$.

We need $f(-0.5) = 14$ so our graph should contain the point $(-0.5, 15)$. We see that we need to shift the graph 1 unit to the left as shown below in order to make that happen.

A horizontal shift of 1 to the left corresponds to $h = -1$ so

$$f(x) = 6 \sin(\pi(x + 1)) + 8.$$  

**Answer:** $f(x) = 6 \sin(\pi(x + 1)) + 8$

---

**4.3.3 Practice Exercise.** Find a sinusoidal function $f$ such that:

- The maximum of $f$ is $-1$.
- The minimum of $f$ is $-6$.
- The period of $f$ is 3.
- $f(1) = -1$.  

Notice that we have not used any negative values of $A$ or $B$ in this section. There is no reason they can’t be negative, but there is also no reason that they need to be negative, either. We have already seen that any sinusoidal function can be written in multiple ways by using different values of $h$. In fact, given any sinusoidal function we can always find a way to write it such that $A$ and $B$ are both positive. As an example, you can check that

$$f(x) = -3 \sin \left( -\frac{\pi}{2} (x - 1) \right) - 2 \quad \text{and} \quad g(x) = 3 \sin \left( \frac{\pi}{2} (x - 1) \right) - 2$$

are the same functions. In these notes we will not use negative values of $A$ and $B$.

Lastly, consider an arbitrary sinusoidal function but distribute the inside of the sine function so that

$$f(x) = A \sin(Bx - Bh) + k.$$

The quantity $Bh$ has some important properties and is called the **phase shift** of $f$. There are certain applications where the phase shift is much more significant than the horizontal shift, but we will not focus on phase shift in these notes.

**Applications**

We want to be able to model some actual situations with sinusoidal functions. In Example 4.3.14 we were given a list of features of a function and we tried to find a sinusoidal function which modeled those features. In the applications that follow, we first try to find that list of features. We can then tackle the modeling similar to the way that we tackled Example 4.3.14.

(4.3.15) **Example.** An engineer measures the oscillating pressure in a certain pipe. He finds that it takes 45 min for the pressure to drop from a high of 5000 kPa to a low of 2000 kPa and then another 45 min for it to get back up to 5000 kPa. (kPa is a unit of pressure called the kilopascal) The function $P$ models the pressure since the first time the pressure was at its peak. That is, $P(t)$ is the pressure (in kPa) in the pipe $t$ minutes after the first time the pressure reached 5000 kPa.
Assuming that $P$ is a sinusoidal function, find a formula for $P$.

**Solution:**

Since $P$ is sinusoidal, we have

$$P(t) = A \sin(B(t - h)) + k.$$ 

We first want to determine all of the features of this graph. Obviously its maximum is 5000 and its minimum is 2000. What about its period? The problem stipulates that it takes 45 min for the pressure to drop from a high of 5000 kPa to a low of 2000 kPa and then another 45 min for it to get back up to 5000 kPa. That’s 45 min to go from the high to the low and then another 45 min to go from the low to the high, which means that the period will be 90. The period of a sinusoidal function is always the distance between two consecutive maximum values or between two consecutive minimum values. Also, the problem stipulates that the pressure is measured from the first time the pressure reaches 5000 kPa so $P(0) = 5000$. We have now accumulated the following four features of $P$:

- The maximum of $P$ is 5000.
- The minimum of $P$ is 2000.
- The period of $P$ is 90.
- $P(0) = 5000$.

It remains only to use these features to find $A$, $B$, $h$, and $k$. We have that $A = \frac{5000 - 2000}{2} = 1500$, $k = \frac{5000 + 2000}{2} = 3500$, and $B = \frac{2\pi}{90} = \frac{\pi}{45}$. Hence

$$P(t) = 1500 \sin\left(\frac{\pi}{45}(t - h)\right) + 3500.$$ 

To find $h$ we graph $y = 1500 \sin\left(\frac{\pi}{45}t\right) + 3500$ as shown below.
We know that \( P(0) = 5000 \). Since 5000 is the maximum value, we need a horizontal shift which shifts the maximum of the graph above to \( t = 0 \). Remember, in a sinusoidal function with no horizontal shift, there is a maximum at a \( t \)-value of one fourth of the period. In this case, when \( t = \frac{1}{4}(90) = 17.5 \). We need to shift the graph so that the point \((17.5, 5000)\) shifts to \((0, 5000)\). This requires a shift to the left by 17.5, so we need \( h = -17.5 \). Hence

\[
P(t) = 1500 \sin\left(\frac{\pi}{45}(t + 17.5)\right) + 3500.
\]

The graph is shown below.

**Answer:** \( P(t) = 1500 \sin(\frac{\pi}{45}(t + 17.5)) + 3500 \)
Example. The power in an outlet in a standard US home is typically supplied with alternating current. The voltage of this current alternates from a high of 155.6 V to a low of $-155.6$ V and back again sixty times every second. (V is a unit of voltage, conveniently called volts.) Find a sinusoidal function \( P \) which models the voltage in the outlet. That is, \( P(t) \) should be the voltage (in volts) \( t \) seconds after the power is turned on. Assume that the voltage is at its lowest the moment the power is turned on.

Solution:

We can write

\[
P(t) = A \sin(B(t - h)) + k,
\]

as usual. We see that the maximum is 155.6 and the minimum is $-155.6$. The voltage goes from its maximum to its minimum and then back again sixty times every second. Hence it takes \( \frac{1}{60} \) seconds to get from its maximum back to its maximum, meaning that the period of \( P \) is \( \frac{1}{60} \). Also, since the voltage is at its lowest when the power is turned on, \( P(0) = -155.6 \). We’ve now found the following four features:

- The maximum of \( P \) is 155.6.
- The minimum of \( P \) is $-155.6$.
- The period of \( P \) is \( \frac{1}{60} \).
- \( P(0) = -155.6 \).

As always, we only need to find \( A \), \( B \), \( h \), and \( k \).

We first have that \( A = 155.6 \), \( k = 0 \), and \( B = \frac{2\pi}{\frac{1}{60}} = 120\pi \) so that

\[
P(t) = 155.6 \sin(120\pi(t - h)).
\]

Below is the graph of \( y = 155.6 \sin(120\pi t) \).

We know that \( P(0) = -155.6 \) so that we need to shift a minimum of the graph above to \( t = 0 \). The minimum occurs at
three fourths of the period, or \( t = \frac{1}{60} \cdot \frac{3}{4} = \frac{1}{80} \). Thus the point \((\frac{1}{80}, -155.6)\) appears as a minimum on the above graph. If we shift this graph to the left by \( \frac{1}{80} \) units we will have the graph of \( y = P(t) \), so \( h = -\frac{1}{80} = -0.0125 \) and

\[
P(t) = 155.6 \sin(120\pi(t + 0.0125)).
\]

The graph of \( y = P(t) \) is shown below.

![Graph of y = P(t)](image)

\[\textbf{Answer:} \quad P(t) = 155.6 \sin(120\pi(t + 0.0125))\]

We now want to explore the Ferris wheel example with which we opened the section. We’ve seen that any time we try to model one of the coordinates of something that moves in a circle as a function of its angle we get a function of the form

\[f(\theta) = A \sin(\theta) + k \quad \text{or} \quad f(\theta) = A \cos(\theta) + k.\]

We can now try to model these applications as a function of time instead of as a function of the angle. As long as the object rotates at a constant rate, \( \theta \) will be a linear function of time and the model will be sinusoidal. In this unique situation, if we know the starting angle then there is a very convenient way to find the value of \( h \).

(4.3.17) Example. Someone is sitting on a Ferris wheel with a radius of 80 ft, whose center is 100 ft off of the ground. He starts the ride at an angle of \(-\frac{\pi}{6}\) from the horizontal, as shown below, and it takes one minute for the Ferris wheel to complete an entire revolution. Find a function \( H \) such that after \( t \) seconds the rider’s height is \( H(t) \) (in ft).
**Solution:**
We will model \( H \) by a sinusoidal function so that

\[ H(t) = A \sin(B(t - h)) + k. \]

As we’ve seen several times before, \( A = 80 \) and \( k = 100 \). Since the Ferris wheel takes one minute to complete an entire rotation, every one minute the rider will be back where he started. It follows that the period of \( H \) will be 1 and \( B = 2\pi \). Hence we have

\[ H(t) = 80 \sin(2\pi(t - h)) + 100. \]

Normally, we would have to use a graph to find \( h \). However, this time we can let the geometry help us. We know that \( 2\pi(t - h) \) is the rider’s angle at time \( t \). We also know that when \( t = 0 \) the rider’s angle is \(-\frac{\pi}{6}\). Hence

\[
\begin{align*}
-\frac{\pi}{6} &= 2\pi(0 - h) \\
-\frac{1}{12} &= -h \\
h &= \frac{1}{12}
\end{align*}
\]

and

\[ H(t) = 80 \sin\left(2\pi\left(t - \frac{1}{12}\right)\right) + 100. \]

**Answer:** \( H(t) = 80 \sin\left(2\pi\left(t - \frac{1}{12}\right)\right) + 100 \)

We’ve modeled things going in circles before. We already know that the value of \( k \) is the distance of the center from wherever we’re measuring and that the value of \( A \) is the radius of the circle. When we model these things as a function of time, the period will be the time that it takes the circle to complete one revolution.
Look at the answer to Example 4.3.17. When we distribute inside of the sine function we have

\[ H(t) = 80 \sin \left(2\pi t - \frac{\pi}{6}\right) + 100. \]

Observe that the inside of sine function looks like \( Bt + \phi \) where \( \phi \) is the rider’s starting angle; this is no coincidence. Incidentally, the value \( \phi = -Bh \) is the negative of the phase shift discussed earlier.

We will look at one more example of a rotation model. Compare Example 4.3.18 with Example 3.4.18.

(4.3.18) Example. The center of a wind turbine is attached to the top of a 60 m tower and it has four spinning blades that are 40 m long. When the wind is blowing, this turbine makes 40 revolutions every minute. We’re trying to track the motion of a particular blade. When the turbine first starts turning the blade makes an angle of \( \frac{\pi}{4} \) with the horizontal. Find a function \( H \) such that \( t \) minutes after the turbine starts turning the tip of this particular blade is at a height of \( H(t) \) feet. The position of the turbine when it first starts turning is shown below.

Solution:

We know that \( H \) is sinusoidal so that

\[ H(t) = A \sin(B(t - h)) + k. \]

We have that \( A = 40 \) and \( k = 60 \). Since the turbine makes 40 revolutions each minute we have that one revolution takes \( \frac{1}{40} \) minutes, which is the period of \( H \). Hence \( B = \frac{2\pi}{1/40} = 80\pi \) and

\[ H(t) = 40 \sin(80\pi(t - h)) + 60. \]
For any given value of \( t \), the blade makes an angle of \( 80\pi(t - h) \) with the horizontal. Since the blade starts at an angle of \( \frac{\pi}{4} \), we know that when \( t = 0 \) we have that \( 80\pi(t - h) = \frac{\pi}{4} \). Hence

\[
\frac{\pi}{4} = 80\pi(0 - h)
\]

\[
\frac{1}{320} = -h
\]

\[
h = -\frac{1}{320}.
\]

It follows that

\[
H(t) = 40 \sin\left(80\pi\left(t + \frac{1}{320}\right)\right) + 60.
\]

**Answer:** \( H(t) = 40 \sin(80\pi(t + \frac{1}{320})) + 60 \)

(4.3.b) **Practice Exercise.** Sonya is riding a Ferris wheel whose maximum height is 225 m and whose minimum height is 25 m. This Ferris wheel takes 4 min to complete an entire rotation and when the ride starts, Sonya’s carriage makes an angle of \( -\frac{\pi}{4} \) with the horizontal. Find a function \( H \) such that after \( t \) minutes on the Ferris wheel, Sonya’s height is \( H(t) \) meters.
4.3.1. Sketch the graph of
\( g(x) = 2 \sin\left(\frac{\pi}{3}(x + 1)\right) + 3. \) Be sure to include at least one entire period, and be sure to label the y-intercept, one “maximum,” and one “minimum.”

4.3.2. Consider the sinusoidal function
\[ f(t) = 2000 \sin\left(\frac{\pi}{12}(t - 3)\right) - 1500. \]
A.) What is the amplitude of \( f \)?
B.) What is the midline of \( f \)?
C.) What is the period of \( f \)?
D.) Sketch the graph of \( f \). Be sure to include at least one entire period.

4.3.3. Consider the function \( f(x) = 3 \sin\left(\frac{\pi}{4}x\right) - 2 \) (where \( x \) is in radians).
A.) What is the amplitude of \( f \)?
B.) What is the midline of \( f \)?
C.) What is the period of \( f \)?
D.) Sketch the graph of \( f \). Be sure to include at least one entire period.

4.3.4. Consider the function
\[ f(x) = 3 \sin\left(\frac{\pi}{4}x - \pi\right) - 5. \]
A.) What is the amplitude of \( f \)?
B.) What is the midline of \( f \)?
C.) What is the period of \( f \)?
D.) Graph \( y = f(x) \). Be sure to include at least one period and to label one maximum and one minimum.

4.3.5. Consider the sinusoidal function
\[ f(t) = 100 \sin\left(\frac{\pi}{6}(t - 9)\right) + 250. \]
A.) What is the amplitude of \( f \)?
B.) What is the midline of \( f \)?
C.) What is the period of \( f \)?
D.) Suppose that \( f(t) \) is the height (in feet) of a rider on a Ferris wheel \( t \) seconds after the ride starts. What is the radius of the Ferris wheel?

4.3.6. Below is the graph of a sinusoidal function, \( y = p(x) \). Find a formula for \( p \) of the form \( p(x) = A \sin(B(x - h)) + k \).

4.3.7. Find a sinusoidal function of the form \( f(x) = A \sin(B(x - h)) + k \) with the following attributes:
- An amplitude of 7.
- A midline of \( y = 10 \).
- A period of 5.
- \( f(0) = 3 \).

4.3.8. Find a sinusoidal function \( f \) with the following attributes:
- an amplitude of 20
- a midline of \( y = 35 \)
- a period of 8
- \( f(2) = 55 \)

4.3.9. Find a sinusoidal function \( f \) with the following attributes:
- An amplitude of 26
- A midline of \( y = 7 \)
- A period of 8
- \( f(0) = 33 \)

4.3.10. Find a sinusoidal function \( f \) with the following attributes:
- An amplitude of 10.
- A midline of \( y = 5 \).
• A period of 20.
• \( f(0) = -5 \).

4.3.11. A ping-pong ball is bouncing on a table. The ball hits the table once a second and it bounces to a height of exactly 15 in off the table. Find a sinusoidal function \( h \) such that \( h(t) \) gives the height of the ball off the table (in inches) \( t \) seconds after the ball first hits the table. Note: This is actually a horrible model for a bouncing ball. Just go with it.

4.3.12. The lowest average temperature in Eugene is 35 °F on February 1st (the 32nd day of the year). The highest average temperature in Eugene is 70 °F. Assuming that average temperature can be modeled by a sinusoidal function, find a function \( f \) such that on the \( d \)-th day of the year, \( f(d) \) gives the average temperature on that day. Note: Assume that every year has 365 days in it and assume that, as expected, the maximum and minimum average temperature are achieved exactly once a year.

4.3.13. The population of a particular beetle can be modeled by a sinusoidal function with a period of one year. The beetle’s population reaches its highest on August 1st (the 214th day of the year) at which point the population is measured at 6000 beetles per square mile. The lowest the population gets in the year is 500 beetles per square mile. Find a function \( P \) such that on the \( d \)-th day of the year, \( P(d) \) gives the beetle’s population.

4.3.14. A friend and I watched all 48 minutes of a basketball game between the Cavs and the Blazers. As the game progressed we kept track of the score using a function, \( d \), which recorded the number of points by which the Cavs were leading (of course, negative values were used when the Cavs were losing). For example, 2 minutes into the game the Blazers were leading by 8 so \( d(2) = -8 \) whereas 6 minutes into the game the Cavs were leading by 4 so \( d(6) = 4 \). The Blazers were able to score two points almost immediately after tipoff so, for convenience, we recorded \( d(0) = -2 \).

A.) We noticed a curious thing. As the game progressed, the function \( d \) was approximately sinusoidal with a period of 8 minutes. If the Cavs’ largest lead was 4 points and the Blazers’ largest lead was 8 points, write an expression for \( d(t) \) as a sinusoidal function. (For some students it may be helpful to do part (b) before this part.)

b.) Draw the graph of the function you found in part (a) over (at least) the first 12 minutes of the game (that is, from \( t = 0 \) to \( t = 12 \)). Be as accurate as you can and label at least one maximum, one minimum, and the \( y \)-intercept.

c.) Which team won the game and by how much did they win? Explain your answer. Note: One sentence should be enough to explain your answer.

d.) How many times over the course of the game was there a change in the team that was leading? Note: At any given time the team with more points is the team that is leading. I do not care at what times there was a lead change; I only want to know how many times the lead changed.

4.3.15. Consider the water wheel again. Remember, it has a radius of 15 ft and its center is 12 ft off the water. Assume that the beetle lands on the very edge of the wheel at a position that makes an angle of \( \frac{2\pi}{3} \) with the horizontal. If the wheel takes 20 seconds to complete one revolution, find a sinusoidal function of the form \( h(t) = A \sin(B(t - h)) + k \) which describes the beetle’s height \( t \) seconds after it lands on the water wheel.
4.3.16. A Ferris wheel has a radius of 50 ft and its highest point is 125 ft off the ground. It takes the Ferris wheel three minutes to complete one rotation. Suppose that the position of Dave’s carriage makes an angle of \(-\frac{\pi}{3}\) with the horizontal when the wheel starts turning. Find a function \(f\) such that \(f(t)\) is Dave’s height (in feet) off the ground \(t\) minutes after the Ferris wheel starts turning.

\textbf{Challenge Problems}

\textbf{(4.3.c1) Challenge Problem.} It is sometimes the case that functions are sinusoidal even when it does not appear that they should be sinusoidal from their definitions. For example, it just so happens that any function \(f(x) = P \sin(x) + Q \cos(x)\) for real numbers \(P\) and \(Q\) is a sinusoidal function. Here we explore that fact further.

a. Let \(g(x) = \cos(x) + \sin(x)\). Find real numbers \(A, B, h,\) and \(k\) such that 
\[g(x) = A \sin(B(x - h)) + k.\]

b. Let \(h(x) = \sqrt{3} \sin(x) + \cos(x)\). Find real numbers \(A, B, h,\) and \(k\) such that 
\[h(x) = A \sin(B(x - h)) + k.\]

c. Let \(f(x) = \sin(x) + \cos(2x)\). Is it the case that \(f\) is sinusoidal? Explain your answer.

\textbf{(4.3.c2) Challenge Problem.} This exercise is meant to explore the concept of a \textbf{real function of two variables}. To this point, you know a function as a machine which inputs numbers and outputs other numbers. If \(f\) is a function and \(x\) is some number (in the domain of \(f\), of course), then \(f(x)\) is a real number. For mathematicians, functions can input and output anything and the special case of a function which inputs a real number and outputs a real number is called a \textbf{real function of one variable}. A real function of two variables is a function which inputs two numbers and outputs a single real number. If \(F\) is a real function of two variables and if \(x\) and \(y\) are two numbers then \(F(x, y)\) is a real number. We often define a real function of one variable using an expression in that variable. For example, \(f(x) = x^2 - 2\) defines a real function of one variable. We can do a similar thing with functions of two variables. For example,

\[F(x, y) = 2x^2 + 3y^2 - \frac{3}{2}xy\]

is a function of two variables. If \(x = 5\) and \(y = 4\) then

\[F(5, 4) = 2(5)^2 + 3(4)^2 - \frac{3}{2}(5)(4) = 50 + 48 - 30 = 68\]
Of course, it is just as easy to define a function of three variables or a function of nineteen variables if that is necessary but we won’t worry about those things now.

I. Consider the function $F(x, y) = y(y - x^2)$. Note that the domain of this function is the set of all pairs of numbers $x$ and $y$, that is all ordered pairs $(x, y)$. This, of course, is the entire $xy$-plane.

A) Calculate $F(1, 3)$, $F(3, 1)$, $F(4, 9)$, and $F(3, 9)$.

b) The roots of a real function of two variables are the points (in the $xy$-plane) at which the function is zero. In this case, roots are points $(x, y)$ such that $F(x, y) = 0$. At this point (if you’ve done the first part correctly) you should know exactly one root of this function. Describe all of the roots of $F$. (There are an infinite number of them.)

c) We can make the graph of a function of two variables by graphing the equation $z = F(x, y)$ (compared to $y = f(x)$ in one variable) and points on this graph are three-dimensional points $(x, y, z)$ such that $z = F(x, y)$. For example, the point $(0, 2, 4)$ is on the graph of $z = F(x, y)$ whereas $(1, 1, 4)$ is not. The graph of this particular function (that is, the graph of $z = y(y - x^2)$) is shown below. Write a sentence or two explaining how this graph works. Note: You do not need to explain the colors. The colors are not really a part of the graph but they are included to show a visual contrast between points which are “high” and points which are “low.”

II. Imagine a Ferris wheel of diameter 200 ft whose center is level with the ground. Then imagine that one of the seats on that Ferris wheel is replaced with a second Ferris wheel of diameter 50 ft (whose center is where the seat used to be). This is shown in the diagram below:
A rider sits on one of the seats of the smaller Ferris wheel. We can describe her position by two angles. The angle $\theta$ describes the position of the center of the smaller circle within the bigger circle and the angle $\phi$ describes the position of the rider within the smaller circle. These angles are shown in the diagram.

A) First you will find a function, $H$, which describes the height of the rider. In this case there are two parameters (the two angles) which describe her position so $H$ must be a function of two variables. If $\theta$ and $\phi$ are the angles described above then $H(\theta, \phi)$ should be her height (in feet) above the ground (which is even with the center of the larger circle). Find a formula for $H(\theta, \phi)$. Hint: Use the geometry to do this. I suggest breaking it up into two parts. First find the height of the center of the smaller circle above the ground and then find the height of the rider above the center of the smaller circle. These two things together should give you what you want. A second (totally useless and irrelevant) hint is that the graph of $z = H(\theta, \phi)$ is drawn below:
b) Imagine that the angle \( \theta \) is fixed at \( \frac{\pi}{4} \). In this case, her position only a function of one parameter - namely \( \phi \). We want to find a function, \( f \), which describes her height in this case. However, as her position is only determined by one parameter, this function will be a function of one variable. Find a formula for \( f \) so that \( f(\phi) \) is her height (in feet). You can start from scratch if you’d like, but it is easier to use the fact that \( f(\phi) = H(\frac{\pi}{4}, \phi) \) (try to justify that for yourself).

c) Imagine now that the angle \( \phi \) is fixed at \( \frac{\pi}{6} \) (and \( \theta \) is no longer fixed at \( \frac{\pi}{4} \)). Again, the rider’s position is now determined by a single parameter - namely \( \theta \). Find a function of one variable, \( g \), such that \( g(\theta) \) is her height (in feet). Hint: Find a way to relate \( g \) to \( H \) similar to how \( f \) and \( H \) were related in the previous part.

d) Imagine that this ride is set up so that \( \phi \) is always three times as large as \( \theta \). In this case, the rider’s position is completely determined by \( \theta \). Find a function, \( h \), such that \( h(\theta) \) is the rider’s height (in feet) in this case. Hint: You should, again, be able to use \( H \) for this.
4.4 **TRIGONOMETRIC EQUATIONS**

This section corresponds to section 8.4 in Functions Modeling Change [4]

In Section 3.6 we learned how to solve trigonometric equations like \( \sin(\theta) = \frac{1}{2} \) given that we knew \( \theta \) was an acute angle. This was extremely helpful in solving right triangles. However, as we know there are a lot of angles \( \theta \) such that \( \sin(\theta) = \frac{1}{2} \) and in this section we attempt to find all of them.

**Motivation**

We’ve been working up to Ferris wheel functions for the better part of two chapters now. Someone is sitting on a Ferris wheel with a radius of 80 ft, whose center is 100 ft off of the ground. He starts the ride at an angle of \(-\frac{\pi}{6}\) from the horizontal, as shown in Figure 4.4.1, and it takes one minute for the Ferris wheel to complete an entire revolution. During Example 4.3.17 in Section 4.3 we found that if \( H \) is the function such that after \( t \) seconds the rider’s height is \( H(t) \) (in ft) then

\[
H(t) = 80 \sin \left( 2\pi \left( t - \frac{1}{12} \right) \right) + 100.
\]

Figure 4.4.1: The Ferris wheel from Example 4.3.17.

We’ve done a lot of work to develop that equation. One thing we like to do with equations is solve them. Suppose that we want to know all of the times during this ride that the carriage reaches a height of 140 ft. That is, we would like to be able to find all of the values of \( t \) such that \( H(t) = 140 \). Let’s try to do that.

\[
140 = 80 \sin \left( 2\pi \left( t - \frac{1}{12} \right) \right) + 100
\]

\[
40 = 80 \sin \left( 2\pi \left( t - \frac{1}{12} \right) \right)
\]

\[
\frac{1}{2} = \sin \left( 2\pi \left( t - \frac{1}{12} \right) \right).
\]
We are left with the question of finding all of the values of \( t \) such that \( \sin(2\pi(t - \frac{1}{12})) = \frac{1}{2} \). Luckily, \( \frac{1}{2} \) is in the interval \([-1, 1]\) so that there should actually be some value of \( t \). Unfortunately, there will be lots of them. There are many angles \( \theta \) such that \( \sin(\theta) = \frac{1}{2} \) so any time \( 2\pi(t - \frac{1}{12}) \) is one of those angles, that will be a value of \( t \) such that \( H(t) = 140 \).

Let’s try to find just one of those angles. Since \( \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6} \), if \( t \) is a value such that \( 2\pi(t - \frac{1}{12}) = \frac{\pi}{6} \) then \( t \) should work for us. Hence

\[
2\pi \left( t - \frac{1}{12} \right) = \frac{\pi}{6} \\
t - \frac{1}{12} = \frac{1}{12} \\
t = \frac{1}{6}.
\]

If we’ve worked through things correctly, it should be the case that \( H \left( \frac{1}{6} \right) = 140 \). We can check this:

\[
H \left( \frac{1}{6} \right) = 80 \sin \left( 2\pi \left( \frac{1}{6} - \frac{1}{12} \right) \right) + 100 \\
= 80 \sin \left( \frac{\pi}{6} \right) + 100 \\
= 80(0.5) + 100 \\
= 140.
\]

We see that \( H \left( \frac{1}{6} \right) = 140 \) which is great, but that’s only one of many values. The solutions to \( H(t) = 140 \) are the values of \( t \) where the graphs of \( y = H(t) \) and \( y = 140 \) intersect. The graphs in Figure 4.4.3 show that there are many more solutions. Remember that \( H \) is periodic with period 1 so \( H \left( \frac{1}{6} + n \right) = 140 \) for any integer \( n \). Hence \( H(t) = 140 \) any time \( t \) is one of the following:

\[
\cdots, -\frac{23}{6}, -\frac{17}{6}, -\frac{11}{6}, -\frac{5}{6}, \frac{1}{6}, \frac{7}{6}, \frac{13}{6}, \frac{19}{6}, \cdots \tag{4.4.2}
\]

We can see these solutions in Figure 4.4.3.

However, the graph suggests that the solutions we’ve already found are only some of them. Let’s try to think about what this means on the Ferris wheel. When the Ferris wheel goes around one complete revolution, that corresponds to one period of \( H \). In one revolution, the carriage reaches a height of 140 ft twice; once on the way up and once on the way down. During the process of solving \( \sin(2\pi(t - \frac{1}{12})) = \frac{1}{2} \) we started by solving \( 2\pi(t - \frac{1}{12}) = \frac{\pi}{6} \). That is, we started by picking an angle \( \theta \) such that \( \sin(\theta) = \frac{1}{2} \) and set \( 2\pi(t - \frac{1}{12}) \) equal to that angle. The angle we picked was \( \frac{\pi}{6} \) which corresponds to the rider’s angle at that point during his ride. That means that the value we found, \( t = \frac{1}{6} \), corresponds to the part of the revolution where the carriage is on the way up.
Figure 4.4.3: The graph of the Ferris wheel function \( y = H(t) \) and the line \( y = 100 \). The \( t \) values where these graphs intersect are the solutions to \( H(t) = 140 \).

To find the rest of the values of \( t \), we need to find another angle \( \theta \) such that \( \sin(\theta) = \frac{1}{2} \). This other angle happens to be \( \theta = \frac{5\pi}{6} \). If we can find a value of \( t \) such that \( 2\pi(t - \frac{1}{12}) = \frac{5\pi}{6} \), we should be able to find the rest of the values of \( t \).

\[
2\pi \left( t - \frac{1}{12} \right) = \frac{5\pi}{6}
\]

\[
t - \frac{1}{12} = \frac{5}{12}
\]

\[
t = \frac{1}{2}.
\]

We can check that \( t = \frac{1}{2} \) is a valid solution to \( H(t) = 140 \).

\[
H \left( \frac{1}{2} \right) = 80 \sin \left( 2\pi \left( \frac{1}{2} - \frac{1}{12} \right) \right) + 100
\]

\[
= 80 \sin \left( \frac{5\pi}{6} \right) + 100
\]

\[
= 80(0.5) + 100
\]

\[
= 140.
\]

Again, since \( t = \frac{1}{2} \) is a solution, we also get solutions of the form \( t = \frac{1}{2} + n \) for integers \( n \). Hence \( H(t) = 140 \) any time \( t \) is one of the following:

\[
\cdots, -\frac{7}{2}, -\frac{5}{2}, -\frac{3}{2}, -1, 1, 3, 5, 7, \cdots
\]

(4.4.4)

Together, the list of values in Equation 4.4.2 and Equation 4.4.4 are all of the solutions to \( H(t) = 140 \).

This is something that we are already capable of doing, but the process was long. In this section we work to generalize the process of solving equations like this.
The first thing that we need to do is reevaluate our inverse trigonometric functions. Let’s consider, for a moment, the inverse sine function. We defined $\sin^{-1}(x)$ for values of $x$ in the interval $(0, 1)$. We did this by restricting the sine function to acute angles and then inverting it. However, this isn’t quite good enough. Suppose we want to know an angle $\theta$ such that $\sin(\theta) = -\frac{1}{2}$. We know there are such angles because the range of the sine function is $[-1, 1]$; in fact, $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$.

However, with our current definition of the inverse sine function, the expression $\sin^{-1}(-\frac{1}{2})$ has no meaning. Of course there are many values of $\theta$ such that $\sin(\theta) = -\frac{1}{2}$ but it would be nice if we could assign one angle to $\sin^{-1}(-\frac{1}{2})$.

Let’s reconsider our domain restriction of the sine function. The graph of $y = \sin(x)$ is shown in Figure 4.4.5. We restricted the domain to $(0, \frac{\pi}{2})$ last time but there was no good reason to stop there. If we only consider the highlighted section in Figure 4.4.5 we have an invertible function. Hence if we restrict the sine function to a domain of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ then we can invert it. More than that, on a domain of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ the sine function attains its entire range. That is, for every value of $x$ in $[-1, 1]$ there is a value of $\theta$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(\theta) = x$. This is the motivation for Definition 4.4.6.

\begin{quote}
\textbf{Definition.} For $x$ in $[-1, 1]$ define $\sin^{-1}(x)$ to be the angle $\theta$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(\theta) = x$.
\end{quote}

Compare this definition with Definition 3.6.5. Note that the new definition of $\sin^{-1}$ respects the old one. That is, for a value of $x$ in $(0, 1)$, the two definitions of $\sin^{-1}(x)$ agree with each other. The computations are straightforward, as shown in Example 4.4.7. Note that your calculator has no problem finding $\sin^{-1}$ for negative values.

\begin{quote}
\textbf{Example.} Find the following values:
\begin{itemize}
  \item A. $\sin^{-1}(1)$
\end{itemize}
\end{quote}
b. \( \sin^{-1}(-1) \)  

c. \( \sin^{-1}(0) \)  

d. \( \sin^{-1}(0.3) \)  

e. \( \sin^{-1}(-0.7) \)  

f. \( \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \)  

g. \( \sin^{-1}\left(-\frac{1}{2}\right) \)  

**Solution:**  
Note that all of these answers should be angles in the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

A. We’re looking for an angle at the top of the unit circle, hence \( \sin^{-1}(1) = \frac{\pi}{2} \).  
**Answer:** \( \sin^{-1}(1) = \frac{\pi}{2} \)  

b. We’re looking for an angle at the bottom of the unit circle, so \( \sin^{-1}(-1) = -\frac{\pi}{2} \).  
**Answer:** \( \sin^{-1}(-1) = -\frac{\pi}{2} \)  

c. The inscription point corresponding to this angle should have a y-coordinate of 0, so \( \sin^{-1}(0) = 0 \).  
**Answer:** \( \sin^{-1}(0) = 0 \)  

d. Using a calculator, \( \sin^{-1}(0.3) = 0.3047 \).  
**Answer:** \( \sin^{-1}(0.3) = 0.3047 \)  

e. Using a calculator, \( \sin^{-1}(-0.7) = -0.7754 \).  
**Answer:** \( \sin^{-1}(-0.7) = -0.7754 \)  

f. Since \( \frac{\pi}{3} \) is one of the special angles, \( \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \).  
**Answer:** \( \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \)  

g. Since \( -\frac{\pi}{6} \) is one of the special angles, \( \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \).  
**Answer:** \( \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \)  

It would be nice to restrict the domain of the cosine function to the same interval that we restricted the sine function. However, Figure 4.4.8 shows the graph of \( y = \cos(x) \) and we see that the cosine function is not invertible when restricted to a domain of \([-\frac{\pi}{2}, \frac{\pi}{2}]\). This is unfortunate, but it is salvageable. If we restrict the domain of cosine
to \([0, \pi]\) as highlighted in the graph we see that the cosine function becomes invertible.

![Figure 4.4.8: The graph of \(y = \cos(x)\).](image)

Definition 4.4.9 below is the analog of Definition 4.4.6 for the cosine function. It allows us to evaluate \(\cos^{-1}(x)\) for negative values of \(x\).

\[(4.4.9) \text{ Definition.} \quad \text{For } x \text{ in } [-1, 1] \text{ define } \cos^{-1}(x) \text{ to be the angle } \theta \text{ in } [0, \pi] \text{ such that } \cos(\theta) = x.\]

Lastly, let’s consider the tangent function. The graph of \(y = \tan(x)\) is shown in Figure 4.4.10 and it suggests that in order to invert it we should restrict the domain of tangent to \((-\frac{\pi}{2}, \frac{\pi}{2})\) as we did with the sine function.

![Figure 4.4.10: The graph of \(y = \tan(x)\).](image)

We can finally redefine \(\tan^{-1}\). Since the tangent function can output any real number, we need to be able to evaluate \(\tan^{-1}(x)\) for any value of \(x\). Definition 4.4.11 does that for us.

\[(4.4.11) \text{ Definition.} \quad \text{For } x \text{ in } (-\infty, \infty) \text{ define } \tan^{-1}(x) \text{ to be the angle } \theta \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}) \text{ such that } \tan(\theta) = x.\]

Let’s recap these three new definitions of inverse trigonometric functions.

- The inverse sine function is defined on \([-1, 1]\) and if \(x\) is in \([-1, 1]\) then \(-\frac{\pi}{2} \leq \sin^{-1}(x) \leq \frac{\pi}{2}\). That is, \(\sin^{-1}\) outputs angles whose inscription points are on the right half of the unit circle.
The inverse cosine function is defined on \([-1, 1]\) and if \(x\) is in \([-1, 1]\) then \(0 \leq \cos^{-1}(x) \leq \pi\). That is, \(\cos^{-1}\) outputs angles whose inscription points are on the top half of the unit circle.

The inverse tangent function is defined on \((-\infty, \infty)\) and if \(x\) is in \((-\infty, \infty)\) then \(-\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}\). That is, \(\tan^{-1}\) outputs angles whose inscription points are on the right half of the unit circle.

Let’s now investigate the equation \(\cos(\theta) = -\frac{1}{2}\). Using the inverse cosine function we can find one such \(\theta\). That is, \(\theta = \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}\) is one value of \(\theta\) such that \(\cos(\theta) = -\frac{1}{2}\). There are, of course, many others. Since \(\cos(\theta) = -\frac{1}{2}\) we know that the inscription point corresponding to \(\theta\) must lie on the line \(x = -\frac{1}{2}\). The unit circle and the line \(x = -\frac{1}{2}\) are shown in Figure 4.4.12.

![Figure 4.4.12: Inscription points for angles \(\theta\) with \(\cos(\theta) = -\frac{1}{2}\)](image)

The angles \(\theta\) such that \(\cos(\theta) = -\frac{1}{2}\) are precisely the angles with inscription points of either \(P\) or \(Q\) in Figure 4.4.12. Of course, using the Pythagorean Theorem we can find that \(P = (-\frac{1}{2}, \frac{\sqrt{3}}{2})\) and \(Q = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})\). It is obvious that the angle of \(\frac{2\pi}{3}\) has \(P\) as an inscription point, but remember that it is always the case that \(\cos^{-1}\) outputs angles on the top half of the unit circle. Notice that any angle whose inscription point is \(P\) must be a multiple of 2\(\pi\) away from \(\frac{2\pi}{3}\). That is, all of the angles \(\theta\) such that \(\cos(\theta) = -\frac{1}{2}\) and \(\sin(\theta) = \frac{\sqrt{3}}{2}\) are of the form \(\theta = \frac{2\pi}{3} + 2\pi n\) for an integer \(n\).

What about the angles whose inscription point is \(Q\)? We can find them by looking at reference triangles for both angles, which are show in Figure 4.4.13. Remember that two angles which have the same value of cosine have the same reference triangle. It follows that one of the angles with an inscription point of \(Q\) is the angle \(\theta = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}\). Again, the rest of the angles whose inscription points are \(Q\) are a multiple of 2\(\pi\) away from \(\frac{4\pi}{3}\). That is, all of the angles \(\theta\) such that \(\cos(\theta) = -\frac{1}{2}\) and \(\sin(\theta) = -\frac{\sqrt{3}}{2}\) are of the form \(\theta = \frac{4\pi}{3} + 2\pi n\) for an integer \(n\).
We have just found all of the solutions to \( \cos(\theta) = -\frac{1}{2} \). They are

\[ \theta = \frac{2\pi}{3} + 2\pi n \quad \text{and} \quad \theta = \frac{4\pi}{3} + 2\pi n \quad \text{for integers, } n. \]

This procedure that we used can be generalized to any trigonometric function and any value in the range of that trigonometric function. That is, we can find all of the solutions to equations like

\[ \cos(\theta) = 0.3, \quad \sin(\theta) = -0.55, \quad \text{and} \quad \tan(\theta) = -4 \]

using this method. The strategy is to find two values of \( \theta \) which satisfy the equation but have different inscription points; one of which comes from an inverse trigonometric function. Once you have those, to find all of the solutions we simply add \( 2\pi n \) to those two values for integers \( n \). The process can be somewhat automated. Instead of drawing a unit circle every time and plotting the two inscription points, we can use Lemma 4.4.14 to find the first two angles.

\begin{lemma}
\begin{itemize}
\item If \( x \) is in \((-1, 1)\) then two values of \( \theta \) which satisfy \( \cos(\theta) = x \) and have different inscription points are \( \theta_1 = \cos^{-1}(x) \) and \( \theta_2 = -\cos^{-1}(x) \).
\item If \( x \) is in \((-1, 1)\) then two values of \( \theta \) which satisfy \( \sin(\theta) = x \) and have different inscription points are \( \theta_1 = \sin^{-1}(x) \) and \( \theta_2 = \pi - \sin^{-1}(x) \).
\item If \( x \) is in \((-\infty, \infty)\) then two values of \( \theta \) which satisfy \( \tan(\theta) = x \) and have different inscription points are \( \theta_1 = \tan^{-1}(x) \) and \( \theta_2 = \pi + \tan^{-1}(x) \).
\end{itemize}
\end{lemma}

Be sure to keep the solutions to the three trigonometric equations straight; they are all three subtly different. Note that Lemma 4.4.14
ignores a few special cases, namely \( \sin(\theta) = x \) and \( \cos(\theta) = x \) when \( x = \pm 1 \). These are easier cases, though, as they are at the extreme edges of the unit circle.

- All of the solutions to \( \cos(\theta) = 1 \) are \( 2\pi n \) for integers, \( n \).
- All of the solutions to \( \cos(\theta) = -1 \) are \( \pi + 2\pi n \) for integers, \( n \).
- All of the solutions to \( \sin(\theta) = 1 \) are \( \frac{\pi}{2} + 2\pi n \) for integers, \( n \).
- All of the solutions to \( \sin(\theta) = -1 \) are \( \frac{3\pi}{2} + 2\pi n \) for integers, \( n \).

**Lemma 4.4.14** above is straightforward to apply. We explore some examples below.

### (4.4.15) Example
Find all angles \( \theta \) such that \( \cos(\theta) = 0.3 \). Round to two decimal places when appropriate.

**Solution:**
According to **Lemma 4.4.14**, the first two such values that we need are

\[
\theta_1 = \cos^{-1}(0.3) = 1.27 \quad \text{and} \quad \theta_2 = -\cos^{-1}(0.3) = -1.27.
\]

To find all solutions to the equation we simply add \( 2\pi n \) to these two values for integers \( n \). Hence all of the solutions are given by

\[
\theta = 1.27 + 2\pi n \quad \text{and} \quad \theta = -1.27 + 2\pi n \quad \text{for integers, } n.
\]

Note that we don’t round \( 2\pi \) in our answer. This is because there’s still an \( n \) in that bit of the expression.

**Answer:** \( \theta = 1.27 + 2\pi n \) and \( \theta = -1.27 + 2\pi n \) for integers, \( n \).

### (4.4.16) Example
Find all angles \( \theta \) such that \( \sin(\theta) = -0.55 \).

**Solution:**
According to **Lemma 4.4.14**, the first two such values that we need are

\[
\theta_1 = \sin^{-1}(-0.55) = -0.58 \quad \text{and} \quad \theta_2 = \pi - \sin^{-1}(-0.55) = 3.72.
\]
To find all solutions to the equation we simply add $2\pi n$ to these two values for integers $n$. Hence all of the solutions are given by

$$\theta = -0.58 + 2\pi n \quad \text{and} \quad \theta = 3.72 + 2\pi n \quad \text{for integers, } n.$$ 

**Answer:** $\theta = -0.58 + 2\pi n$ and $\theta = 3.72 + 2\pi n$ for integers, $n$.

(4.4.17) **Example.** Find all angles $\theta$ such that $\tan(\theta) = -4$.

**Solution:**
According to Lemma 4.4.14, the first two such values that we need are

$$\theta_1 = \tan^{-1}(-4) = -1.33 \quad \text{and} \quad \theta_2 = \pi + \tan^{-1}(-4) = 1.82.$$ 

To find all solutions to the equation we simply add $2\pi n$ to these two values for integers $n$. Hence all of the solutions are given by

$$\theta = -1.33 + 2\pi n \quad \text{and} \quad \theta = 1.82 + 2\pi n \quad \text{for integers, } n.$$ 

You might notice that together these solutions are the same as $\theta = 1.82 + \pi n$ for integers, $n$. This only happens with solutions to the tangent function. However, it is usually more convenient to leave the answer as two separate solutions.

**Answer:** $\theta = -1.33 + 2\pi n$ and $\theta = 1.82 + 2\pi n$ for integers, $n$.

(4.4.A) **Practice Exercise.** Find all of the angles $\theta$ such that $\sin(\theta) = 0.35$.

Sometimes the equation that we’re trying to solve gets more complicated, but as long the inside of the trigonometric function is just $\theta$, we can use a similar technique. This is shown in Example 4.4.18.

(4.4.18) **Example.** Find all angles $\theta$ such that $100 \cos(\theta) + 38 = 100$. 

Solution:
The first thing to do here is to solve for \( \cos(\theta) \):

\[
100\cos(\theta) + 38 = 100
\]
\[
100\cos(\theta) = 62
\]
\[
\cos(\theta) = 0.62
\]

At this point, we can solve \( \cos(\theta) = 0.62 \) as we did in the last three examples. The first two solutions are

\[
\theta = \arccos(0.62) = 0.90 + 2\pi n \quad \text{and} \quad \theta = -\arccos(0.62) = -0.90 + 2\pi n
\]

for integers, \( n \).

Answer: \( \theta = 0.90 + 2\pi n \) and \( \theta = -0.90 + 2\pi n \) for integers, \( n \).

It is also useful to ask for solutions to trigonometric equations in a certain interval. That is, we might want to know the values of \( \theta \) in the interval \([0, 4\pi]\) such that \( \sin(\theta) = -0.55 \). Of course, Example 4.4.16 tells us that all of the solutions to this equation are of the form

\[
\theta = -0.58 + 2\pi n \quad \text{and} \quad \theta = 3.72 + 2\pi n
\]

for integers, \( n \).

Hence we only need to figure out which of these solutions fall in the interval \([0, 4\pi]\). This can be done by making a table. The table below shows some of these angles for different values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( -0.58 + 2\pi n )</th>
<th>( 3.72 + 2\pi n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-6.87</td>
<td>-2.57</td>
</tr>
<tr>
<td>0</td>
<td>-0.58</td>
<td>3.72</td>
</tr>
<tr>
<td>1</td>
<td>5.70</td>
<td>10.00</td>
</tr>
<tr>
<td>2</td>
<td>11.98</td>
<td>16.29</td>
</tr>
<tr>
<td>3</td>
<td>18.27</td>
<td>22.57</td>
</tr>
</tbody>
</table>

To find all values of \( \theta \) such that \( \sin(\theta) = -0.55 \) and \( \theta \) is in the interval \([0, 4\pi]\) we simply need to find all of the values of \( \theta \) in the previous table which lie in that interval. Since \( 4\pi = 12.57 \), we see that the desired solutions are 3.72, 5.70, 10.00, and 11.98; values larger than 12.57 and smaller than 0 can be discarded since they are outside of \([0, 4\pi]\).

Note that only some of the values of \( n \) are included in the table. Our table should, theoretically, include each of the infinite number of integers, \( n \). It is not possible to make a table with an infinite number
of values in it, so how do we know that leaving out so many values of \( n \) does not cause us to miss any values of \( \theta \)? Well, if you look at the table it is clear that any smaller values of \( n \) (such as \(-2, -3, \) etc.) will yield negative angles which are not in the interval \([0, 4\pi]\) and any larger values of \( n \) (such as 4, 5, etc.) will yield angles larger than 22.57 which are not in the interval \([0, 4\pi]\). With a little bit of thought you should be able to decide where it is safe to stop your table.

We explore more of these types of equations in the next two examples.

\[\text{(4.4.19) Example.} \quad \text{Find all angles } \theta \text{ such that } \cos(\theta) = 0.3 \text{ and } -4\pi \leq \theta \leq 2\pi. \text{ Round to two decimal places.} \]

\[\text{Solution:} \]

We know from Example 4.4.15 that all of the angles \( \theta \) such that \( \cos(\theta) = 0.3 \) are

\[\theta = 1.27 + 2\pi n \quad \text{and} \quad \theta = -1.27 + 2\pi n \quad \text{for integers, } n.\]

To find the solutions \( \theta \) with \(-4\pi \leq \theta \leq 2\pi\), let’s make a table with some of the values of \( n \) and the corresponding values of \( \theta \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 1.27 + 2\pi n )</th>
<th>(-1.27 + 2\pi n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-17.58</td>
<td>-20.12</td>
</tr>
<tr>
<td>-2</td>
<td>-11.30</td>
<td>-13.83</td>
</tr>
<tr>
<td>-1</td>
<td>-5.02</td>
<td>-7.55</td>
</tr>
<tr>
<td>0</td>
<td>1.27</td>
<td>-1.27</td>
</tr>
<tr>
<td>1</td>
<td>7.55</td>
<td>5.02</td>
</tr>
<tr>
<td>2</td>
<td>13.83</td>
<td>11.30</td>
</tr>
</tbody>
</table>

Since \(-4\pi = -12.57 \) and \(2\pi = 6.28\) we see that the desired values of \( \theta \) are \(-11.30, -7.55, -5.02, -1.27, 1.27, \) and \( 5.02 \).

\[\text{Answer:} \quad -11.30, -7.55, -5.02, -1.27, 1.27, \text{ and } 5.02\]

\[\text{(4.4.20) Example.} \quad \text{Find all angles } \theta \text{ such that } \tan(\theta) = -4 \text{ and } 2\pi \leq \theta \leq 5\pi. \text{ Round to two decimal places.} \]

\[\text{Solution:} \]

We know from Example 4.4.17 that all of the angles \( \theta \) such that \( \tan(\theta) = -4 \) are

\[\theta = -1.33 + 2\pi n \quad \text{and} \quad \theta = 1.82 + 2\pi n \quad \text{for integers, } n.\]
To find the solutions $\theta$ with $2\pi \leq \theta \leq 5\pi$, let’s make a table with some of the values of $n$ and the corresponding values of $\theta$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$-1.33 + 2\pi n$</th>
<th>$1.82 + 2\pi n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-1.33$</td>
<td>$1.82$</td>
</tr>
<tr>
<td>1</td>
<td>4.96</td>
<td>8.10</td>
</tr>
<tr>
<td>2</td>
<td>11.24</td>
<td>14.38</td>
</tr>
<tr>
<td>3</td>
<td>17.52</td>
<td>20.67</td>
</tr>
</tbody>
</table>

Since $2\pi = 6.28$ and $5\pi = 15.71$ we see that the desired values of $\theta$ are 8.10, 11.24, and 14.39.

**Answer:** 8.10, 11.24, and 14.39

We finally have the tools to solve the types of equations we get from our Ferris wheel functions. Remember the function

$$H(t) = 80 \sin\left(2\pi\left(t - \frac{1}{12}\right)\right) + 100$$

from our Ferris wheel in Figure 4.4.1. Suppose we want to find all values of $t$ such that $H(t) = 80$. We start with the easy stuff:

$$80 \sin\left(2\pi\left(t - \frac{1}{12}\right)\right) + 100 = 80$$

$$80 \sin\left(2\pi\left(t - \frac{1}{12}\right)\right) = -20$$

$$\sin\left(2\pi\left(t - \frac{1}{12}\right)\right) = -0.25$$

Now, this last equation relates that any time $2\pi\left(t - \frac{1}{12}\right)$ is an angle $\theta$ such that $\sin(\theta) = -0.25$ then $t$ is a solution to $H(t) = 80$. We know how to find all of the angles $\theta$ such that $\sin(\theta) = -0.25$. The first two solutions are

$$\theta_1 = \sin^{-1}(-0.25) = -0.25 \quad \text{and} \quad \theta_2 = \pi - \sin^{-1}(-0.25) = 3.39$$

Then all of the angles $\theta$ such that $\sin(\theta) = -0.25$ are

$$\theta = -0.25 + 2\pi n \quad \text{and} \quad \theta = 3.39 + 2\pi n \quad \text{for integers, } n.$$

Finally, to find the values of $t$, we set each of these angles equal to $2\pi\left(t - \frac{1}{12}\right)$ and solve. Considering the first collection of angles, we get

$$2\pi\left(t - \frac{1}{12}\right) = -0.25 + 2\pi n$$

$$t - \frac{1}{12} = \frac{-0.25 + 2\pi n}{2\pi}$$

$$t - \frac{1}{12} = -0.04 + n$$

$$t = 0.04 + n.$$
Hence we have a collection of solutions which look like \( t = 0.04 + n \) for integers, \( n \). Considering the second collection of angles, we get

\[
2\pi \left( t - \frac{1}{12} \right) = 3.39 + 2\pi n
\]

\[
t - \frac{1}{12} = \frac{3.39 + 2\pi n}{2\pi}
\]

\[
t - \frac{1}{12} = 0.54 + n
\]

\[
t = 0.62 + n.
\]

Hence we have another collection of solutions which look like \( t = 0.62 + n \) for integers, \( n \). Together, all of the values of \( t \) such that \( H(t) = 80 \) are

\[
t = 0.04 + n \quad \text{and} \quad t = 0.62 + n \quad \text{for integers, } n.
\]

This process is straightforward to replicate. The last two examples here solve similar equations.

(4.4.21) **Example.** Find all values of \( x \) such that

\[
2 \sin(3x + 1) + 4\sqrt{3} = 5\sqrt{3}.
\]

**Solution:**

We start by solving the equation for \( \sin(3x + 1) \) as follows:

\[
2 \sin(3x + 1) + 4\sqrt{3} = 5\sqrt{3}
\]

\[
2 \sin(3x + 1) = \sqrt{3}
\]

\[
\sin(3x + 1) = \frac{\sqrt{3}}{2}
\]

We now set \( \theta = 3x + 1 \) and find all values of \( \theta \) such that \( \sin(\theta) = \frac{\sqrt{3}}{2} \). The first two values are \( \theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \) and \( \theta = \pi - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} \). It follows that all of the values of \( \theta \) such that \( \sin(\theta) = \frac{\sqrt{3}}{2} \) are

\[
\theta = \frac{\pi}{3} + 2\pi n \quad \text{and} \quad \theta = \frac{2\pi}{3} + 2\pi n \quad \text{for integers, } n.
\]
Then we substitute \( \theta = 3x + 1 \) back into those two equations for \( \theta \). The first yields

\[
3x + 1 = \frac{\pi}{3} + 2\pi n
\]

\[
3x = \frac{\pi}{3} - 1 + 2\pi n
\]

\[
x = \frac{\pi}{9} - \frac{1}{3} + \frac{2\pi}{3} n.
\]

The second yields

\[
3x + 1 = \frac{2\pi}{3} + 2\pi n
\]

\[
3x = \frac{2\pi}{3} - 1 + 2\pi n
\]

\[
x = \frac{2\pi}{9} - \frac{1}{3} + \frac{2\pi}{3} n.
\]

**Answer:** \( x = \frac{\pi}{9} - \frac{1}{3} + \frac{2\pi}{3} n \) and \( x = \frac{2\pi}{9} - \frac{1}{3} + \frac{2\pi}{3} n \) for integers, \( n \).

---

**Example.** Find all values of \( t \) such that

\[
8 \cos\left(\frac{\pi}{4}(t - 5)\right) + \frac{10}{3} = 2
\]

**Solution:**

We first solve for \( \cos\left(\frac{\pi}{4}(t - 5)\right) \) as follows:

\[
8 \cos\left(\frac{\pi}{4}(t - 5)\right) + 10 = 2
\]

\[
8 \cos\left(\frac{\pi}{4}(t - 5)\right) = 6
\]

\[
\cos\left(\frac{\pi}{4}(t - 5)\right) = -\frac{1}{2}
\]

We now let \( \theta = \frac{\pi}{4}(t - 5) \) and solve \( \cos(\theta) = -\frac{1}{2} \). The first two values of \( \theta \) are \( \theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \) and \( -\cos^{-1}\left(-\frac{1}{2}\right) = -\frac{2\pi}{3} \). It follows that all of the values of \( \theta \) such that \( \cos(\theta) = -\frac{1}{2} \) are

\[
\theta = \frac{2\pi}{3} + 2\pi n \quad \text{and} \quad \theta = -\frac{2\pi}{3} + 2\pi n
\]

for integers, \( n \).
Then we substitute $\theta = \frac{\pi}{4}(t - 5)$ back in to those two equations for $\theta$. The first yields

$$\frac{\pi}{4}(t - 5) = \frac{2\pi}{3} + 2\pi n$$

$$t - 5 = \frac{8}{3} + 8n$$

$$t = \frac{23}{3} + 8n$$

The second yields

$$\frac{\pi}{4}(t - 5) = -\frac{2\pi}{3} + 2\pi n$$

$$t - 5 = -\frac{8}{3} + 8n$$

$$t = \frac{7}{3} + 8n$$

**Answer:** $t = \frac{23}{3} + 8n$ and $t = \frac{7}{3} + 8n$ for integers, $n$.

The final thing to mention is that it is often useful to restrict the domain in examples like the last two. That is, in Example 4.4.22 it might not be reasonable to look for all of the values of $t$. Suppose we wanted to solve the equation

$$\frac{8 \cos \left( \frac{\pi}{4}(t - 5) \right) + 10}{3} = 2$$

for values of $t$ in the interval $[-20, 20]$. We know that all of the values of $t$ are $t = \frac{23}{3} + 8n$ and $t = \frac{7}{3} + 8n$ for integers, $n$. To find the values which fall in the interval $[-20, 20]$ we can make a table as we did before.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\frac{23}{3} + 8n$</th>
<th>$\frac{7}{3} + 8n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>$-\frac{73}{3}$</td>
<td>$-\frac{89}{3}$</td>
</tr>
<tr>
<td>-3</td>
<td>$-\frac{49}{3}$</td>
<td>$-\frac{65}{3}$</td>
</tr>
<tr>
<td>-2</td>
<td>$-\frac{25}{3}$</td>
<td>$-\frac{41}{3}$</td>
</tr>
<tr>
<td>-1</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{17}{3}$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{23}{3}$</td>
<td>$\frac{7}{3}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{47}{3}$</td>
<td>$\frac{31}{3}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{71}{3}$</td>
<td>$\frac{55}{3}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{95}{3}$</td>
<td>$\frac{79}{3}$</td>
</tr>
</tbody>
</table>

We only have to take the values out of the table that are in the interval $[-20, 20]$. Hence the solutions we’re looking for are

$$-\frac{49}{3}, -\frac{41}{3}, -\frac{25}{3}, -\frac{17}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{7}{3}, \frac{23}{3}, \frac{31}{3}, \frac{47}{3}, \frac{55}{3}, \frac{79}{3}, \frac{95}{3}.$$
Applications

The types of applications in this section should not be surprising. We will be finding solutions to the models that we’ve developed to this point.

(4.4.23) Example. The tire of a particular car is 620 mm in diameter and the valve stem on the wheel is at a radius of 200 mm. The diagram below demonstrates the geometry; the dashed line represents the ground, the large circle represents the outside diameter of the tire, and the point \( v \) represents the position of the valve stem.

If the valve stem makes an angle of \( \theta \) with the horizontal then it is \( f(\theta) \) mm off of the ground where

\[
f(\theta) = 200 \sin(\theta) + 310.
\]

Find all of the angles \( \theta \) at which the valve stem is 400 mm off of the ground. Note: This model was developed in an exercise in Section 3.4.

Solution:

We are looking to find all angles \( \theta \) such that \( f(\theta) = 400 \). We start with

\[
200 \sin(\theta) + 310 = 400
\]

\[
200 \sin(\theta) = 90
\]

\[
\sin(\theta) = \frac{9}{20}.
\]

The first solution is \( \theta_1 = \sin^{-1}\left(\frac{9}{20}\right) = 0.47 \) and the second solution is \( \theta_2 = \pi - \sin^{-1}\left(\frac{9}{20}\right) = 2.67 \). Then all of the solutions are

\[
\theta = 0.47 + 2\pi n \quad \text{and} \quad \theta = 2.67 + 2\pi n \quad \text{for integers, } n.
\]
(4.4.24) Example. An engineer measures the oscillating pressure in a certain pipe. He finds that it takes 45 min for the pressure to drop from a high of 5000 kPa to a low of 2000 kPa and then another 45 min for it to get back up to 5000 kPa. (kPa is a unit of pressure called the kilopascal.) The function $P$ models the pressure since the first time the pressure was at its peak. That is, $P(t)$ is the pressure (in kPa) in the pipe $t$ minutes after the first time the pressure reached 5000 kPa. Given that

$$P(t) = 1500 \sin \left( \frac{\pi}{45} (t + 17.5) \right) + 3500,$$

find all of the times between 0 min and 250 min at which the pressure is 3000 kPa. Note: This model was developed in Example 4.3.15.

Solution:

We are looking for values of $t$ in $[0,250]$ such that $P(t) = 3000$. We start with

$$1500 \sin \left( \frac{\pi}{45} (t + 17.5) \right) + 3500 = 3000$$

$$1500 \sin \left( \frac{\pi}{45} (t + 17.5) \right) = -500$$

$$\sin \left( \frac{\pi}{45} (t + 17.5) \right) = -\frac{1}{3}.$$

Next, we let $\theta = \frac{\pi}{45} (t + 17.5)$ and solve $\sin(\theta) = -\frac{1}{3}$. The first two values are $\theta_1 = \sin^{-1}(-\frac{1}{3}) = -0.34$ and $\theta = \pi - \sin^{-1}(-\frac{1}{3}) = 3.48$. Then all of the angles $\theta$ such that $\sin(\theta) = -\frac{1}{3}$ are

$$\theta = -0.34 + 2\pi n \quad \text{and} \quad \theta = 3.48 + 2\pi n \quad \text{for integers, } n.$$

We then set $\frac{\pi}{45} (t + 17.5)$ equal to those two equations for $\theta$. The first one yields

$$\frac{\pi}{45} (t + 17.5) = -0.34 + 2\pi n$$

$$t + 17.5 = -4.87 + 90n$$

$$t = -22.37 + 90n.$$
The second one yields

\[ \frac{\pi}{45} (t + 17.5) = 3.48 + 2\pi n \]
\[ t + 17.5 = 49.87 + 90n \]
\[ t = 67.37 + 90n. \]

Now that we know all of the \( t \) such that \( P(t) = 3000 \), we need only find those \( t \) in the interval \([0, 250]\). After making a table, we find that the values in the interval \([0, 250]\) are 12.63, 67.37, 102.63, 157.37, 192.63, and 247.37 (all units in minutes).

**Answer:** 12.63, 67.37, 102.63, 157.37, 192.63, and 247.37 (all units in minutes)

(4.4.b) **Practice Exercise.** In Example 4.3.16 we found a function which described the voltage in an outlet. That is, \( P(t) \) is the voltage (in volts) in the outlet \( t \) seconds after the power was turned on. Given that

\[ P(t) = 155.6 \sin(120\pi(t + 0.0125)), \]

find all values of \( t \) at which the voltage in the outlet is 100 V.
4.4.1. What is the domain and range of $\sin^{-1}$ and $\cos^{-1}$?

4.4.2. Find all possible solutions to $\sin(\theta) = \frac{4}{7}$. Round to two decimal places.

4.4.3. Find all solutions to $\cos(\theta) = -\frac{2}{5}$. Round to two decimal places.

4.4.4. Find all solutions to $\sin(\theta) = \frac{1}{3}$. Round to two decimal places.

4.4.5. Find all solutions to $\tan(\theta) = -2.5$. Round to two decimal places.

4.4.6. Find all solutions to $\sin(\theta) = -\frac{\sqrt{2}}{2}$ such that $\pi \leq \theta \leq 4\pi$. Leave your answers in exact form.

4.4.7. Find all of the solutions to $\tan(\theta) = \sqrt{3}$ such that $-2\pi \leq \theta \leq 2\pi$. Leave your answers in exact form.

4.4.8. Find all of the solutions to $\cos(\theta) = 0.24$ such that $-5\pi \leq \theta \leq 0$. Round to two decimal places.

4.4.9. Find all values of $\theta$ such that $4\sin(\theta) + 3 = 1$. Leave your answers in exact form.

4.4.10. Find all possible solutions to $\frac{2\cos(\theta) - 1}{3} = -1$. Leave your answers in exact form.

4.4.11. Find all of the solutions to $\cos^2(\theta) = \frac{3}{4}$ such that $0 \leq \theta \leq 2\pi$. Leave your answers in exact form.

4.4.12. Find all solutions to $4\sin(2\theta + \frac{\pi}{4}) - 1 = 1$. Leave your answers in exact form.

4.4.13. Find all solutions to $3\tan(3x + 5) + 8 = 15$. Round to two decimal places.

4.4.14. Consider the function $f(\theta) = \cos(2\theta + \frac{\pi}{6})$.

4.4.15. A.) Find all possible values of $\theta$ such that $f(\theta) = -\frac{\sqrt{3}}{2}$. Leave your answers in exact form.

B.) Find all values of $\theta$ such that $0 \leq \theta \leq 3\pi$ and $f(\theta) = -\frac{\sqrt{3}}{2}$. Leave your answers in exact form.

4.4.16. Elmer is on a Ferris wheel which has a diameter of 140 ft and whose center is 80 ft off the ground. Find all of the angles $\theta$ such that $0 \leq \theta \leq 3\pi$ and such that Elmer’s carriage is at a height of 115 ft when it makes an angle of $\theta$ with the horizontal. Leave your answers in exact form.

4.4.17. In a previous exercise we discussed a water wheel of radius 15 ft which was submerged 3 ft in flowing water. A beetle was sitting on the very edge of the water wheel and we found that when the beetle made an angle of $\theta$ with the horizontal when the beetle was $f(\theta)$ ft above the water where $f(\theta) = 15\sin(\theta) + 12$. Assuming that $\theta$ is in radians now (the same equation still works), find all of the angles at which the beetle is exactly at the surface of the water. Round to two decimal places when appropriate. Hint: What is the beetle’s height at this point?

4.4.18. A population of a particular town is modeled by a function $P$. That is, $P(t)$ is
approximately the population \( t \) years after 2000. Given that

\[
P(t) = 500 \sin \left( \frac{\pi}{5} (t - 3) \right) + 8000,
\]

find all of the values of \( t \) in the interval \([0, 20]\) at which the population reaches 8400. Round to two decimal places.

4.4.19. The height of a particular blade of a wind turbine is modeled by the function \( H \). That is, \( t \) minutes after the turbine is turned on, the height of the end of this blade is \( H(t) \) meters where

\[
H(t) = 40 \sin \left( 80\pi \left( t + \frac{1}{320} \right) \right) + 60.
\]

Find all of the values of \( t \) at which the end of the blade is at a height of 50 m. Round your answers to five decimal places. This model was developed in examples in previous sections.

**Challenge Problems**

(4.4.c1) **Challenge Problem.** Each of the parts of this problem asks you to find solutions to some equation. If there are an infinite number of solutions then be sure to choose a good way to indicate all of them. Be sure to show your work.

A. Find all angles \( \theta \) such that

\[
\sin^2(\theta) - 0.65 \sin(\theta) - 0.66 = 0.
\]

Round your answers to two decimal places.

B. Find all angles \( \theta \) such that

\[
\cos^2(\theta) + \left( \frac{\sqrt{3} - 1}{2} \right) \cos(\theta) = \frac{\sqrt{3}}{4}.
\]

Leave your answers in exact form.

C. Find the smallest value of \( x \) such that

\[
\sin \left( \frac{\pi}{3} (\sqrt{x} + 10) \right) = \frac{\sqrt{3}}{2}.
\]

Leave your answer in exact form.
4.5 RELATIONSHIPS AND GRAPHS

This section corresponds to Section 8.3 in Functions Modeling Change [4].

The family of trigonometric functions is fascinating and complex. There are a lot of relationships hidden within them. We have already found some of these relationships and in this section we explore some of the others. This section has no motivation and no applications; at least none that are going to be available immediately. However, in more complex areas of trigonometry it is crucial to be able to rewrite trigonometric expressions in convenient forms.

For example, in calculus we study something called integration. It turns out that the function \( f(x) = 2 \sin(x) \cos(x) \) is difficult to integrate when written in that form. However, it just so happens that \( 2 \sin(x) \cos(x) = \sin(2x) \) and the function \( f(x) = \sin(2x) \) is easy to integrate. It is possible that you never need the techniques in this section, but if you find yourself working with trigonometry they are absolutely imperative.

The three trigonometric functions that we’ve discussed so far are the most common. However, there are three more that are very useful at times. These are defined in Definition 4.5.1.

(4.5.1) Definition.

- The secant function, sec, is defined such that
  \[
  \sec(\theta) = \frac{1}{\cos(\theta)}.
  \]

- The cosecant function, csc, is defined such that
  \[
  \csc(\theta) = \frac{1}{\sin(\theta)}.
  \]

- The cotangent function, cot, is defined such that
  \[
  \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}.
  \]

Note that, most of the time, \( \cot(\theta) = \frac{1}{\tan(\theta)} \). However, we don’t want to define it like that. Note that \( \tan\left(\frac{\pi}{2}\right) \) is undefined which means that \( \frac{1}{\tan\left(\frac{\pi}{2}\right)} \) is undefined. However,

\[
\cot\left(\frac{\pi}{2}\right) = \frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \frac{0}{1} = 0
\]

and so \( \cot\left(\frac{\pi}{2}\right) \) is defined. We have that \( \cot(\theta) = \frac{1}{\tan(\theta)} \) whenever \( \tan(\theta) \) is defined.
Sometimes these three new functions are referred to as reciprocal trigonometric functions. Their graphs are shown in Figure 4.5.2.

Figure 4.5.2: The graph of \( y = \sec(x) \), \( y = \csc(x) \), and \( y = \cot(x) \).

(4.5.3) Example. Calculate the following:

A. \( \csc \left( \frac{\pi}{4} \right) \)

B. \( \cot \left( \frac{7\pi}{6} \right) \)

C. \( \sec(0.32) \)

Solution:

A. Using Definition 4.5.1,

\[
csc \left( \frac{\pi}{4} \right) = \frac{1}{\sin \left( \frac{\pi}{4} \right)} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} = 2\sqrt{2}.
\]

**Answer:** \( 2\sqrt{2} \)

B. Using Definition 4.5.1,

\[
cot \left( \frac{7\pi}{6} \right) = \frac{\cos \left( \frac{7\pi}{6} \right)}{\sin \left( \frac{7\pi}{6} \right)} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}.
\]

**Answer:** \( \sqrt{3} \)

C. Using Definition 4.5.1,

\[
\sec(0.32) = \frac{1}{\cos(0.32)} = 1.05.
\]

**Answer:** \( 1.05 \)
Note that most calculators do not have specific buttons for reciprocal trigonometric functions. To calculate them, you should use Definition 4.5.1 directly.

Lemma 4.5.4 gives the very long list of identities that we’re going to be using. We have used the trigonometric functions to relate values before. The Law of Sines and Law of Cosines use the sine and cosine function to relate quantities in triangles. The identities in Lemma 4.5.4 differ from those things in that they express relationships between the trigonometric functions themselves.

(4.5.4) Lemma. In the formulas below, $\alpha$, $\beta$, and $\theta$ can be any angle.

**Basic Formulas**
1. $\sin^2(\theta) + \cos^2(\theta) = 1$
2. $\cos(-\theta) = \cos(\theta)$
3. $\sin(-\theta) = -\sin(\theta)$
4. $\tan(-\theta) = -\tan(\theta)$
5. $\sin(\theta) = \cos(\theta - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - \theta) = -\sin(\theta + \pi)$
6. $\cos(\theta) = \sin(\theta + \frac{\pi}{2}) = \sin(\frac{\pi}{2} - \theta) = -\cos(\theta + \pi)$

**Half Angle and Double Angle Formulas**
7. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
8. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
9. $\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$
10. $\cos(\frac{\theta}{2}) = \pm \sqrt{\frac{1 + \cos(\theta)}{2}}$
11. $\sin(\frac{\theta}{2}) = \pm \sqrt{\frac{1 - \cos(\theta)}{2}}$
12. $\tan(\frac{\theta}{2}) = \frac{1 - \cos(\theta)}{\sin(\theta)} = \frac{\sin(\theta)}{1 + \cos(\theta)}$

**Sum and Difference Formulas**
13. $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
14. $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
15. $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
16. $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
17. $\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$
The basic formulas are things that we’ve seen before but the rest of them are new and exciting. These identities can be used for many things. The rest of the section gives examples of such applications. First, we can calculate some exact values of trigonometric functions if we need to do so.

(4.5.5) Example. Find an exact value for \( \sin(22.5^\circ) \).

Solution:
This isn’t one of the special angles but it is half of one of the special angles. That is, we know everything about the trigonometric functions at \( 45^\circ \). Using Equation 11 in Lemma 4.5.4 with \( \theta = 45^\circ \) we have

\[
\sin(22.5^\circ) = \sin\left(\frac{45^\circ}{2}\right) = \pm \sqrt{\frac{1 - \cos(45^\circ)}{2}} = \pm \sqrt{\frac{1 - \sqrt{2}/2}{2}} = \pm \sqrt{\frac{2 - \sqrt{2}}{4}} = \pm \sqrt{\frac{2 - \sqrt{2}}{2}}.
\]

Of course, \( \sin(22.5^\circ) \) cannot be two different values. Since \( 22.5^\circ \) has an inscription point in the fist quadrant, we know that \( \sin(22.5^\circ) \) must be positive.

Answer: \( \sin(22.5^\circ) = \sqrt{\frac{2 - \sqrt{2}}{2}} \)

(4.5.6) Example. Find an exact value for \( \cos(75^\circ) \).

Solution:
Again, this isn’t one of the special angles, but we do know that \( 75^\circ = 45^\circ + 30^\circ \) and \( 45^\circ \) and \( 30^\circ \) are special angles. Then
we can use Equation 13 in Lemma 4.5.4 with \( \alpha = 45^\circ \) and \( \beta = 30^\circ \) as follows:

\[
\cos(75^\circ) = \cos(45^\circ + 30^\circ)
= \cos(45^\circ) \cos(30^\circ) - \sin(45^\circ) \sin(30^\circ)
= \left( \frac{\sqrt{2}}{2} \right) \left( \frac{\sqrt{3}}{2} \right) - \left( \frac{\sqrt{2}}{2} \right) \left( \frac{1}{2} \right)
= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}
= \frac{\sqrt{6} - \sqrt{2}}{4}
\]

**Answer:** \( \cos(75^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4} \)

(4.5.a) **Practice Exercise.** Use Example 4.5.6 and the fact that \( 37.5^\circ = \frac{75^\circ}{2} \) to find an exact value for \( \sin(37.5^\circ) \).

These are interesting, but it is hardly the most useful application. In practice we don’t usually need exact values this precise; approximations are generally fine. The next two examples, however, show how rewriting a trigonometric expression can help us graph a function.

(4.5.7) **Example.** Sketch a graph of the function \( f(x) = (\sin(x) + \cos(x))^2 \).

**Solution:**

The right side of this function is pretty complicated. We can sketch the graph of \( y = (\sin(x) + \cos(x)) \) using the techniques in Section 2.1, but then sketching the square of that will be complicated. Instead of dealing with that mess, we try to simplify the expression \( (\sin(x) + \cos(x))^2 \). We’ll do this by distributing and seeing what we get.

\[
(\sin(x) + \cos(x))^2 = \sin^2(x) + 2 \sin(x) \cos(x) + \cos^2(x)
= 2 \sin(x) \cos(x) + (\sin^2(x) + \cos^2(x))
= 2 \sin(x) \cos(x) + 1
= \sin(2x) + 1
\]

Note that we used the fact that \( \sin^2(x) + \cos^2(x) = 1 \) from Equation 1 in Lemma 4.5.4 and the fact that \( 2 \sin(x) \cos(x) = \sin(2x) \). We now have that \( f(x) = \sin(2x) + 1 \). This function is
much easier to graph. It is a sinusoidal function and, using the techniques in Section 4.3, we have the following graph:

\[
\begin{align*}
\text{Graph of } f(x) &= \cos^4(x) - \sin^4(x) \\
\end{align*}
\]

**Answer:** On graph above.

(4.5.8) **Example.** Sketch a graph of the function \( f(x) = \cos^4(x) - \sin^4(x) \).

**Solution:**

As in Example 4.5.7 we want to start by simplifying the function as much as possible.

\[
\begin{align*}
\cos^4(x) - \sin^4(x) &= (\cos^2(x) - \sin^2(x))(\cos^2(x) + \sin^2(x)) \\
&= (\cos^2(x) - \sin^2(x))(1) \\
&= \cos(2x)
\end{align*}
\]

The first line is just factoring of the form 

\[
\begin{align*}
a^4 - b^4 &= (a^2 - b^2)(a^2 + b^2)
\end{align*}
\]

The next two lines use the fact that \( \cos^2(x) + \sin^2(x) = 1 \) from Equation 1 in Lemma 4.5.4 and the fact that \( \cos^2(x) - \sin^2(x) = \cos(2x) \) from Equation 7 in Lemma 4.5.4. We now have that \( f(x) = \cos(2x) \), which is simple enough to graph, as shown below:
The rest of the examples that we investigate are rudimentary versions of what mathematicians call **proofs**. That is, you will be given something that is already known to be true and asked to verify it.Lemma 4.5.4 gives a list of trigonometric identities. Each of the following example gives an additional identity and we are asked to derive it using the identities in Lemma 4.5.4. It is not necessarily the identities themselves that are are useful here, but the skills used in deriving them.

\[
\frac{\sin(\alpha + \beta)}{\cos(\alpha) \cos(\beta)} = \tan(\alpha) + \tan(\beta) \quad (4.5.10)
\]

**Solution:**

We are going to start with one side of Equation 4.5.10 and use the equations in Lemma 4.5.4 to see if we can get to the other side. In general, it is usually a better strategy to start with the side that “looks more complicated.” If that side doesn’t work, switch to the other side and start over.

\[
\frac{\sin(\alpha + \beta)}{\cos(\alpha) \cos(\beta)} = \frac{\sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)}{\cos(\alpha) \cos(\beta)}
\]

\[
= \frac{\sin(\alpha) \cos(\beta)}{\cos(\alpha) \cos(\beta)} + \frac{\cos(\alpha) \sin(\beta)}{\cos(\alpha) \cos(\beta)}
\]

\[
= \frac{\sin(\alpha)}{\cos(\alpha)} + \frac{\sin(\beta)}{\cos(\beta)}
\]

\[
= \tan(\alpha) + \tan(\beta)
\]
The equality between the first and last step of this equation show that
\[
\frac{\sin(\alpha + \beta)}{\cos(\alpha) \cos(\beta)} = \tan(\alpha) + \tan(\beta),
\]
which is precisely what we wanted to show in Equation 4.5.10. All we really used was Equation 15 in Lemma 4.5.4 and some algebra.

**Example.** For all values of $\theta$, show that
\[
\sin(2\theta) = \frac{2 \cot(\theta)}{1 + \cot^2(\theta)} \quad (4.5.12)
\]

**Solution:**
When these equations involve the reciprocal trigonometric functions, it is often the best strategy to write them all in terms of the sine and cosine functions. We are going to start with the right hand side of Equation 4.5.12.

\[
\begin{align*}
\frac{2 \cot(\theta)}{1 + \cot^2(\theta)} &= \frac{2 \cos(\theta)}{\sin(\theta)} \\
&= \frac{2 \cos(\theta)}{1 + \frac{\cos^2(\theta)}{\sin^2(\theta)}} \\
&= \frac{2 \cos(\theta)}{\sin^2(\theta) + \cos^2(\theta)} \\
&= \frac{2 \cos(\theta)}{\sin(\theta)} \cdot \frac{\sin^2(\theta)}{\sin^2(\theta) + \cos^2(\theta)} \\
&= \frac{2 \cos(\theta)}{\sin(\theta)} \cdot \frac{1}{1} \\
&= 2 \cos(\theta) \sin(\theta) \\
&= \sin(2\theta)
\end{align*}
\]
After rewriting every instance of $\cot(\theta)$ as $\frac{\cos(\theta)}{\sin(\theta)}$ and manipulating the algebra, we used Equation 8 in Lemma 4.5.4. We have now shown Equation 4.5.12, as desired.
Example. For all values of $\theta$, show that
\[ \csc(2\theta) = \frac{1}{2} (\cot(\theta) + \tan(\theta)). \] (4.5.14)

Solution:
Again, it is usually the best idea to start with the more complicated side of the identity, which is the right side.

\[
\frac{1}{2} (\cos(\theta) + \tan(\theta)) = \frac{1}{2} \left( \frac{\cos(\theta)}{\sin(\theta)} + \frac{\sin(\theta)}{\cos(\theta)} \right) \\
= \frac{\cos^2(\theta) + \sin^2(\theta)}{2 \sin(\theta) \cos(\theta)} \\
= \frac{1}{2 \sin(\theta) \cos(\theta)} \\
= \frac{1}{\sin(2\theta)} \\
= \csc(2\theta).
\]

We used Equation 1 and Equation 8 in Lemma 4.5.4. This shows Equation 4.5.14, as desired.

Example. For all values of $\theta$, show that
\[ 2 \sin(\alpha + \beta) \cos(\alpha - \beta) = \sin(2\alpha) + \sin(2\beta) \] (4.5.16)

Solution:
This one takes a lot of steps, but it is simply a matter of expanding everything and simplifying. We start with the right side of Equation 4.5.16.

\[
2 \sin(\alpha + \beta) \cos(\alpha - \beta)
\]
\[
= 2 \left( \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta) \right)
\]
\[
\left( \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) \right)
\]
\[
= 2 \left( \sin(\alpha) \cos(\alpha) \cos^2(\beta) + \sin(\alpha) \cos(\alpha) \sin^2(\beta)
\right.
\]
\[
+ \sin(\beta) \cos(\beta) \sin^2(\alpha) + \sin(\beta) \cos(\beta) \cos^2(\alpha) \bigg)
\]
\[
= 2 \left( \sin(\alpha) \cos(\alpha) \left( \sin^2(\beta) + \cos^2(\beta) \right)
\right.
\]
\[
+ \sin(\beta) \cos(\beta) \left( \sin^2(\alpha) + \cos^2(\alpha) \right) \bigg)
\]
\[
= 2 \left( \sin(\alpha) \cos(\alpha) + \sin(\beta) \cos(\beta) \right)
\]
\[
= 2 \sin(\alpha) \cos(\alpha) + 2 \sin(\beta) \cos(\beta)
\]
\[
= \sin(2\alpha) + \sin(2\beta)
\]
4.5.1. If \( \cos(\theta) = \frac{1}{2} \) and \( \sin(\theta) < 0 \), find the following and give exact answers:

- A.) \( \sin(\theta) \)
- B.) \( \tan(\theta) \)
- C.) \( \cot(\theta) \)
- D.) \( \sec(\theta) \)
- E.) \( \csc(\theta) \)

4.5.2. If \( \csc(\theta) = -\frac{3}{2} \) and \( \cos(\theta) < 0 \), find the following and give exact answers:

- A.) \( \sin(\theta) \)
- B.) \( \cos(\theta) \)
- C.) \( \tan(\theta) \)
- D.) \( \sec(\theta) \)
- E.) \( \cot(\theta) \)

4.5.3. If \( \csc(\theta) = 3 \) and \( \frac{\pi}{2} \leq \theta < \pi \), find the following and give exact answers:

- A.) \( \sin(\theta) \)
- B.) \( \cos(\theta) \)
- C.) \( \tan(\theta) \)
- D.) \( \sec(\theta) \)
- E.) \( \cot(\theta) \)

4.5.4. Calculate the following and give exact answers:

- A.) \( \sec\left(\frac{11\pi}{6}\right) \)
- B.) \( \cot\left(-\frac{3\pi}{4}\right) \)
- C.) \( \csc\left(\frac{4\pi}{3}\right) \)
- D.) \( \sec\left(\frac{2\pi}{3}\right) \)
- E.) \( \csc\left(\frac{3\pi}{4}\right) \)
- F.) \( \cot\left(\frac{\pi}{4}\right) \)
- G.) \( \sec\left(\frac{7\pi}{3}\right) \)
- H.) \( \csc\left(\frac{5\pi}{4}\right) \)
- I.) \( \cot\left(\frac{11\pi}{6}\right) \)

4.5.5. Find an angle \( \theta \) such that \( \csc(\theta) = \frac{\pi}{2} \) and \( \sec(\theta) < 0 \) where \( 0 \leq \theta < 2\pi \). Round your answer to two decimal places.

4.5.6. Suppose that \( 0 \leq \theta < 2\pi \), \( \csc(\theta) = 5/2 \) and \( \sec(\theta) < 0 \). Find \( \theta \) and round to two decimal places.

4.5.7. Find an angle \( \theta \) such that \( 0 \leq \theta < 2\pi \), \( \sec(\theta) = 2 \), and \( \csc(\theta) < 0 \). Round to two decimal places.

4.5.8. Find an angle \( \theta \) such that \( \csc(\theta) = -\frac{27}{13} \), \( \sec(\theta) < 0 \), and \( 0 \leq \theta < 2\pi \). Round your answer to two decimal places.

4.5.9. Suppose \( \theta \) is an angle such that \( \csc(\theta) = -\frac{7}{3} \) and \( \frac{\pi}{2} \leq \theta < \frac{3\pi}{2} \).

- A.) Find \( \cos(\theta) \).
- B.) Find \( \theta \) and round to two decimal places.

4.5.10. Find all of the angles \( \theta \) such that \( \csc(\theta) = -\frac{2}{\sqrt{3}} \). Leave your answer in exact form.

4.5.11. Find an exact value for \( \tan\left(\frac{\pi}{12}\right) \). It may be helpful to use the fact that \( \frac{\pi}{12} = \frac{\pi}{6}/2 \).

4.5.12. Find an exact value for \( \sin\left(\frac{17\pi}{12}\right) \). It may be helpful to use the fact that \( \frac{17\pi}{12} = \frac{7\pi}{6} + \frac{\pi}{4} \).

4.5.13. Show that

\[
\left(\sin(\theta) + \cos(\theta)\right) \left(\tan(\theta) + \cot(\theta)\right) = \sec(\theta) + \csc(\theta).
\]

Carefully show each step.

4.5.14. Show that

\[
\sec^2(\theta) - \tan^2(\theta) = 1.
\]

Carefully show each step.

4.5.15. Show that

\[
\frac{1}{1 - \cos(\theta)} + \frac{1}{1 + \cos(\theta)} = 2\csc^2(\theta).
\]

Carefully show each step.
4.5.16. Show that
\[ \cot(\alpha + \beta) = \frac{\cot(\alpha) \cot(\beta) - 1}{\cot(\beta) + \cot(\alpha)}. \]
Carefully show each step.

4.5.17. Show that
\[ \cos^2\left(\frac{\theta}{2}\right) = \frac{\sec(\theta) + 1}{2\sec(2\theta)} \]
Carefully show each step.

4.5.18. Show that
\[ \frac{\sin(3x)}{\sin(x)} - \frac{\cos(3x)}{\cos(x)} = 2. \]
Carefully show each step.

**Challenge Problems**

(4.5.c1) Challenge Problem.

A. Find an exact value for \( \cos(36^\circ) = \cos\left(\frac{\pi}{5}\right) \). Do not use any other known values of sine and cosine. Note: This is the most difficult part of this problem. Once you get this part, the rest are easier. There is a way to do this problem without any of the tools from this section. That is, it can be done geometrically using only the techniques up to Section 3.2. I will accept that solution, but it is much more difficult to do it that way.

B. Use the previous part to find \( \sin(36^\circ) = \sin\left(\frac{\pi}{5}\right) \).

C. Use the previous part(s) to find \( \cos(18^\circ) = \cos\left(\frac{\pi}{10}\right) \).

D. Use the previous part(s) to find \( \sin(18^\circ) = \sin\left(\frac{\pi}{10}\right) \).
VECTORS

The material in this chapter serves as a very elementary introduction to an area of mathematics called linear algebra. We focus on the most rudimentary forms of vectors and their applications. This chapter covers the content of Chapter 12 in College Algebra and Trigonometry for the University of Oregon, by Connally et al. [4]
5.1 INTRODUCTION TO VECTORS

This section corresponds to section 12.1 in Functions Modeling Change [4]

This section serves as a rough introduction to vectors. We define vectors theoretically and pictorially and we learn how to manipulate them. In the next section we will develop a more useful notation to facilitate these notions, but for now we want to focus on understanding how vectors conceptually.

**Motivation**

Imagine that you want to describe to a friend how to travel from Eugene, Oregon to Milwaukee, Wisconsin. You could tell them that it is 1750 mi away, but that’s not enough information. If your friend just starts in Eugene and goes 1750 mi there is almost no chance they will end up in Milwaukee. You need to tell them to go 1750 mi east. That is, the distance is not enough. In order to describe how to get from one place to another you need to report both distance and direction.

This is one example of how sometimes a simple quantity is not enough to describe a situation accurately. In this section we will introduce vectors which will encode information like distance and direction.

**Theory**

A vector can be a difficult thing to define in the situation that we’ll be using them. What we want to do is have a quantity which describes displacement. Imagine two points in a plane, P and Q, as shown in Figure 5.1.1. We want a vector to be something that describes the displacement from P to Q. In its most rudimentary form, this can be done with an arrow.

![Figure 5.1.1: The vector \( \vec{v} \) from a point P to a point Q.](image)

What are the attributes of an arrow? Remember, we’re trying to describe the displacement. That is, a vector doesn’t care about its starting and ending point; it only cares about the displacement. The
attributes of the arrow itself are the direction that it points and its length. In vector language, the length of the arrow is called its magnitude.

\[(5.1.2) \text{ Definition. A \textbf{vector} is a measurement of displacement. This quantity consists of a \textbf{direction} and a \textbf{magnitude}.}\]

The notation that we use for a vector is a letter with an arrow above it. That is, instead of writing \( v \) to denote a vector we will write \( \vec{v} \). Then, in order to denote the magnitude of \( \vec{v} \) we write \( \| \vec{v} \| \). Remember that when we draw a vector as an arrow, the length of the arrow is its magnitude. A vector \( \vec{v} \) is shown in Figure 5.1.1. According to the figure, the magnitude of \( \vec{v} \) is 7 so \( \| \vec{v} \| = 7 \).

Note that the magnitude of a vector can never be negative under any circumstances. The length of an arrow can never be negative and a displacement in a certain direction must be positive; that is, the distance that something moves cannot be negative.

It is important to understand that a vector has nothing to do with its starting and ending point. Consider Figure 5.1.3. Each arrow is a vector, but vectors of the same color are actually the same vector. Since vectors only encode their displacement, we can translate a vector anywhere we’d like without changing the vector. As long as the direction that the arrow points doesn’t change and the length of the arrow doesn’t change, the vector doesn’t change.

Since a vector remains unchanged when it is translated, we say that two vectors are equal if one can be translated to the other.

Now, we think of vectors as displacement. Consider two vectors, \( \vec{u} \) and \( \vec{v} \), as shown in Figure 5.1.4. Each of them corresponds to a displacement. As drawn, if we start at \( P \) and then displace by \( \vec{u} \), we land at \( Q \). If we then displace by \( \vec{v} \), we finally land at the point \( R \). This idea of performing displacements consecutively like that corresponds to the notion of adding vectors. That is, when things are set up as in figure Figure 5.1.4, we say \( \vec{u} + \vec{v} = \vec{w} \).

We make the notion of vector addition precise in Definition 5.1.5.

\[(5.1.5) \text{ Definition. Let } \vec{u} \text{ and } \vec{v} \text{ be two vectors. By translating, we can arrange them so that } \vec{v} \text{ starts at the end of } \vec{u}. \text{ When the vectors are arranged that way, the vector } \vec{u} + \vec{v} \text{ is defined to be the vector which starts at the beginning of } \vec{u} \text{ and ends at the end of } \vec{v}. \text{ As a displacement, } \vec{u} + \vec{v} \text{ is the overall displacement corresponding to first displacing by } \vec{u} \text{ and then displacing by } \vec{v}.\]

Suppose you walk 3 mi east and then you walk 4 mi north, as shown in Figure 5.1.6. These two parts of your walk correspond to
two different vectors. Let \( \vec{e} \) be a vector whose magnitude is 3 mi and whose direction is “east” and let \( \vec{n} \) be a vector whose magnitude is 4 mi and whose direction is “north.” Your total trip consists of walking the vector \( \vec{e} \) and then walking the vector \( \vec{n} \), hence your total displacement is \( \vec{v} = \vec{e} + \vec{n} \).

Notice that, although you’ve walked a total of 7 mi during this walk, you are not 7 mi from where you started. Since the directions of “north” and “east” are perpendicular, the triangle formed in Figure 5.1.6 is a right triangle. By the Pythagorean Theorem,

\[
\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.
\]

It follows that

\[
\|\vec{u} + \vec{v}\| = \sqrt{3^2 + 4^2} = 5.
\]
We define $\vec{u} + \vec{v}$ by arranging $\vec{u}$ and $\vec{v}$ so that $\vec{v}$ starts at the same place that $\vec{u}$ ends. We expect addition to be commutative, so it would be nice if $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. Of course, $\vec{v} + \vec{u}$ is defined by arranging the vectors so that $\vec{u}$ starts where $\vec{v}$ ends. Geometrically, these happen to be the same. Figure 5.1.7 shows the situation for a particular choice of $\vec{u}$ and $\vec{v}$. Since $\vec{u} + \vec{v}$ and $\vec{v} + \vec{u}$ lie on top of one another, they are equal.

It should make more sense that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ when we think about displacement. If you think about the 5 mi trip in Figure 5.1.6, it shouldn’t matter whether we go 3 mi east first and then go 4 mi north or if we go 4 mi north first and then go 3 mi east. Either path should land you in the same place.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure516.png}
\caption{The total displacement obtained after travelling by $\vec{e}$ and then $\vec{n}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure517.png}
\caption{The vectors $\vec{u} + \vec{v}$ and $\vec{v} + \vec{u}$ for a choice of vectors $\vec{u}$ and $\vec{v}$.}
\end{figure}

\textbf{(5.1.8) Example.} The vectors $\vec{v}$ and $\vec{w}$ are shown below. Draw the vector $\vec{v} + \vec{w}$. 
Solution:
In order to add vectors we must first draw them in the right way. That is, we need to draw them so that the beginning of \( \vec{w} \) is at the end of \( \vec{v} \). This is shown below:

Now, once the vectors are arranged in this manner, drawing in the vector \( \vec{v} + \vec{w} \) is a simple matter of drawing an arrow from the start of \( \vec{v} \) to the end of \( \vec{w} \). This is shown below:

**Answer:** Shown in previous diagram.

Note that in Example 5.1.8 it is perfectly acceptable to draw \( \vec{w} + \vec{v} \) since it is the same as \( \vec{v} + \vec{w} \). Also, when drawing these vectors by hand, consider them a sketch; just be as accurate as you can manage.

(5.1.9) Example. The vectors \( \vec{a} \) and \( \vec{b} \) are shown below where \( \| \vec{a} \| = 3 \) and \( \| \vec{b} \| = 2 \).

a. Draw \( \vec{a} + \vec{b} \).

b. Find \( \| \vec{a} + \vec{b} \| \).
Solution:

A. First we shift the vector $\vec{b}$ so that it starts at the end of $\vec{a}$.

Note that we have simply translated $\vec{b}$ horizontally without changing its direction so the angle is preserved. To draw $\vec{a} + \vec{b}$ we draw an arrow starting at the beginning of $\vec{a}$ and ending at the end of $\vec{b}$.

**Answer:** Shown in the previous diagram.

B. From the last diagram, we have the following triangle:
We can easily find the value of $\|\vec{a} + \vec{b}\|$ using the Law of Cosines.

\[
\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\left(\frac{5\pi}{9}\right)
\]

\[
\|\vec{a} + \vec{b}\|^2 = 3^2 + 2^2 - 2(3)(2)\cos\left(\frac{5\pi}{9}\right)
\]

\[
\|\vec{a} + \vec{b}\| = \sqrt{3^2 + 2^2 - 2(3)(2)\cos\left(\frac{5\pi}{9}\right)}
\]

\[
\|\vec{a} + \vec{b}\| = 3.88
\]

**Answer:** $\|\vec{a} + \vec{b}\| = 3.88$

Before we can continue, there is an important vector that needs to be defined. Vectors describe displacement and there is one extremely unique kind of displacement: the vector which corresponds to no displacement at all.

**(5.1.10) Definition.** The **zero vector** is the unique vector which has a magnitude of zero. As a displacement, the zero vector is the vector which has no displacement at all. We write $\vec{0}$ to denote the zero vector.

It is difficult to draw the zero vector as it has a magnitude of zero, meaning it is an arrow with no length at all. It may seem rather boring, but the zero vector has some very important properties. One of which is that $\vec{v} + \vec{0} = \vec{v}$ for any vector $\vec{v}$.

Vector addition is one of two types of vector manipulation introduced in this section. The second type is multiplication by a number. When dealing with vectors, the word **scalar** is used to represent numbers. There is a good reason for this but it is beyond the scope of this course. Know that the word *scalar* is used to differentiate real numbers from vectors.

**(5.1.11) Definition.** Let $c$ be a scalar and let $\vec{v}$ be a vector. The vector $c\vec{v}$ is defined as follows:

- If $c > 0$ then $c\vec{v}$ is the vector in the same direction as $\vec{v}$ but with a magnitude of $c\|\vec{v}\|$.
- If $c < 0$ then $c\vec{v}$ is the vector in the opposite direction of $\vec{v}$ but with a magnitude of $|c|\|\vec{v}\|$.
- If $c = 0$ then $c\vec{v} = \vec{0}$. 

If \( c \) is a positive integer then Definition 5.1.11 is easy. For example,

\[
2\vec{v} = \vec{v} + \vec{v} \quad \text{and} \quad 3\vec{v} = \vec{v} + \vec{v} + \vec{v}.
\]

Definition 5.1.11 is best shown by examples. Figure 5.1.12 gives a vector \( \vec{v} \) as well as some of its scalar multiples. Multiplying a vector by \( c \) simply multiplies the magnitude of the vector by \( c \). The exception is when the scalar is negative. In that case, we multiply the magnitude by \( |c| \) but switch the direction. Multiplying a vector by 0 always results in the zero vector.

\[
\vec{v} \quad 2\vec{v} \quad 2\vec{v} \quad -\vec{v} \quad -3\vec{v} \quad 0.5\vec{v}
\]

Figure 5.1.12: A vector \( \vec{v} \) and some scalar multiples of \( \vec{v} \).

Example 5.1.13 gives an example of using scalar multiplication and vector addition together.

Example 5.1.13

Vectors \( \vec{u} \) and \( \vec{v} \) are drawn below. Draw the vector \( \vec{u} + 2.5\vec{v} \).

\[
\begin{array}{c}
\vec{v} \\
\vec{u} \\
\end{array}
\]

Solution:
The first thing we need to do is to draw the vector \( 2.5\vec{v} \) so that we can add it to \( \vec{u} \). To do this, we draw a vector in the same direction as \( \vec{v} \) which is 2.5 times as long. This is shown below:
Then, in order to add them, we first shift $2.5\vec{v}$ so that $2.5\vec{v}$ starts at the end of $\vec{u}$. This is shown below:

Finally, we draw a vector which starts at the beginning of $\vec{u}$ and ends at the end of $2.5\vec{v}$. This vector will be $\vec{u} + 2.5\vec{v}$.

**Answer:** On the previous diagram.

Note that there is one thing we cannot do under any circumstances: we cannot multiply vectors together. If $c$ is a scalar and $\vec{v}$ is a vector then we’ve defined $c\vec{v}$. If $\vec{v}$ and $\vec{w}$ are two vectors then we’ve defined $\vec{v} + \vec{w}$. However, we have no meaning for $\vec{v}\vec{w}$.

Now that we know how to add vectors and how to multiply by scalars we can define vector subtraction.

(5.1.14) **Definition.** If $\vec{v}$ and $\vec{w}$ are vectors, define the vector $\vec{v} - \vec{w}$ such that

$$\vec{v} - \vec{w} = \vec{v} + (-1)\vec{w}.$$ 

In order to subtract a vector, we simply add its negative, much like we do with real numbers. When we compute $7 - 4$ what we really
compute is $7 + (-1)4$. We do the same thing with vectors. In order to compute $\vec{v} - \vec{w}$ we compute $\vec{v} + (-1)\vec{w}$.

(5.1.15) **Example.** Vectors $\vec{v}$ and $\vec{w}$ are shown below. Draw the vector $\vec{v} - \vec{w}$.

![Vectors v and w](image)

**Solution:**
In order to draw $\vec{v} - \vec{w}$ we will draw $\vec{v} + (-1)\vec{w}$. To do this, we first need to draw $-\vec{w}$. This is easy; we simply reverse the direction of the arrow.

![Vectors v minus w](image)

Then, to add those two vectors, we first shift $-\vec{w}$.

![Vectors v minus w](image)

Finally, we draw in the vector $\vec{v} - \vec{w}$ which is the same as $\vec{v} + (-1)\vec{w}$.

(5.1.A) **Practice Exercise.** Vectors $\vec{u}$ and $\vec{v}$ are shown below. Assume $|\vec{u}| = 4$ and $|\vec{v}| = 2$.

(i) Sketch the vector $\vec{u} + \vec{v}$ and find $|\vec{u} + \vec{v}|$.

(ii) Sketch the vector $\vec{u} - \vec{v}$ and find $|\vec{u} - \vec{v}|$. 
We can now add and subtract vectors as well as multiply them by scalars. We spent a few paragraphs justifying that addition of vectors is commutative; that is, that \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \). There is a whole list of other properties that we get from these new operations. We’re not going to spend any time verifying these, but they are not difficult to verify and they are very convenient.

\[(5.1.16) \text{ Lemma.} \quad \text{In the list of properties below, } \vec{u}, \vec{v}, \text{ and } \vec{w} \text{ are vectors while } c \text{ and } d \text{ are scalars.} \]

- \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \)
- \( (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \)
- \( \vec{u} + \vec{0} = \vec{u} = \vec{0} + \vec{u} \)
- \( -\vec{u} + \vec{u} = \vec{0} = \vec{u} - \vec{u} \)
- \( \vec{0} \vec{u} = \vec{0} \)
- \( 1 \vec{u} = \vec{u} \)
- \( (cd)\vec{u} = c(d\vec{u}) \)
- \( c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v} \)
- \( (c + d)\vec{u} = c\vec{u} + d\vec{u} \)
- \( \|\vec{u}\| \text{ is only zero when } \vec{u} = \vec{0} \)
- \( \|c\vec{u}\| = |c|\|\vec{u}\| \)

Don’t worry about memorizing the properties in Lemma 5.1.16. In short, these properties tell us that addition and scalar multiplication work exactly like we expect. Anything that we expect to be able to do with addition - like commute or associate - is allowed. The expected properties of multiplication hold; multiplication by zero is the zero vector and multiplication by one does nothing. Also, these operations interact the way we expect. We are allowed to distribute and subtracting a vector from itself yields the zero vector.
Note that throughout this whole section so far we’ve been drawing vectors \textit{in the plane}. This is unavoidable, as paper is a plane. These vectors that we’ve been drawing represent two-dimensional displacement. That is, they represent displacements that can be done on a map. However, it is just as easy to make three-dimensional vectors. Imagine if you could “draw” a vector in space. For instance, the vector from a point on the ground to some point on the top of a building would be a three-dimensional vector. All of the same rules apply to three-dimensional vectors. The only reason they have not been addressed until now is that we can’t draw them on paper.

\textbf{Applications}

Displacements are not the only thing that can be represented as a vector. Any quantity which needs both a magnitude and direction is a vector quantity.

For example, imagine that you’re flying an airplane and someone on the radio asks about your path. That is, they can see where you are, but they want to know about how you’re moving. You could report your speed, but that doesn’t quite tell them the whole story. To describe your motion you really need to tell them the direction in which you’re flying, as well. The data of “300 mi/h” is called speed and it is a scalar quantity. The data of “300 mi/h in the eastern direction” is called velocity and it is a vector quantity. Of course, magnitude of a vector is always a scalar and the magnitude of a velocity vector is speed.

Another example of a vector quantity is any force. If you push on something and you want to describe the way in which you pushed, it is not enough simply to say how hard you pushed. You also need to say the direction in which you pushed. We will use forces a lot in the upcoming sections. The units used on force will be Newtons (N). Newtons are the S.I. unit for force and are technically the same as \( \text{kg m/s}^2 \) but these details won’t come into play for us. These units are units on the magnitude of the force, so “200 N” is a scalar quantity, while “200 N north” is a vector quantity.

Here are some other examples of vector quantities:

\begin{itemize}
  \item Momentum
  \item Lift
  \item Drag
  \item Weight
  \item Friction
  \item Magnetism
\end{itemize}
• Wind Patterns

In later sections we will also look at some more abstract applications of vectors. Listed below are some scalar quantities, i.e. quantities that should not be vectors:

• Length
• Area
• Volume
• Mass
• Density
• Pressure
• Temperature
• Energy

Note that for each of these the notion of “direction” is meaningless. It can be tempting to think of a change in a scalar quantity as a vector. That is, if we are examining the change in mass over time we might want to think of “5 g larger” as a vector whose direction is “larger” and “5 g smaller” as a vector in the opposite direction. Though this is not unreasonable, it is rarely helpful. In these notes we will not consider quantities like that to be vectors.

The first application of vectors that we will explore is resultant forces. Suppose that two people are pushing on a large statue and imagine that you are looking down on them from above. This is shown in Figure 5.1.17. The statue is shown as a circle and the forces that the two people are using to push the statue are represented by \( \vec{F}_1 \) and \( \vec{F}_2 \).

![Figure 5.1.17: A mass and two forces, \( \vec{F}_1 \) and \( \vec{F}_2 \) which are acting on it.](image)

You can imagine that as these two people pushing together will result in the statue moving roughly downward and to the right (relative to the orientation in the figure). However, it is not exactly that simple. If one person is pushing much harder than the other person, then
that person’s force will dominate the other person’s force. That is, if the magnitude of $\vec{F}_2$ is much larger than the magnitude of $\vec{F}_1$, then the statue will move more downward than to the right. The statue will act as if one large force is pushing on it. This force is called the **resultant force** and it is obtained by the sum of the two force vectors acting on it. The resultant force on our statue is shown in Figure 5.1.18.

![Force Diagram](image)

**Figure 5.1.18**: The resultant force, $\vec{F}_R$, from two forces acting on a point.

A helpful tool in physics is a **force diagram**. When analyzing the forces acting on a certain object, we draw the object as a single point in the center of the diagram and then draw all of the forces acting on the object as arrows emanating from that point. This diagram is a force diagram, much like the one in Figure 5.1.18. The motion of the object is then determined by a single force vector called the resultant vector which is obtained by adding all of the forces acting on the object. Sometimes the resultant force is drawn on the force diagram when it is convenient.

When we push on an object with a force, that object moves. Its motion is explained by Newton’s laws of motion. Even if multiple forces are acting on an object, it can only move in one direction. The value of the resultant force is that when multiple forces are acting on an object then that object’s motion is exactly the same as if only the resultant force was acting on it. Our first example of finding a resultant force is shown in Example 5.1.19 where we analyze our statue a little further.

(5.1.19) **Example**. Two forces, $\vec{F}_1$ and $\vec{F}_2$ are acting on an object as shown below. For convenience, their resultant vector, $\vec{F}_R$ is also shown below. Assume that $\|\vec{F}_1\| = 180$ N and $\|\vec{F}_2\| = 190$ N.

a. Find $\|\vec{F}_R\|$.

b. Find $\theta$. 
Solution:

A. The first thing that we can do is arrange the vectors into a triangle as shown below.

The angle of $\frac{11\pi}{18}$ comes from $\frac{\pi}{2} + \frac{\pi}{9}$. This is because when we translate $\vec{F}_1$ downward, the angle with the horizontal is preserved. We can now use the Law of Cosines to find $\|\vec{F}_R\|$.

$$\|\vec{F}_R\|^2 = \|\vec{F}_1\|^2 + \|\vec{F}_2\|^2 - 2\|\vec{F}_1\|\|\vec{F}_2\| \cos\left(\frac{11\pi}{18}\right)$$

$$\|\vec{F}_R\|^2 = 180^2 + 190^2 - 2(180)(190) \cos\left(\frac{11\pi}{18}\right)$$

$$\|\vec{F}_R\| = \sqrt{180^2 + 190^2 - 2(180)(190) \cos\left(\frac{11\pi}{18}\right)}$$

$$\|\vec{F}_R\| = 303.14.$$

**Answer:** $\|\vec{F}_R\| = 303.14$ N

B. We can’t find $\theta$ directly, but using the triangle we can find the angle $\phi$ that is shown below.
Using the Law of Sines we have
\[
\frac{\sin(\phi)}{\|\vec{F}_1\|} = \frac{\sin\left(\frac{11\pi}{18}\right)}{\|\vec{F}_R\|} \quad \text{and} \quad \sin(\phi) = 0.558
\]
Since \(\phi\) is an acute angle, \(\phi = \sin^{-1}(0.558) = 0.59\). It then follows that
\[
\theta = \frac{\pi}{2} - \phi = 0.98.
\]
**Answer:** \(\theta = 0.98\)

(5.1.B) **Practice Exercise.** There are two forces on an airplane during takeoff: the force of lift, \(\vec{F}_L\), and the force of thrust, \(\vec{F}_T\). Thrust acts horizontally to propell the plane forward and lift acts vertically to bring the plane off of the ground as shown below.

Notably, these two forces are perpendicular to one another. For a small model airplane the force of lift is 10 N and the force of thrust is 50 N. Draw the resultant force on this model airplane and find its magnitude.

**Example 5.1.20** below is going to seem more like an exercise in trigonometry than an exercise in vectors. However, it should at least be believable that these quantities will be represented as displacement vectors. In the next section we will develop a notation for vectors which will make vectors extremely useful in examples like these.
Example. An observatory is tracking a GPS satellite. They find a displacement vector, $\vec{d}$, from the tower to the satellite, and the displacement vector, $\vec{r}$, from the center of the earth to the tower. These are shown in the diagram below where $\|\vec{r}\| = 4000$ mi and $\|\vec{r}\| = 17,000$ mi. Draw the displacement vector, $\vec{v}$, from the center of the earth to the satellite and use it to find the height of the satellite’s orbit above the surface of the earth. (Assume that the earth is a perfect sphere.)

Solution:

Drawing the vector $\vec{v}$ is simple. It happens to be the same as $\vec{r} + \vec{d}$, but we don’t need to think of it that way. It is just the vector starting at the center of the earth and ending at the satellite.

From the triangle above, we can use the Pythagorean Theorem to find $\|\vec{v}\|$.

\[
\|\vec{v}\|^2 = \|\vec{r}\|^2 + \|\vec{d}\|^2
\]
\[
\|\vec{v}\|^2 = 4000^2 + 17000^2
\]
\[
\|\vec{v}\| = 17464.
\]
The satellite’s altitude should be the distance between the satellite and the surface of the earth. Since the radius of the earth is \( ||\vec{r}|| \), we have

\[
||\vec{v}|| - ||\vec{r}|| = 17464 - 4000 = 13464.
\]

Hence that the satellite’s altitude is about 13,464 mi.

\[\text{Answer:} \quad 13,464 \text{ mi}\]

The applications become a lot more accessible in the next section when we learn a more comprehensive notation for vectors. When we have a more succinct way of adding vectors and computing their magnitudes, vectors will become an invaluable tool in examples like Example 5.1.20.
5.1.1. True or False: \( \vec{u} - \vec{v} = \vec{v} - \vec{u} \) for all vectors \( \vec{u} \) and \( \vec{v} \).

5.1.2. True or False: \( 3\vec{v} + \vec{v} = 4\vec{v} \) for all vectors \( \vec{v} \).

5.1.3. Consider the vectors below:

Of the following vectors, which one is not drawn? This is a multiple choice question; there is only one correct answer.

A.) \( -2\vec{u} \)
B.) \( -\vec{v} \)
C.) \( \vec{u} + \vec{v} \)
D.) \( \vec{u} - \vec{v} \)
E.) \( \vec{v} - \vec{u} \).

5.1.4. The vectors \( \vec{u} \) and \( \vec{v} \) are drawn below. Sketch \( \vec{u} + \vec{v} \), \( \vec{u} - \vec{v} \), and \(-3\vec{u}\). Be sure to label your answers.

5.1.5. Vectors \( \vec{u}, \vec{v}, \) and \( \vec{w} \) are shown below. Sketch \( \vec{u} + \vec{v} + \vec{w} \).

5.1.6. Suppose that \( \|\vec{v}\| = 5 \) and \( \|\vec{w}\| = 12 \). Suppose also that, when drawn starting at the same point, \( \vec{v} \) and \( \vec{w} \) make an angle of \( \pi/4 \).

A.) Find \( \|\vec{w} + \vec{v}\| \) and round to two decimal places.
B.) Find \( \|\vec{w} - \vec{v}\| \) and round to two decimal places.

5.1.7. The vector \( \vec{v} \) represents the displacement vector from Portland, Oregon to Houston, Texas. The vector \( \vec{w} \) represents the displacement vector from Houston, Texas to Akron, Ohio. Use the vectors \( \vec{v} \) and \( \vec{w} \) to answer the following questions:

A.) What is the displacement vector of a plane which flies from Akron, Ohio to Houston, Texas?

B.) In order to fly home from Portland to Akron I have a layover in Houston. Find a vector which describes my total displacement over the whole trip.

C.) It is approximately 3000 km from Portland to Houston. It is approximately 5500 km from Anchorage, Alaska to Houston. A plane flies in a straight line from Houston to Anchorage and during the flight it passes directly over Portland. Find a vector which describes the displacement of the plane.
5.1.8. The forces \( \vec{f} \) and \( \vec{g} \) are acting on an object with \( \| \vec{f} \| = 40 \text{ N} \) and \( \| \vec{g} \| = 25 \text{ N} \) as shown below. Draw the resultant force and find its magnitude. Round to two decimal places.

\[
\begin{align*}
\vec{f} & \\
\vec{g} & \end{align*}
\]

**Challenge Problems**

**(5.1.c1) Challenge Problem.** An important mathematical feature of vectors is what mathematicians call the **Triangle Inequality**. This is the fact that for any vectors \( \vec{v} \) and \( \vec{w} \),

\[
\| \vec{v} + \vec{w} \| \leq \| \vec{v} \| + \| \vec{w} \|.
\]

A. Using a picture, explain why the triangle inequality holds. **Hint:** It is called the “triangle” inequality for a reason.

B. Under what circumstances does it happen that \( \| \vec{v} + \vec{w} \| = \| \vec{v} \| + \| \vec{w} \| \)?

C. Since \( \| \vec{v} + \vec{w} \| \) is never larger than \( \| \vec{v} \| + \| \vec{w} \| \) we call \( \| \vec{v} \| + \| \vec{w} \| \) an **Upper Bound** for \( \| \vec{v} + \vec{w} \| \). Magnitudes are always positive so \( 0 \leq \| \vec{v} + \vec{w} \| \). For this reason we call 0 a **Lower Bound** for \( \| \vec{v} + \vec{w} \| \). However, using \( \| \vec{v} \| \) and \( \| \vec{w} \| \), we can actually find a more useful lower bound for \( \| \vec{v} + \vec{w} \| \). Use the picture that you drew earlier in the problem to see if you can determine a lower bound for \( \| \vec{v} + \vec{w} \| \). That is, if you know \( \| \vec{v} \| \) and \( \| \vec{w} \| \), what is the smallest that \( \| \vec{v} + \vec{w} \| \) can possibly be?
5.2 COMPONENTS OF A VECTOR

This section corresponds to section 12.2 in Functions Modeling Change [4]

Now that we’ve obtained a conceptual understanding of vectors and their operations, this section introduces a convenient notation for vectors.

Motivation

In the last section we defined vectors and we learned how to add them, subtract them, and multiply them by a constant. However, working with them was extremely inconvenient.

One of the primary applications of vectors was to describe planar displacement. We talked about describing a path on a map from one place to another. Imagine three cities on a map, as shown in Figure 5.2.1. As shown in the figure, the vector \( \vec{u} \) is the vector from city A to city B and \( \vec{v} \) is the vector from city B to city C. As we learned in the last section, the vector from city A to city C is \( \vec{u} + \vec{v} \). However, if we actually wanted to calculate anything about \( \vec{u} + \vec{v} \) we would need to know all of the angles shown and we would have to use some trigonometry as we did in Example 5.1.19.

Think about how you would describe these displacements to a friend. Instead of reporting angles, you could just as easily tell them how far north and how far east to go. That is, you could say “city B is 3 mi east and 1 mi north of city A.” Each of “3 mi east” and “1 mi north” are vectors. We can write the vector \( \vec{v} \) as a sum of those two vectors. If \( \vec{e} \) is the vector “3 mi east” and \( \vec{n} \) is the vector “1 mi north” then \( \vec{u} = \vec{e} + \vec{n} \).

We can use constant multiplication to do even a little bit better than that. If \( \vec{E} \) is the vector “1 mi east” and \( \vec{N} \) is the vector “1 mi north” then
we have $\vec{u} = 3\vec{E} + \vec{N}$. We should be able to write $\vec{v}$ similarly. City C is 1 mi east and 4 mi north of city B so $\vec{v} = \vec{E} + 4\vec{N}$. This is what we will come to call the decomposition of $\vec{u}$ and $\vec{v}$ and it is shown in Figure 5.2.2.

![Figure 5.2.2: Vector decomposition of the vectors in Figure 5.2.1.](image)

When we write the vectors this way, it becomes very easy to add them using the properties in Lemma 5.1.16.

\[
\vec{u} + \vec{v} = (3\vec{E} + \vec{N}) + (\vec{E} + 4\vec{N}) \\
= (3\vec{E} + \vec{E}) + (\vec{N} + 4\vec{N}) \\
= 4\vec{E} + 5\vec{N}.
\]

Equation 5.2.3 can also be seen from Figure 5.2.2. The vector $3\vec{E} + \vec{N} + \vec{E} + 4\vec{N}$ is clearly the displacement from city A to city C. We see that city C is 4 mi east and 5 mi north of city A.

Writing vectors in this way is extremely convenient. We see that it makes them easy to add and it also makes it easy to find their magnitudes. For example, since $3\vec{E}$ has a length of 3 mi and $\vec{N}$ has a length of 1 mi we see that

\[
\|\vec{u}\| = \sqrt{\|3\vec{E}\| + \|\vec{N}\|} = \sqrt{3^2 + 1^2} = \sqrt{10}.
\]

This is very similar to the notation that we will use throughout the rest of the section.

**Theory**

For the time being we are going to concentrate on vectors in the plane. It was mentioned at the end of Section 5.1 that we can define vectors in three dimensions just as easily as we can in two dimensions but in the beginning of this section we will focus on two dimensional vectors.
From this point forward we are going to assume that there is an $xy$-coordinate system underlying every plane that we draw. Unless stated otherwise, the $x$-axis will be horizontal and the $y$-axis will be vertical. This means that any time we draw a vector we can move its starting point to the origin of this coordinate system. Figure 5.2.4 shows a vector $\vec{v}$ drawn in this manner.

Like we did with the vectors $\vec{E}$ and $\vec{N}$ in the beginning of the section, it is convenient to have vectors which go horizontally and vertically. Define the vector $\vec{i}$ to be the vector with a length of 1 unit in the same direction as the positive $x$-axis and define the vector $\vec{j}$ to be the vector with a length of 1 unit in the same direction as the positive $y$-axis. Using these two vectors we can rewrite the vector in Figure 5.2.4 as $5\vec{i} + 2\vec{j}$. This is shown in Figure 5.2.5.
When we say that $\vec{i}$ is a vector of length 1, what we mean is that it has a length of 1 unit according to the $xy$-coordinate system. For example, the distance between the point $(0, 0)$ and $(1, 0)$ is 1 unit. It is easy to see that we can write any vector as we wrote $\vec{v}$ above. Some examples of vectors written in this way are shown in Figure 5.2.6.

![Figure 5.2.6: Some examples of plane vectors and their unit vector decompositions.](image)

These examples motivate Definition 5.2.7 below.

(5.2.7) **Definition.** Any vector of length 1 is called a **unit vector**. Define the vector $\vec{i}$ to be the unit vector in the direction of the positive $x$-axis and define the vector $\vec{j}$ to be the unit vector in the direction of the positive $y$-axis. Every plane vector $\vec{v}$ can be written as $\vec{v} = x\vec{i} + y\vec{j}$ for real (possibly zero) numbers $x$ and $y$. This is called a **unit vector decomposition** for $\vec{v}$. The number $x$ is called the $\vec{i}$-**component** of $\vec{v}$ and the number $y$ is called the $\vec{j}$-**component** of $\vec{v}$.

It should be clear from the examples above that a unit vector decomposition is unique. That is, you cannot find two different unit
vector decompositions for the same vector. It follows that two vectors are equal when they have the same unit vector decomposition.

When we write vectors in this manner, it becomes straightforward to perform operations on them using the properties in Lemma 5.1.16. For example, suppose \( \vec{u} = 3\vec{i} + 2\vec{j} \) and \( \vec{v} = -4\vec{i} + 6\vec{j} \) and suppose that we want to find \( \vec{u} + \vec{v} \).

\[
\vec{u} + \vec{v} = (3\vec{i} + 2\vec{j}) + (-4\vec{i} + 6\vec{j}) \\
= (3\vec{i} - 4\vec{i}) + (2\vec{j} + 6\vec{j}) \\
= (-1\vec{i}) + (8\vec{j}) \\
= -\vec{i} + 8\vec{j}.
\]

You can see what happened there. The unit vector decomposition for \( \vec{u} + \vec{v} \) was just obtained by adding the \( \vec{i} \)-components together and adding the \( \vec{j} \)-components together. Now suppose we want to find \( 7\vec{u} \).

\[
7\vec{u} = 7(3\vec{i} + 2\vec{j}) = 21\vec{i} + 14\vec{j}.
\]

The unit vector decomposition of \( 7\vec{u} \) was just obtained by multiplying each component of \( \vec{u} \) by 7.

Example 5.2.8 explores some more examples of vector operations using unit vector decompositions.

---

(5.2.8) Example. Let \( \vec{u}, \vec{v}, \) and \( \vec{w} \) be defined as follows:

\[
\vec{u} = -5\vec{i} + 11\vec{j} \quad \vec{v} = \vec{i} - \vec{j} \quad \vec{w} = -3\vec{i} - 7\vec{j}
\]

Find a unit vector decomposition for each of the following:

A. \( \vec{u} + \vec{v} \)
B. \( \vec{v} - \vec{w} \)
C. \( \vec{u} + \vec{v} + \vec{w} \)
D. \( 3\vec{w} \)
E. \( 6\vec{u} - 2\vec{v} \)

Solution:

A. We have

\[
\vec{u} + \vec{v} = (-5\vec{i} + 11\vec{j}) + (\vec{i} - \vec{j}) \\
= (-5 + 1)\vec{i} + (11 - 1)\vec{j} \\
= -4\vec{i} + 10\vec{j}.
\]
5.2 Components of a Vector

Answer: \(-4\mathbf{i} + 10\mathbf{j}\)

b. We have
\[
\vec{v} - \vec{w} = (\mathbf{i} - \mathbf{j}) - (-3\mathbf{i} - 7\mathbf{j}) \\
= (1 - (-3))\mathbf{i} + (-1 - (-7))\mathbf{j} \\
= 4\mathbf{i} + 6\mathbf{j}.
\]

Answer: \(4\mathbf{i} + 6\mathbf{j}\)

c. We have
\[
\vec{u} + \vec{v} + \vec{w} = (-5\mathbf{i} + 11\mathbf{j}) + (\mathbf{i} - \mathbf{j}) + (-3\mathbf{i} - 7\mathbf{j}) \\
= (-5 + 1 - 3)\mathbf{i} + (11 - 1 - 7)\mathbf{j} \\
= -7\mathbf{i} + 3\mathbf{j}.
\]

Answer: \(-7\mathbf{i} + 3\mathbf{j}\)

d. We have
\[
3\vec{w} = 3(-3\mathbf{i} - 7\mathbf{j}) = -9\mathbf{i} - 21\mathbf{j}.
\]

Answer: \(-9\mathbf{i} - 21\mathbf{j}\)

e. We have
\[
6\vec{u} - 2\vec{v} = 6(-5\mathbf{i} + 11\mathbf{j}) - 2(\mathbf{i} - \mathbf{j}) \\
= (-30\mathbf{i} + 66\mathbf{j}) + (-2\mathbf{i} + 2\mathbf{j}) \\
= (-30 - 2)\mathbf{i} + (66 + 2)\mathbf{j} \\
= -32\mathbf{i} + 68\mathbf{j}.
\]

Answer: \(-32\mathbf{i} + 68\mathbf{j}\)

We can also use a unit vector decomposition to find the magnitude of a vector. We defined \(\mathbf{i}\) and \(\mathbf{j}\) such that \(\|\mathbf{i}\| = 1\) and \(\|\mathbf{j}\| = 1\) so that \(\|x\mathbf{i}\| = |x|\) and \(\|y\mathbf{j}\| = |y|\) for any \(x\) and \(y\) as discussed in the previous section. Consider the vector in Figure 5.2.5. We found that \(\vec{v} = 5\mathbf{i} + 2\mathbf{j}\). Using the triangle formed in the figure and the fact that \(\|5\mathbf{i}\| = 5\) and \(\|2\mathbf{j}\| = 2\) we have
\[
\|\vec{v}\|^2 = \|5\mathbf{i}\|^2 + \|2\mathbf{j}\|^2 \\
\|\vec{v}\| = \sqrt{5^2 + 2^2} \\
\|\vec{v}\| = \sqrt{29}.
\]

You can see in Figure 5.2.6 that the unit vector decomposition of any vector forms a triangle similar to the one in Figure 5.2.5 so we can...
always calculate the magnitude of a vector in the way that we calculated $\|\vec{v}\|$. That is, if $\vec{u} = x\vec{i} + y\vec{j}$ then

$$\|\vec{u}\| = \sqrt{x^2 + y^2}.$$

The calculations that we’ve explored so far are summarized in Lemma 5.2.9. From this point forward, any time we deal with vectors we will deal with their unit vector decomposition. Any time you are asked to “find” a vector, you should find its unit vector decomposition.

(5.2.9) **Lemma.** Suppose that $\vec{u} = u_1\vec{i} + u_2\vec{j}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j}$ are vectors. Suppose that $c$ is a scalar. Then

1. $\vec{u} + \vec{v} = (u_1 + v_1)\vec{i} + (u_2 + v_2)\vec{j}$.
2. $\vec{u} - \vec{v} = (u_1 - v_1)\vec{i} + (u_2 - v_2)\vec{j}$.
3. $c\vec{u} = (cu_1)\vec{i} + (cu_2)\vec{j}$.
4. $\|\vec{u}\| = \sqrt{(u_1)^2 + (u_2)^2}$.

(5.2.10) **Example.** Find the magnitude of the following vectors:

A. $\vec{u} = 3\vec{i} + 4\vec{j}$
B. $\vec{v} = -7\vec{i} + 3\vec{j}$
C. $\vec{w} = -2\vec{i} - 13\vec{j}$

**Solution:**

A. Using Lemma 5.2.9,

$$\|\vec{u}\| = \sqrt{3^2 + 4^2} = 5$$

**Answer:** $\|\vec{u}\| = 5$

B. Be careful when the components of vectors are negative. The *square* of a component of a vector is always positive. Using Lemma 5.2.9,

$$\|\vec{v}\| = \sqrt{(-7)^2 + 3^2} = \sqrt{49 + 9} = \sqrt{58}$$

**Answer:** $\|\vec{v}\| = \sqrt{58}$
c. Again, using Lemma 5.2.9,
\[
\|\vec{w}\| = \sqrt{(-2)^2 + (-13)^2} = \sqrt{4 + 169} = \sqrt{173}
\]

**Answer:** \(\|\vec{w}\| = \sqrt{173}\)

---

(5.2.A) **Practice Exercise.** Let \(\vec{u} = 2\vec{i} - 3\vec{j}\) and \(\vec{v} = -3\vec{i} - 7\vec{j}\).

(i) Find \(-4\vec{u}\).

(ii) Find \(\|\vec{v}\|\) and round to two decimal places.

(iii) Find \(3\vec{u} - 2\vec{v}\).

In **Section 5.1** we specified the direction of our vectors by using an angle. If we are given a vector which is specified in this manner it is convenient to be able to find its unit vector decomposition. This can be done using the triangle formed by the component vectors. Consider the vector \(\vec{v}\) shown in **Figure 5.2.11** which has been drawn as \(\vec{v} = x\vec{i} + y\vec{j}\). Of course, the length of the hypotenuse of the triangle in **Figure 5.2.11** is \(\|\vec{v}\|\). Using right triangle trigonometry from **Section 3.2** we have that

\[
x = \|\vec{v}\| \cos(\theta) \quad \text{and} \quad y = \|\vec{v}\| \sin(\theta).
\]

![Figure 5.2.11: Some examples of plane vectors and their unit vector decompositions.](image)

This result can certainly be generalized to arbitrary vectors as in **Lemma 5.2.12**.
(5.2.12) Lemma. Let \( \mathbf{v} \) be a vector. Let \( \theta \) be the angle that \( \mathbf{v} \) makes counterclockwise from the positive x-axis. Then the unit vector decomposition for \( \mathbf{v} \) is
\[
\mathbf{v} = (||\mathbf{v}|| \cos(\theta))\mathbf{i} + (||\mathbf{v}|| \sin(\theta))\mathbf{j}.
\]

(5.2.13) Example. Find a unit vector decomposition for \( \mathbf{w} \) drawn below where \( ||\mathbf{w}|| = 4 \).

\[\text{Solution:}\]
The angle \( \frac{2\pi}{3} \) is measured clockwise from the x-axis so we can apply Lemma 5.2.12 with \( \theta = \frac{2\pi}{3} \). We have
\[
\mathbf{w} = (||\mathbf{w}|| \cos(\theta))\mathbf{i} + (||\mathbf{w}|| \sin(\theta))\mathbf{j}
\]
\[
= (4 \cos\left(\frac{2\pi}{3}\right))\mathbf{i} + (4 \sin\left(\frac{2\pi}{3}\right))\mathbf{j}
\]
\[
= \left(4\left(-\frac{1}{2}\right)\right)\mathbf{i} + \left(4\left(\frac{\sqrt{3}}{2}\right)\right)\mathbf{j}
\]
\[
= -2\mathbf{i} + 2\sqrt{3}\mathbf{j}.
\]

Answer: \( \mathbf{w} = -2\mathbf{i} + 2\sqrt{3}\mathbf{j} \)

We can check our answer in Example 5.2.13. If we did things correctly, it should be that \( ||\mathbf{w}|| = 4 \) and, from Lemma 5.2.9, we have
\[
||\mathbf{w}|| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4.
\]

In Example 5.2.13 it was imperative that the angle given was the angle counterclockwise from the positive x-axis. If it’s not, then we must find that angle before we use Lemma 5.2.12.
Example. Find a unit vector decomposition for \( \vec{u} \) drawn below where \( \|\vec{u}\| = 5 \).

Solution:

It is tempting to use Lemma 5.2.12 with \( \theta = \frac{\pi}{3} \), but that isn’t going to work. The angle used in Lemma 5.2.12 needs to be the angle between \( \vec{u} \) and the positive x-axis measured counterclockwise. That is, we need the angle \( \theta \) shown below:

\[
\theta = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}.
\]

Then, using Lemma 5.2.12, we have

\[
\vec{u} = (\|\vec{u}\| \cos(\theta))\hat{i} + (\|\vec{u}\| \sin(\theta))\hat{j} = (5 \cos(\frac{5\pi}{6}))\hat{i} + (5 \sin(\frac{5\pi}{6}))\hat{j} = (5\left(-\frac{\sqrt{3}}{2}\right))\hat{i} + (5\left(\frac{1}{2}\right))\hat{j} = -\frac{5\sqrt{3}}{2}\hat{i} + \frac{5}{2}\hat{j}.
\]

Answer: \( \vec{u} = -\frac{5\sqrt{3}}{2}\hat{i} + \frac{5}{2}\hat{j} \)
It is important that you be careful to use the correct angle when dealing with problems like Example 5.2.13 and Example 5.2.14.

It is also useful to be able to define vectors using two points in a plane. That is, if \( P \) and \( Q \) are two points then we might want to know the vector which describes the displacement from \( P \) to \( Q \). Suppose that \( P = (-2, 1) \) and \( Q = (3, 3) \) as shown in Figure 5.2.15 and let \( \vec{v} \) be the vector which starts at \((-2, 1)\) and ends at \((3, 3)\). We can see that the horizontal displacement between these two points is \( (3 - (-2)) = 5 \) and the vertical displacement between them is \( (3 - 1) = 2 \). It follows that

\[
\vec{v} = (3 - (-2))\vec{i} + (3 - 1)\vec{j} = 5\vec{i} + 2\vec{j}.
\]

This procedure can be performed for arbitrary points \( P \) and \( Q \) in the same manner, which is described in Lemma 5.2.16. As a matter of notation, we write \( \overrightarrow{PQ} \) to denote the vector which starts at \( P \) and ends at \( Q \). Note that this is not the same as \( \overrightarrow{QP} \). You can see from Figure 5.2.15 that \( \overrightarrow{PQ} = -\overrightarrow{QP} \).

\[
(5.2.16) \text{ Lemma.} \text{ Let } P = (x_1, y_1) \text{ and } Q = (x_2, y_2) \text{ be two points in the xy-plane. Then the vector } \overrightarrow{PQ} \text{ is the vector which starts at } P \text{ and ends at } Q. \text{ The unit vector decomposition for } \overrightarrow{PQ} \text{ is }
\]

\[
\overrightarrow{PQ} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}.
\]

\[
(5.2.17) \text{ Example.} \text{ Find a unit vector decomposition for } \vec{v} \text{ drawn below.}
\]
Solution:
We see that \( \vec{v} \) is the vector drawn between two points. In the language of Lemma 5.2.16 write \( P = (3, 5) \) and \( Q = (-4, -2) \).
Then \( \vec{v} = \overrightarrow{PQ} \) since \( \vec{v} \) starts at \( P \) and ends at \( Q \). Then, using Lemma 5.2.16, we have
\[
\vec{v} = \overrightarrow{PQ} = (-4 - 3)\vec{i} + (-2 - 5)\vec{j} = -7\vec{i} - 7\vec{j}.
\]
**Answer:** \( \vec{v} = -7\vec{i} - 7\vec{j} \)

(5.2.b) Practice Exercise. Find a unit vector decomposition for the two vectors shown below assuming that \( \| \vec{u} \| = 4 \).
Note: The vector \( \vec{u} \) is not \( -3\vec{i} + 3\vec{j} \).
Throughout the whole section we’ve been using two dimensional vectors. If we’re working in three dimensions then we can use a similar notation. If we assume that our three dimensional space has an $xyz$-coordinate system then we let $\mathbf{i}$ be the vector of length 1 in the direction of the positive $x$-axis, $\mathbf{j}$ be the vector of length 1 in the direction of the positive $y$-axis, and $\mathbf{k}$ be the vector of length 1 in the direction of the positive $z$-axis. We can then write any vector $\mathbf{v}$ as

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

for real numbers $x$, $y$, and $z$. We can calculate magnitudes of three dimensional vectors according to

$$||\mathbf{v}|| = \sqrt{x^2 + y^2 + z^2}.$$ 

Addition and scalar multiplication of these vectors is done on the components, just as it was with two dimensional vectors.

In these notes we will focus primarily on two dimensional vectors but we will occasionally explore three dimensional vectors in a limited capacity.

(5.2.18) Example. A weather balloon has floated 6 mi north, 2 mi east, and 7 mi above the laboratory where it was released. What is its distance from the laboratory?

Solution:

We want to find a displacement vector, $\mathbf{d}$, for the balloon’s journey. In order to do this, we must implement a coordinate system. Let $\mathbf{i}$ be the vector pointing 1 mi east, let $\mathbf{j}$ be the vector pointing 1 mi north, and let $\mathbf{k}$ be the vector which points 1 mi upward. According to this coordinate system we have

$$\mathbf{d} = 2\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}.$$ 

To find the balloon’s distance from the laboratory we simply find $||\mathbf{d}||$.

$$||\mathbf{d}|| = \sqrt{2^2 + 6^2 + 7^2} = \sqrt{4 + 36 + 49} = \sqrt{89} = 9.43$$

Answer: 9.43 mi

The last thing to discuss is the coordinate system that we’ve been using. Each time we implement coordinate so that we can use a unit vector decomposition, this coordinate system involves a choice. We’ve always drawn the $x$-axis horizontally and the $y$-axis vertically, but there is no need to do that. As long as they meet at a right angle, there is no reason we can’t rotate the coordinates and shift the origin. Changing the coordinate system does not change the vectors,
but it definitely changes the unit vector decomposition. Fortunately this will not cause us any problems and so it is not something we will explore in these notes.

Applications

We will look at a lot more applications of vectors later in the chapter. For now, though, we will focus on the force diagrams and displacement vectors that we discussed in Section 5.1. Conceptually, the examples don’t change, but now that we have unit vector decompositions the examples become more practical. In order to use a unit vector decomposition, though, we need to implement a coordinate system.

Consider the force diagram shown in Figure 5.2.19a. Assume that $\|\vec{F}_1\| = 4\text{ N}$ and $\|\vec{F}_2\| = 8\text{ N}$. We can certainly draw the resultant force without a coordinate system and we can even find the length of the resultant force without a coordinate system. However, those things become much easier with a coordinate system. This coordinate system is shown in Figure 5.2.19b.

![Figure 5.2.19: A force diagram with and without a coordinate system.](image)

In order to calculate the resultant force we can find a unit vector decomposition for $\vec{F}_1$ and $\vec{F}_2$. Since $\vec{F}_1$ is in the direction of $-\hat{j}$ with a length of 4 we have that $\vec{F}_1 = -4\hat{j}$. Using Lemma 5.2.12 we have that

$$\vec{F}_2 = \left(\|\vec{F}_2\| \cos\left(\frac{5\pi}{18}\right)\right)\hat{i} + \left(\|\vec{F}_2\| \sin\left(\frac{5\pi}{18}\right)\right)\hat{j}$$

$$= 5.14\hat{i} + 6.13\hat{j}.$$
We can then find the resultant force, \( \vec{F}_R \), by

\[
\vec{F}_R = \vec{F}_1 + \vec{F}_2 \\
= (-4\vec{j}) + (5.14\vec{i} + 6.13\vec{j}) \\
= 5.14\vec{i} + 2.13\vec{j}.
\]

In order to find the magnitude of \( \vec{F}_R \) we compute

\[
\|\vec{F}_R\| = \sqrt{5.14^2 + 2.13^2} = 5.57
\]

so that \( \|\vec{F}_R\| = 5.57 \text{ N} \).

Look back at the coordinate system that we used in Figure 5.2.19b. Notice that the actual axes were never used at any point in our calculation. The only thing that we used were the vectors \( \vec{i} \) and \( \vec{j} \) that we got from that coordinate system. From this point forward, you won’t be given an actual xy-plane to find unit vector decomposition. Instead, we will simply define the vectors \( \vec{i} \) and \( \vec{j} \) in the corner of a diagram like the ones in Figure 5.2.19b.

\[(5.2.20) \text{ Example.} \quad \text{Consider the force diagram shown below where } \|\vec{F}_1\| = 600 \text{ N}, \|\vec{F}_2\| = 200 \text{ N, and } \|\vec{F}_3\| = 300 \text{ N.}
\]

\textbf{a.} Find a unit vector decomposition for each of the three force vectors.

\textbf{b.} Find a unit vector decomposition for the resultant force.

\textbf{c.} Find the magnitude of the resultant force.

\[\vec{F}_3\]

\[\frac{3\pi}{4}\]

\[\vec{F}_2\]

\[\vec{i}\]

\[\vec{j}\]

\[\vec{F}_1\]

\textbf{Solution:}

\textbf{a.} It is simple to find a unit vector decomposition for any vector which is vertical or horizontal. Since \( \vec{F}_1 \) is length
600 and parallel to \( \vec{i} \) we have that \( \vec{F}_1 = 600 \vec{i} \). Since \( \vec{F}_2 \) is length 200 and parallel to \( -\vec{j} \) we have that \( \vec{F}_2 = -200 \vec{j} \). To find \( \vec{F}_3 \) we can use Lemma 5.2.12.

\[
\vec{F}_3 = \left( \|\vec{F}_3\| \cos \left( \frac{3\pi}{4} \right) \right) \vec{i} + \left( \|\vec{F}_3\| \sin \left( \frac{3\pi}{4} \right) \right) \vec{j}
\]

\[
= \frac{300\sqrt{2}}{2} \vec{i} + \frac{300\sqrt{2}}{2} \vec{j}
\]

\[
= -212.13 \vec{i} + 212.13 \vec{j}.
\]

**Answer:** \( \vec{F}_1 = 600 \vec{i} \), \( \vec{F}_2 = -200 \vec{j} \), and \( \vec{F}_3 = -212.13 \vec{i} + 212.13 \vec{j} \).

b. To find the resultant vector, \( \vec{F}_R \), we simply add all of the force vectors.

\[
\vec{F}_R = \vec{F}_1 + \vec{F}_2 + \vec{F}_3
\]

\[
= (600\vec{i}) + (-200\vec{j}) + (-212.13\vec{i} + 212.13\vec{j})
\]

\[
= 387.87\vec{i} + 12.13\vec{j}.
\]

**Answer:** \( \vec{F}_R = 387.87 \vec{i} + 12.13 \vec{j} \)

c. To calculate the magnitude of the resultant force,

\[
\|\vec{F}_R\| = \sqrt{387.87^2 + 12.13^2} = 388.06
\]

**Answer:** \( \|\vec{F}_R\| = 388.06 \)

In Example 5.2.20 we assumed that \( \vec{F}_1 \) was parallel to \( \vec{i} \) because the small vector \( \vec{i} \) in the corner of the diagram makes it look as though they are parallel. We made a similar assumption that \( \vec{F}_2 \) was parallel to \( \vec{j} \). You are allowed to make these assumptions in exercises unless stated otherwise.

(5.2.c) **Practice Exercise.** Find the magnitude of the resultant vector in the force diagram below. Assume that \( \|\vec{F}_1\| = 450 \text{ N} \) and \( \|\vec{F}_2\| = 600 \text{ N} \)
(5.2.21) Example. An astronomer finds a map of a part of the solar system at a certain moment as shown below. He also finds the following displacement vectors:

\[
\vec{r}_e = 114.6\hat{i} + 96.2\hat{j}
\]
\[
\vec{r}_m = 96.3\hat{i} - 206.6\hat{j}
\]
\[
\vec{r}_v = -101.7\hat{i} + 37\hat{j}
\]

All of the units on these vectors are in gigameters (Gm), which are billions of meters.

A. Find the distance from Earth to Mars.

B. Find the distance from Mars to Venus.

C. Find the distance from Venus to Earth.
Solution:

A. Let \( \vec{u} \) be the vector from Earth to Mars. This vector is shown in the diagram at the end of the solution. From the diagram we have

\[
\vec{r}_e + \vec{u} = \vec{r}_m
\]

and hence

\[
\vec{u} = \vec{r}_m - \vec{r}_e
\]

\[
= (96.3\hat{i} - 206.6\hat{j}) - (114.6\hat{i} + 96.2\hat{j})
\]

\[
= -18.3\hat{i} - 302.8\hat{j}.
\]

It follows that the distance from Earth to Mars is \( \|\vec{u}\| \) and

\[
\|\vec{u}\| = \sqrt{(-18.3)^2 + (-302.8)^2} = 303.35
\]

**Answer:** 303.35 Gm

B. Let \( \vec{v} \) be the vector from Mars to Venus. This vector is shown in the diagram at the end of the solution. From the diagram we have

\[
\vec{r}_m + \vec{v} = \vec{r}_v
\]

and hence

\[
\vec{v} = \vec{r}_v - \vec{r}_m
\]

\[
= (-101.7\hat{i} + 37\hat{j}) - (96.3\hat{i} - 206.6\hat{j})
\]

\[
= -198\hat{i} + 243.6\hat{j}.
\]

It follows that the distance from Mars to Venus is \( \|\vec{v}\| \) and

\[
\|\vec{v}\| = \sqrt{(-198)^2 + (243.6)^2} = 313.92
\]

**Answer:** 260.71 Gm

C. Let \( \vec{w} \) be the vector from Venus to Earth. This vector is shown in the diagram at the end of the solution. From the diagram we have

\[
\vec{r}_v + \vec{w} = \vec{r}_e
\]

and hence

\[
\vec{w} = \vec{r}_e - \vec{r}_v
\]

\[
= (114.6\hat{i} + 96.2\hat{j}) - (-101.7\hat{i} + 37\hat{j})
\]

\[
= 216.3\hat{i} + 59.2\hat{j}.
\]
It follows that the distance from Venus to Earth is $\|\vec{w}\|$ and

$$\|\vec{w}\| = \sqrt{(216.3)^2 + (59.2)^2} = 224.26$$

**Answer:** 224.26 Gm
Homework Assignment

5.2.1. Let \( \vec{u} = 2\hat{i} + 3\hat{j} \), let \( \vec{v} = -3\hat{i} + \hat{j} \), and let \( \vec{w} = -6\hat{i} - 5\hat{j} \).
   a.) Find \( \vec{w} - 3\vec{v} \).
   b.) Find \( 3(\vec{u} + \hat{i}) + \vec{v} \).
   c.) Find \( \vec{u} + \vec{v} - \vec{w} \).

5.2.2. Let \( \vec{u} = 3\hat{i} + \hat{j} \) and let \( \vec{v} = -\hat{i} + 4\hat{j} \). Find a unit vector decomposition for \( 5\vec{u} - 2\vec{v} \).

5.2.3. Let \( \vec{u} = 4\hat{j} \), let \( \vec{v} = -6\hat{i} + 7\hat{j} \), and let \( \vec{w} = 2\hat{i} - 5\hat{j} \).
   a.) Find \( \vec{u} + \vec{v} \).
   b.) Find \( 2\vec{w} - 3\vec{v} \).
   c.) Find \( \vec{u} - 4(\vec{v} - \vec{w}) \).

5.2.4. Consider the vector \( \vec{v} = 5\hat{i} - \hat{j} \).
   a.) Find the magnitude of \( \vec{v} \) and leave your answer in exact form.
   b.) Find the angle that \( \vec{v} \) makes with the vector \( \hat{i} \) (assume that this angle is measured in the counterclockwise direction, like usual). Round your answer to two decimal places. *Hint: It may be helpful to draw a picture.*

5.2.5. The vector \( \vec{v} \) is graphed below. If \( \theta = \pi/3 \) and \( ||\vec{v}|| = 150 \), find its unit vector decomposition. Leave your answer in exact form.

5.2.6. Three vectors, \( \vec{u}, \vec{v}, \) and \( \vec{w} \) are graphed below. Write the unit vector decomposition for each of them.

5.2.7. The vectors \( \vec{u} \) and \( \vec{v} \) are drawn below.
   a.) If \( ||\vec{u}|| = 8 \), find a unit vector decomposition for \( \vec{u} \). Leave your answer in exact form.
   b.) Find a unit vector decomposition for \( \vec{v} \). Leave your answer in exact form.
   *Note: In case it is not clear, \( \vec{v} \) is the vector from \( (2, -3) \) to \( (4, 5) \).*

5.2.8. The vectors \( \vec{u} \) and \( \vec{v} \) are drawn below. If \( \vec{v} = 4\hat{i} + 4\hat{j}, \theta = 2\pi/3, \) and \( ||\vec{u}|| = 5 \), find a unit vector decomposition for \( \vec{u} \). Round your answers to two decimal places.
5.2.9. The vectors \( \vec{u} \) and \( \vec{v} \) are shown below. Find a unit vector decomposition for both of them.

5.2.10. Consider the diagram below. If \( \theta = \frac{\pi}{3} \) and \( ||\vec{v}|| = 20 \), find a unit vector decomposition for \( \vec{v} \). Leave your answer in exact form.

5.2.11. Consider the diagram below. If \( \theta = \frac{\pi}{6} \) and \( ||\vec{v}|| = 3000 \), find a unit vector decomposition for \( \vec{v} \). Leave your answer in exact form.

5.2.12. Consider a weather balloon floating in the air. There are three forces acting on this balloon: the force of gravity is labeled \( \vec{F}_G \), the lift from the balloon is labeled \( \vec{F}_L \), and the force from the wind is labeled \( \vec{F}_W \). The orientation of these forces along with a coordinate system is given below:

Assume that \( ||\vec{F}_G|| = 20 \text{ N} \), \( ||\vec{F}_L|| = 25 \text{ N} \), and \( ||\vec{F}_W|| = 15 \text{ N} \). Find the magnitude of the resultant force acting on the weather balloon and round your answer to two decimal places.

5.2.13. An \( xy \)-plane is placed on a map of the city of Mystic Falls such that town’s post office is positioned at the origin, the positive \( x \)-axis points east, and the positive \( y \)-axis points north. The Salvatores’ house is located at the point \((1, 4)\) on the map and the Gilber’s house is located at the point \((-1, -2)\). A pigeon flies from the Salvatores’ house to the Gilber’s house. Find the displacement vector which describes the pigeon’s journey.

5.2.14. A box is being pushed up an incline of \( 30^\circ \) with a force of 100 N (which is parallel to
the incline) and the force of gravity on the box is 50 N (gravity acts straight downward). Find the magnitude of the resultant force and round to two decimal places.

5.2.15. The force diagram below displays the forces acting on some object. Let $\vec{F}$ be the resultant force on the object.

A.) Sketch $\vec{F}$ in the diagram.

---

5.2.16. The circle below is an object on which three forces are acting as shown:

Given that $\|\vec{F}_L\| = 300 \text{ N}$, that $\|\vec{F}_G\| = 100 \text{ N}$, and that $\|\vec{F}_W\| = 50 \text{ N}$, find the resultant force vector, $\vec{F}$, which is acting on the object. Round to two decimals when necessary.

---

**Challenge Problems**

5.2.1 Challenge Problem. The value of the vectors $\vec{i}$ and $\vec{j}$ are that every vector $\vec{v}$ can be written as

$$\vec{v} = x\vec{i} + y\vec{j}$$

for unique real numbers $x$ and $y$. Because of this, we call the set $\{\vec{i}, \vec{j}\}$ a **basis**. However, these are not the only pair of vectors for which this is true. Assume that $\vec{i}$ and $\vec{j}$ are fixed.

A. Explain how the vectors $\vec{a}_1 = \vec{i} + \vec{j}$ and $\vec{b}_1 = -\vec{i} + \vec{j}$ form a basis.

To do this, explain how a vector of the form $\vec{v} = x\vec{i} + y\vec{j}$ can be written as

$$\vec{v} = \alpha \vec{a}_1 + \beta \vec{b}_1$$

for real numbers $\alpha$ and $\beta$. That is, find these numbers in terms of $x$ and $y$.

B. Explain why the vectors $\vec{a}_2 = \vec{i}$ and $\vec{b}_2 = 2\vec{i}$ do not form a basis.

What sorts of vectors cannot be written as

$$\alpha \vec{a}_2 + \beta \vec{b}_2$$

for real numbers $\alpha$ and $\beta$?
5.3 THE DOT PRODUCT

This section corresponds to section 12.4 in Functions Modeling Change [4]

The dot product is a valuable tool in dealing with vectors even in the limited capacity that we are using them. However, in vector calculus and beyond the dot product becomes essential.

Motivation

We have used angles to relate vectors several times in previous sections. Lemma 5.2.12 tells us that the vector \( \vec{u} \) in Figure 5.3.1a can be written as

\[
\vec{u} = (\|\vec{u}\| \cos(\theta)) \hat{i} + (\|\vec{u}\| \sin(\theta)) \hat{j}.
\]

We also saw that if we are given the angle \( \phi \) as well as \( \|\vec{v}\| \) and \( \|\vec{w}\| \) then we can find \( \|\vec{v} + \vec{w}\| \) and \( \|\vec{v} - \vec{w}\| \) in Figure 5.3.1b.

However, what we didn’t do in earlier sections was use vectors to find angles. That is, imagine that we know the vectors \( \vec{v} \) and \( \vec{w} \) in Figure 5.3.1b and we want to find \( \phi \). How would you go about doing that? The natural choice is to use the Law of Cosines. In Figure 5.3.3 we draw \( \vec{w} - \vec{v} \) and we see that

\[
\|\vec{w} - \vec{v}\|^2 = \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\|\vec{w}\|\|\vec{v}\| \cos(\phi).
\] (5.3.2)

Of course, if we already know \( \vec{v} \) and \( \vec{w} \) then we certainly know \( \vec{w} - \vec{v} \) so this equation will allow us to find \( \theta \).

Using Equation 5.3.2 is feasible but complicated. In this section we will develop a more convenient method.
Theory

We start with the definition of the dot product.

\[(5.3.4) \text{ Definition.}\] Let \(\vec{u}\) and \(\vec{v}\) be vectors with
\[\vec{u} = u_1 \vec{i} + u_2 \vec{j} \quad \text{and} \quad \vec{v} = v_1 \vec{i} + v_2 \vec{j}.\]

Then their \textbf{dot product} is denoted \(\vec{u} \cdot \vec{v}\) and is defined by
\[\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2\]

Note that the dot product of two vectors is defined to be a real number. That is, \(\vec{u} \cdot \vec{v}\) is a number and not a vector.

It was emphasized in Section 5.1 that two vectors cannot be multiplied and it seems that we have now defined a way to multiply two vectors. This isn’t really the case, though. Think about anything else in the world that you know how to multiply. When you multiply two numbers together you always get another number. When you multiply two polynomials you get another polynomial. When you multiply two functions you get another function. When you multiply two matrices you get another matrix. Any time we multiply two \textit{things} together we get another \textit{thing}. However, \(\vec{u} \cdot \vec{v}\) is not a vector so the dot product doesn’t really work the way that multiplication should work.

Example 5.3.5 shows some basic dot product calculations.

\[(5.3.5) \text{ Example.}\]
\[\text{a. Find } \vec{u} \cdot \vec{v} \text{ where } \vec{u} = 2\vec{i} + 3\vec{j} \text{ and } \vec{v} = 7\vec{i} - 5\vec{j}.\]
\[\text{b. Find } \vec{v} \cdot \vec{w} \text{ where } \vec{v} = -4\vec{i} - 9\vec{j} \text{ and } \vec{w} = 2\vec{i} - 2\vec{j}.\]
c. Find $\vec{a} \cdot \vec{b}$ where $\vec{a} = 4\vec{i} + 5\vec{j}$ and $\vec{b} = 2\vec{i}$.

d. Find $\vec{i} \cdot \vec{j}$.

e. Find $\vec{0} \cdot (\vec{i} + \vec{j})$.

Solution:

a. From Definition 5.3.4,

$$\vec{u} \cdot \vec{v} = (2)(7) + (3)(-5) = -1.$$  

**Answer:** $\vec{u} \cdot \vec{v} = -1$

b. From Definition 5.3.4,

$$\vec{v} \cdot \vec{w} = (-4)(2) + (-9)(-2) = 10.$$  

**Answer:** $\vec{v} \cdot \vec{w} = 10$

c. We can write $\vec{b} = 2\vec{i} + 0\vec{j}$. Then, according to Definition 5.3.4,

$$\vec{a} \cdot \vec{b} = (4)(2) + (5)(0) = 8.$$  

**Answer:** $\vec{a} \cdot \vec{b} = 8$

d. We can write $\vec{i} = 1\vec{i} + 0\vec{j}$ and $\vec{j} = 0\vec{i} + 1\vec{j}$. Then, according to Definition 5.3.4,

$$\vec{i} \cdot \vec{j} = (1)(0) + (0)(1) = 0$$  

**Answer:** $\vec{i} \cdot \vec{j} = 0$

e. We can write $\vec{0} = 0\vec{i} + 0\vec{j}$ and $\vec{i} + \vec{j} = 1\vec{i} + 1\vec{j}$. Then, from Definition 5.3.4,

$$\vec{0} \cdot (\vec{i} + \vec{j}) = (0)(1) + (0)(1) = 0$$  

**Answer:** $\vec{0} \cdot (\vec{i} + \vec{j}) = 0$

Lemma 5.3.6 gives us some elementary properties of the dot product. They are all easily verified. Take, for example, the claim that $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$. Assume that $\vec{u} = u_1\vec{i} + u_2\vec{j}$. By definition,

$$\vec{u} \cdot \vec{u} = (u_1)(u_1) + (u_2)(u_2) = (u_1)^2 + (u_2)^2.$$  

Using Lemma 5.2.9,

$$\|\vec{u}\|^2 = \left(\sqrt{(u_1)^2 + (u_2)^2}\right)^2 = (u_1)^2 + (u_2)^2.$$
Combining these two things, we see that $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$. The rest of the properties are verified similarly.

\[ (5.3.6) \text{Lemma.} \] Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be any vectors and $c$ be any scalar.

- $\mathbf{0} \cdot \mathbf{u} = 0$
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$

We are not going to use the properties in Lemma 5.3.6 much in these notes. They come in handy occasionally with calculations but we do not need to do any examples with them. Theorem 5.3.7, on the other hand, will be immensely useful.

\[ (5.3.7) \text{Theorem.} \] Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors. If these two vectors are drawn originating from the same point then they form two angles at that point. Let $\theta$ be the smaller of those two vectors. Then

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos(\theta).$$

The difficult thing about Theorem 5.3.7 can be understanding $\theta$. Figure 5.3.8 shows two angles, $\mathbf{u}$ and $\mathbf{v}$ which are drawn from the same point, $p$. Since we are allowed to translate vectors wherever we want, we are always allowed to translate them until they begin at the same point. It certainly does not matter which point since translation preserves angles. The two angles formed by the two vectors are obviously $\theta$ and $\phi$. Theorem 5.3.7 demands that the equation uses the smaller of the two, so

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos(\theta).$$

From this point forward when we refer to the angle between $\mathbf{u}$ and $\mathbf{v}$ we will always mean the angle formed like $\theta$ in Figure 5.3.8. Suppose that we have a unit vector decomposition for both $\mathbf{u}$ and $\mathbf{v}$ and we would like to find $\theta$. Then we can calculate $\mathbf{u} \cdot \mathbf{v}$, $||\mathbf{u}||$, and $||\mathbf{v}||$ and it follows that

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}.$$
By using the techniques in Section 4.4 we can find solutions to this equation. We happen to get a little bit lucky, though. We know that \( \theta \) should be an angle with \( 0 \leq \theta \leq \pi \) and, fortunately, \( \cos^{-1} \) will output angles precisely between 0 and \( \pi \). It follows that

\[
\theta = \cos^{-1}\left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right).
\]

This motivates Corollary 5.3.9 below.

\begin{corollary}
If \( \vec{u} \) and \( \vec{v} \) are non-zero vectors and \( \theta \) is the angle between them then

\[
\theta = \cos^{-1}\left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right).
\]
\end{corollary}

We will now look at some examples.

\begin{example}
Find the angle, \( \theta \), between \( \vec{u} = 3\vec{i} + 4\vec{j} \) and \( \vec{v} = \vec{i} + 7\vec{j} \).
\end{example}

**Solution:**
First we calculate

\[
\|\vec{u}\| = \sqrt{3^2 + 4^2} = 5
\]
\[
\|\vec{v}\| = \sqrt{1^2 + 7^2} = \sqrt{50} = 5\sqrt{2}
\]
\[
\vec{u} \cdot \vec{v} = (3)(1) + (4)(7) = 31.
\]
Using Theorem 5.3.7, we have

\[ \|\vec{u}\|\|\vec{v}\| \cos(\theta) = \vec{u} \cdot \vec{v} \]

(5)(5\sqrt{2}) \cos(\theta) = 31

\[ \cos(\theta) = \frac{31}{25\sqrt{2}} \]

\[ \theta = \cos^{-1}\left(\frac{31}{25\sqrt{2}}\right) \]

\( \theta = 0.50 \).

Remember, the equation \( \cos(\theta) = \frac{31}{25\sqrt{2}} \) can be dangerous, but since \( \theta \) is the angle between two vectors we are allowed to use \( \cos^{-1} \) to find it.

**Answer:** \( \theta = 0.50 \)

---

(5.3.11) **Example.** Find the angle, \( \theta \), between \( \vec{a} = -2\vec{i} + 9\vec{j} \) and \( \vec{b} = 5\vec{i} - \vec{j} \).

**Solution:**

First we calculate

\[ \|\vec{a}\| = \sqrt{(-2)^2 + 9^2} = \sqrt{85} \]

\[ \|\vec{b}\| = \sqrt{5^2 + (-1)^2} = \sqrt{26} \]

\[ \vec{a} \cdot \vec{b} = (-2)(5) + (9)(-1) = -19 \]

Using Theorem 5.3.7, we have

\[ \|\vec{a}\|\|\vec{b}\| \cos(\theta) = \vec{a} \cdot \vec{b} \]

\[ \sqrt{85}\sqrt{26} \cos(\theta) = -19 \]

\[ \cos(\theta) = -\frac{19}{\sqrt{85}\sqrt{26}} \]

\[ \theta = \cos^{-1}\left(\frac{-19}{\sqrt{85}\sqrt{26}}\right) \]

\( \theta = 1.99 \).

**Answer:** \( \theta = 1.99 \)

---

(5.3.a) **Practice Exercise.** Let \( \vec{u} = -2\vec{i} + 5\vec{j} \) and \( \vec{v} = -6\vec{i} + 4\vec{j} \).
(i) Find $\vec{u} \cdot \vec{v}$.

(ii) Find the angle $\theta$ between $\vec{u}$ and $\vec{v}$. Round to two decimal places.

Examples like Example 5.3.10 and Example 5.3.11 are pretty predictable. If you are given a unit vector decomposition for two vectors, the process used in these two examples will find the angle between them. Example 5.3.12 adds a small complication but it is roughly the same type of problem.

(5.3.12) Example. Find the angle $\theta$ below.

Solution:

We first must find a unit vector decomposition for $\vec{u}$ and $\vec{v}$. Using Lemma 5.2.16 we have

$$\vec{u} = 4\vec{i} + \vec{j} \quad \text{and} \quad \vec{v} = -3\vec{i} + 2\vec{j}.$$  

Now the process becomes just like Example 5.3.10 and Example 5.3.11. First we calculate

$$\|\vec{u}\| = \sqrt{4^2 + 1^2} = \sqrt{17}$$  

$$\|\vec{v}\| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$$  

$$\vec{u} \cdot \vec{v} = (4)(-3) + (1)(2) = -10.$$  

Using Theorem 5.3.7, we have

\[ \|\vec{u}\| \|\vec{v}\| \cos(\theta) = \vec{u} \cdot \vec{v} \]
\[ (\sqrt{17})(\sqrt{13}) \cos(\theta) = -10 \]
\[ \cos(\theta) = \frac{-10}{\sqrt{17}\sqrt{13}} \]
\[ \theta = \cos^{-1}\left(\frac{-10}{\sqrt{17}\sqrt{13}}\right) \]
\[ \theta = 2.31. \]

**Answer:** \( \theta = 2.31 \)

There is one time where this procedure fails. If we’re looking for the angle between \( \vec{u} \) and \( \vec{v} \) and one of these vectors is \( \vec{0} \) then things become complicated. It makes no sense to talk about the angle between the zero vector and another vector. The reason that \( \vec{u} \) and \( \vec{v} \) need to be non-zero in Corollary 5.3.9 is because that would force \( \|\vec{u}\| \|\vec{v}\| \) to be zero which means that \( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \) would be undefined. The moral of the story is that we will ignore the concept of the angle between a vector and the zero vector.

The equation in Theorem 5.3.7 can be quite enlightening. If \( \vec{u} \) and \( \vec{v} \) are two non-zero vectors then

\[ \vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta) \]

where \( \theta \) is the angle between them. Notice that \( \|\vec{u}\| \) and \( \|\vec{v}\| \) are both positive. It follows that \( \vec{u} \cdot \vec{v} = 0 \) exactly when \( \cos(\theta) = 0 \). Otherwise, \( \vec{u} \cdot \vec{v} \) and \( \cos(\theta) \) have the same sign.

Think about what it means when \( \cos(\theta) = 0 \). We know that \( \theta \) is in the interval \([0, \pi]\) and the only angle in that interval whose cosine is zero is \( \frac{\pi}{2} \). It follows that the vectors \( \vec{u} \) and \( \vec{v} \) are perpendicular exactly when their dot product is zero. What about when \( \cos(\theta) \neq 0 \)? If \( \cos(\theta) > 0 \) then \( 0 \leq \theta < \frac{\pi}{2} \) and if \( \cos(\theta) < 0 \) then \( \frac{\pi}{2} < \theta < \pi \). Hence if \( \vec{u} \cdot \vec{v} > 0 \) then \( \vec{u} \) and \( \vec{v} \) meet at an acute angle whereas if \( \vec{u} \cdot \vec{v} < 0 \) then \( \vec{u} \) and \( \vec{v} \) meet at an obtuse angle. This is summarized in Lemma 5.3.13.

\[ (5.3.13) \text{ Lemma.} \text{ Let } \vec{u} \text{ and } \vec{v} \text{ be two non-zero vectors.} \]
\[ \bullet \text{ If } \vec{u} \cdot \vec{v} = 0 \text{ then } \vec{u} \text{ and } \vec{v} \text{ are perpendicular.} \]
\[ \bullet \text{ If } \vec{u} \cdot \vec{v} > 0 \text{ then } \vec{u} \text{ and } \vec{v} \text{ meet at an acute angle.} \]
\[ \bullet \text{ If } \vec{u} \cdot \vec{v} < 0 \text{ then } \vec{u} \text{ and } \vec{v} \text{ meet at an obtuse angle.} \]

The condition for \( \vec{u} \) and \( \vec{v} \) to be perpendicular in Lemma 5.3.13 is immensely powerful. It is very useful to work with perpendicular
vectors and we now have an easy condition to check if two vectors are perpendicular. We now explore some examples of how to use this lemma.

(5.3.14) Example. True or False: \( \vec{u} = 2\vec{i} + \vec{j} \) and \( \vec{v} = -4\vec{i} + 8\vec{j} \) are perpendicular.

Solution:
According to Lemma 5.3.13, all we need to check is the dot product. Observe that
\[
\vec{u} \cdot \vec{v} = (2)(-4) + (1)(8) = 0.
\]
Since \( \vec{u} \cdot \vec{v} = 0 \) it follows that \( \vec{u} \) and \( \vec{v} \) are perpendicular.
Answer: True

(5.3.15) Example. Find all of the vectors which are perpendicular to \( \vec{u} = -3\vec{i} + 2\vec{j} \).

Solution:
The only way that we know how to check if something is perpendicular to \( \vec{u} \) is by using Lemma 5.3.13. Let’s assume that \( \vec{v} \) is an arbitrary vector which is perpendicular to \( \vec{u} \) and write \( \vec{v} = x\vec{i} + y\vec{j} \). Since they are perpendicular, \( \vec{u} \cdot \vec{v} = 0 \). Hence
\[
\vec{u} \cdot \vec{v} = 0 \\
(-3\vec{i} + 2\vec{j}) \cdot (x\vec{i} + y\vec{j}) = 0 \\
-3x + 2y = 0 \\
y = \frac{3}{2}x.
\]
Since \( \vec{v} = x\vec{i} + y\vec{j} \) it must be that any vector \( \vec{v} \) which is perpendicular to \( \vec{u} \) must be of the form
\[
\vec{v} = x\vec{i} + \left( \frac{3}{2}x \right)\vec{j}
\]
for a real number \( x \). The only time such a vector won’t be perpendicular to \( \vec{u} \) is when \( x = 0 \) because then \( \vec{v} = \vec{0} \). Thus all of the vectors perpendicular to \( \vec{u} \) are \( x\vec{i} + \left( \frac{3}{2}x \right)\vec{j} \) for non-zero real numbers \( x \).
Answer: \( x\vec{i} + \left( \frac{3}{2}x \right)\vec{j} \) for every non-zero real number \( x \).
(5.3.16) **Example.** Find all values of \( t \) (if any) such that \( \vec{a} = (6t)\vec{i} + (t + 6)\vec{j} \) is perpendicular to \( \vec{b} = t\vec{i} - 6\vec{j} \).

**Solution:**

We know from Lemma 5.3.13 that \( \vec{a} \) and \( \vec{b} \) are perpendicular precisely when their dot product is zero. Hence

\[
\vec{a} \cdot \vec{b} = 0
\]

\[
(6t)(t) + (t + 6)(-6) = 0
\]

\[
6t^2 - 6t - 36 = 0
\]

\[
t^2 - t - 6 = 0
\]

\[
(t - 3)(t + 2) = 0.
\]

We see that \( \vec{a} \cdot \vec{b} = 0 \) exactly when \( (t - 3)(t + 2) = 0 \) which happens exactly when \( t = 3 \) and \( t = -2 \). The only thing left to check is that \( \vec{a} \) and \( \vec{b} \) are non-zero when \( t = 3 \) and \( t = -2 \). We see that

\[
\vec{a} = 18\vec{i} + 9\vec{j} \quad \text{and} \quad \vec{b} = 3\vec{i} - 6\vec{j} \quad \text{when} \ t = 3
\]

and

\[
\vec{a} = -12\vec{i} + 4\vec{j} \quad \text{and} \quad \vec{b} = -2\vec{i} - 6\vec{j} \quad \text{when} \ t = -2
\]

Since \( \vec{a} \) and \( \vec{b} \) are non-zero when \( t = 3 \) and \( t = -2 \) we have that \( \vec{a} \) and \( \vec{b} \) are perpendicular for those values of \( t \).

**Answer:** \( t = 3 \) and \( t = -2 \)

As a last note on the dot product, suppose \( \vec{u} \) and \( \vec{v} \) are two three dimensional vectors. Then

\[
\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k} \quad \text{and} \quad \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}.
\]

We define the dot product in this case similar to the two dimensional case. That is,

\[
\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.
\]

All of the same properties that we’ve learned still hold. That is, all of the results in Lemma 5.3.6 and the result in Theorem 5.3.7 are still valid.

**Applications**

The first application of the dot product is to large scale travel of things like boats and planes. We say that something is **traveling along a vector** \( \vec{p} \) if it is traveling in the direction of \( \vec{p} \). Normally when we use
vectors to describe motion we use a displacement vector. However it might be that we don’t know the entire displacement at a particular moment. Maybe the pilot doesn’t know how far he will be going but he knows the direction in which he’s going at a particular moment. He could describe this direction with a vector.

We can use these vectors to find information about the paths. In the following examples we are going to make the assumption that the vector $\vec{i}$ points to the east and the vector $\vec{j}$ points to the north in our coordinate system. The units on $\vec{i}$ and $\vec{j}$ will depend on the units in the particular problem. That is, if the problem deals with miles then the length of $\vec{i}$ will be $1\text{ mi}$, if the problem deals with meters then the length of $\vec{i}$ will be $1\text{ m}$, etc. This is somewhat irrelevant, though. It doesn’t matter whether we’re talking about inches, miles, or lightyears, the vectors $\vec{i}$ and $\vec{j}$ always point east and north, respectively. If something is traveling along the vector $\vec{p}$ then only the direction of $\vec{p}$ matters and the units don’t affect the direction.

**Example.** Two small aircraft leave an airport. The first is traveling along a vector of $\vec{p} = -\vec{i} + 2\vec{j}$ and the second is traveling along a vector of $\vec{q} = 4\vec{i} + 3\vec{j}$. Find the angle between their flight paths.

**Solution:**

In order to find the angle between the two paths we need only find the angle between the vectors $\vec{p}$ and $\vec{q}$. First, we have

\[
||\vec{p}|| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}
\]
\[
||\vec{q}|| = \sqrt{4^2 + 3^2} = 5
\]
\[
\vec{p} \cdot \vec{q} = (-1)(4) + (2)(3) = 2.
\]

Using Theorem 5.3.7 we have

\[
(5)(\sqrt{5}) \cos(\theta) = 2
\]
\[
\cos(\theta) = \frac{2}{5\sqrt{5}}
\]
\[
\theta = \cos^{-1}\left(\frac{2}{5\sqrt{5}}\right)
\]
\[
\theta = 1.57
\]

where $\theta$ is the angle between the two vectors. It follows that the angle between the two paths is $1.57$.

**Answer:** 1.57
Example. A ship is traveling along a vector of \( \vec{p} = \vec{i} + 7\vec{j} \), which is very close to due north. How close is it? That is, what is the angle between this ship’s path of travel and due north.

Solution:

All we need to do is find the angle between \( \vec{p} \) and a vector which points due north. It doesn’t matter which of these angles we pick, so we might as well pick \( \vec{j} \). First, we have

\[
\|\vec{j}\| = 1 \\
\|\vec{p}\| = \sqrt{1^2 + 7^2} = \sqrt{50} = 5\sqrt{2} \\
\vec{j} \cdot \vec{p} = (0)(1) + (1)(7) = 7.
\]

Using Theorem 5.3.7, we have

\[
(1)(5\sqrt{2}) \cos(\theta) = 7 \\
\cos(\theta) = \frac{7}{5\sqrt{2}} \\
\theta = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right) \\
\theta = 0.14
\]

where \( \theta \) is the angle between \( \vec{j} \) and \( \vec{p} \). It follows that the ship’s path is 0.14 radians away from north.

Answer: 0.14

An application of the dot product to physics is the calculation of work. Roughly speaking, work is a measure of how helpful a force is in moving an object. Suppose that you and your friend are both pushing on a refrigerator. You’re trying to move the refrigerator but your friend, who is a jerk, is pushing on the other side of the refrigerator to try and hinder you. This is shown in the force diagram in Figure 5.3.19. You are pushing with a force of \( \vec{F}_1 \) and your friend is pushing with a force of \( \vec{F}_2 \).

![Figure 5.3.19: The force diagram for two people pushing on a refrigerator in opposite directions.](image)

Luckily, you are stronger than your friend so that \( \|\vec{F}_1\| = 400 \text{ N} \) and \( \|\vec{F}_2\| = 200 \text{ N} \). Since the force being applied on the left is stronger than the force being applied on the right, the refrigerator is moving from...
left to right. Let’s assume that while these forces are being applied the refrigerator moves 5 m. Now, we said that work is a measure of how helpful a force is to the motion. In this case, the force on the left is being very helpful but the force on the other side is just a hindrance to the motion. According to our work calculation, the work done by $\vec{F}_1$ will be positive which signifies that it is helping and the work done by $\vec{F}_2$ will be negative which signifies that it is hindering.

Definition 5.3.20 below defines how to calculate work.

\[(5.3.20) \text{ DEFINITION.} \] Let $\vec{F}$ be a force acting on an object. Suppose that this object is displaced by a vector of $\vec{d}$ while under the influence of this force. Then the work done by $\vec{F}$ is $W$ where

$$W = \vec{F} \cdot \vec{d}.$$ 

The units on $W$ are the units on $\|\vec{F}\|$ times the units on $\|\vec{d}\|$.

Let’s try to calculate the work done on our refrigerator by the two forces. It is easy to write down the two force vectors since they are both parallel to $\vec{i}$. We see from the diagram that

$$\vec{F}_1 = 400\vec{i} \quad \text{and} \quad \vec{F}_2 = -200\vec{i}.$$ 

In order to calculate work we must also find a vector, $\vec{d}$, which describes its displacement. We know that $\|\vec{d}\| = 5$ m since we assumed that the refrigerator moved 5 m. We also know that it moved directly to the right, so

$$\vec{d} = 5\vec{i}.$$ 

We can now perform a work calculation for each force. Let $W_1$ be the work done by $\vec{F}_1$ and let $W_2$ be the work done by $\vec{F}_2$. According to Definition 5.3.20 we have

$$W_1 = \vec{F}_1 \cdot \vec{d} = (400\vec{i}) \cdot (5\vec{i}) = 2000$$

$$W_2 = \vec{F}_2 \cdot \vec{d} = (-200\vec{i}) \cdot (5\vec{i}) = -1000$$

Now, the last thing described by Definition 5.3.20 is the units. Since the units on force are Newtons and the units on distance are meters, the units on work are simply Newton-meters, which are denoted N m. Hence the work done by $\vec{F}_1$ is 2000 N m while the work done by $\vec{F}_2$ is $-1000$ N m. These units seem like nonsense, but in physics this is a reasonable thing. When calculating work, simply multiply the units on the force by the units on the distance to obtain the units on the work.

The force diagram in Figure 5.3.19 used for our refrigerator is obviously a very simple one. We will explore some more difficult computations in the remaining examples.
Example. Consider an object that moves a displacement of \( \vec{d} = 7\hat{i} + 3\hat{j} \) with units in meters. There are three forces acting on this object:

\[
\begin{align*}
\vec{F}_1 &= 500\hat{i} \\
\vec{F}_2 &= -300\hat{i} + 50\hat{j} \\
\vec{F}_3 &= 300\hat{i} - 700\hat{j}.
\end{align*}
\]

A. Find the work, \( W_1 \), done by \( \vec{F}_1 \).

B. Find the work, \( W_2 \), done by \( \vec{F}_2 \).

C. Find the work, \( W_3 \), done by \( \vec{F}_3 \).

D. Find the work, \( W \), done by the resultant force.

Solution:

A. By Definition 5.3.20,

\[
W_1 = \vec{F}_1 \cdot \vec{d} = (500)(7) = 3500.
\]

Answer: 3500 N m

B. By Definition 5.3.20,

\[
W_2 = \vec{F}_2 \cdot \vec{d} = (-300)(7) + (50)(3) = -1950.
\]

Answer: -1950 N m

C. By Definition 5.3.20,

\[
W_3 = \vec{F}_3 \cdot \vec{d} = (300)(7) + (-700)(3) = 0.
\]

Answer: 0 N m

D. We first need to find the resultant force, \( \vec{F} \). We have

\[
\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 500\hat{i} - 650\hat{j}.
\]

Then, by Definition 5.3.20,

\[
W = \vec{F} \cdot \vec{d} = (500)(7) + (-650)(3) = 1550.
\]

Answer: 1550 N m

Let’s look at some things in Example 5.3.21. What does it mean that \( W_3 = 0 \)? Well, \( W_3 \) should be a measure of how helpful the force \( \vec{F}_3 \)
is on moving the object, so we can conclude that $\vec{F}_3$ was no help at all. It also means that $\vec{F}_3 \cdot \vec{d} = 0$ which, as we found in Lemma 5.3.13, means that $\vec{F}_3$ and $\vec{d}$ are perpendicular. Imagine something that is moving horizontally and a force that is pushing vertically (so that the force and the displacement are perpendicular). As you can imagine, this force is not assisting in the motion at all.

Also notice in Example 5.3.21 that $W_1 + W_2 + W_3 = W$. This is not an accident. If several forces act on an object then the sum of the work done by each of them will equal the work done by the resultant force.

The calculations in Example 5.3.21 are relatively straightforward. The difficult part of examples involving work is often figuring out the vectors involved. We will have the same difficulties that we faced in previous force diagrams plus the difficulties involved in finding the displacement vector.

(5.3.22) Example. A box slides 20 m down a hill which has an incline of $\frac{\pi}{9}$ as shown below. The force of gravity, $\vec{F}_g$, has a magnitude of 600 N and the force of friction, $\vec{F}_f$ (which is parallel to the hill), has a magnitude of 450 N.

a. Find the work, $W_g$, done by gravity.

b. Find the work, $W_f$, done by friction.

Solution:

Before we handle the desired calculations, we need to find the box’s displacement vector. Obviously $||\vec{d}|| = 20$. To find $\vec{d}$, though, we need to know it’s angle with the horizontal. Since the box is sliding down the hill, we know that $\vec{d}$ points downward and to the left as shown below:
We see that
\[ \vec{d} = \left( 20 \cos \left( \frac{10\pi}{9} \right) \right) \hat{i} + \left( 20 \sin \left( \frac{10\pi}{9} \right) \right) \hat{j} = -18.79\hat{i} - 6.84\hat{j} \]
by Lemma 5.2.12.

A. From the diagram we have that \( \vec{F}_g = -600\hat{j} \) so
\[ W_g = \vec{F}_g \cdot \vec{d} = (-600)(-6.84) = 4104.24 \]

**Answer:** 4104.24 N m

B. From Lemma 5.2.12 and the diagram we have
\[ \vec{F}_f = \left( 450 \cos \left( \frac{\pi}{9} \right) \right) \hat{i} + \left( 450 \sin \left( \frac{\pi}{9} \right) \right) \hat{j} = 422.86\hat{i} + 153.91\hat{j} \]
and hence
\[ W_f = \vec{F}_f \cdot \vec{d} \\
= (422.86)(-18.79) + (153.91)(-6.84) \\
= -9000 \]

**Answer:** -9000 N m

We will have several examples like the one in Example 5.3.22 where we deal with something on a hill. The hill in Example 5.3.22 has an incline of \( \frac{\pi}{9} \). We can define an incline of any angle similarly. Whenever we say something goes “up” or “down” a hill, we mean the direction it would go under the influence of gravity.
(5.3.23) Example. A boy is pulling a small wagon along the ground as shown below. He pulls it a total of 100 m to the right, parallel to the ground. Assume that
\[ ||\vec{F}_g|| = 300 \text{ N}, \quad ||\vec{F}_f|| = 200 \text{ N}, \quad \text{and} \quad ||\vec{F}_p|| = 400 \text{ N}. \]

A. Find the work, \( W_g \), done by \( \vec{F}_g \).

B. Find the work, \( W_f \), done by \( \vec{F}_f \).

C. Find the work, \( W_p \), done by \( \vec{F}_p \).

D. Find the work, \( W \), done by the resultant force.

\[ \vec{F}_g \]
\[ \vec{F}_f \]
\[ \vec{F}_p \]

Solution:
We first need to find the displacement vector, \( \vec{d} \). Since the wagon moves 100 m to the right, \( \vec{d} = 100\vec{i} \).

A. From the diagram, \( \vec{F}_g = -300\vec{j} \) so
\[ W_g = \vec{F}_g \cdot \vec{d} = 0. \]

\[ \boxed{\text{Answer:} \ 0 \text{ N m}} \]

B. From the diagram, \( \vec{F}_f = -200\vec{i} \) so
\[ W_f = \vec{F}_f \cdot \vec{d} = (-200)(100) = -20000. \]

\[ \boxed{\text{Answer:} \ -20,000 \text{ N m}} \]
c. We can use Lemma 5.2.12 to find \( \bar{F}_p \) with \( \theta = \frac{3}{5} \) and \( ||\bar{F}_p|| = 400 \).

\[
\bar{F}_p = \left( 400 \cos \left( \frac{\pi}{5} \right) \right) \hat{i} + \left( 400 \sin \left( \frac{\pi}{5} \right) \right) \hat{j} \\
= 323.61 \hat{i} + 235.11 \hat{j}
\]

Then

\[
W_p = \bar{F}_p \cdot \bar{d} = (323.61)(100) = 32360.7.
\]

| Answer: | 32,360.7 N m |

The resultant force, \( \bar{F} \), is

\[
\bar{F} = \bar{F}_g + \bar{F}_f + \bar{F}_p = 123.61 \hat{i} - 64.89 \hat{j},
\]

hence

\[
W = \bar{F} \cdot \bar{d} = (123.61)(100) = 12360.7.
\]

Of course, if you remembered that the sum of the work done by the individual forces should equal the sum of the work done by the resultant force then instead of finding \( \bar{F} \) you could have simply calculated

\[
W = W_g + W_f + W_p = 12360.7.
\]

Either way is fine.

| Answer: | 12,360.7 N m |

Notice in Example 5.3.23 that we didn’t really need to calculate \( W_g \). We see that \( \bar{F}_g \) is perpendicular to the displacement vector, \( \bar{d} \), so we know that \( W_g = \bar{F}_g \cdot \bar{d} = 0 \) without any real calculation. What we used implicitly there was the fact that

\[
W_g = \bar{F}_g \cdot \bar{d} = ||\bar{F}_g|| ||\bar{d}|| \cos(\theta)
\]

where \( \theta \) is the angle between \( \bar{F}_g \) and \( \bar{d} \) as dictated by Theorem 5.3.7. When it is convenient, we can actually use Theorem 5.3.7 for any work calculation. For instance, consider \( W_p \) in Example 5.3.23. We see from the diagram that the angle between \( \bar{d} \) and \( \bar{F}_p \) is \( \frac{\pi}{5} \) and it was given in the problem that \( ||\bar{F}_p|| = 400 \) and \( ||\bar{d}|| = 100 \). Hence

\[
W_p = \bar{F}_p \cdot \bar{d} = ||\bar{F}_p|| ||\bar{d}|| \cos \left( \frac{\pi}{5} \right) = (400)(100) \cos \left( \frac{\pi}{5} \right) = 32360.7
\]

so that \( W_p = 32,360.7 \) N m which agrees with what we got in Example 5.3.23. Note that this calculation did not require us to find a unit vector decomposition for either \( \bar{F}_p \) or \( \bar{d} \).
This is a very handy trick at times. Compare Example 5.3.24 below with Example 5.3.22. In Example 5.3.24 we calculate the desired work without actually find unit vector decompositions for the vectors involved.

<table>
<thead>
<tr>
<th>Example 5.3.24</th>
<th>A box is on a hill which has an incline of $\frac{\pi}{7}$. The box slides 20 m down the hill under the force of gravity which has a magnitude of 30 N. Find the work, $W$, done by gravity.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution:</td>
<td>The first thing we should do is draw a force diagram. The force of gravity always points straight downward and the displacement force points down the hill. If $\vec{F}$ is the force of gravity and $\vec{d}$ is the displacement then we get the following force diagram:</td>
</tr>
<tr>
<td></td>
<td><img src="force_diagram.png" alt="" /></td>
</tr>
<tr>
<td></td>
<td>We can see from the force diagram that the angle between $\vec{F}$ and $\vec{d}$ is $\frac{\pi}{2} - \frac{\pi}{7} = \frac{5\pi}{14}$. It was given in the problem that $</td>
</tr>
<tr>
<td>Answer:</td>
<td>$260.33$ N m</td>
</tr>
</tbody>
</table>

| Example 5.3.8 | Practice Exercise. Three forces are acting on an object as shown below with $||\vec{F}_1|| = 25$ N, $||\vec{F}_2|| = 20$ N, and $||\vec{F}_3|| = 30$ N. Assume that while under the influence of these three forces the object moves 12 m straight upward. |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|               | (i) Find the work, $W_1$, done by $\vec{F}_1$. Round to two decimal places.                                                                                                                    |
(ii) Find the work, $W_2$, done by $\vec{F}_2$. Round to two decimal places.

(iii) Find the work, $W_3$, done by $\vec{F}_3$. Round to two decimal places.
5.3.1. Let $\vec{u} = -5\vec{i} + 3\vec{j}$, let $\vec{v} = 4\vec{i} + \vec{j}$, and let $\vec{w} = 2\vec{i} - 8\vec{j}$.
   a.) Calculate $\vec{u} \cdot \vec{v}$.
   b.) Calculate $\vec{v} \cdot \vec{w}$.
   c.) Calculate $\vec{w} \cdot \vec{u}$.

5.3.2. Let $\vec{u} = 3\vec{i} - 4\vec{j}$ and let $\vec{v} = -\vec{i} - 5\vec{j}$. Calculate the following:
   a.) $\vec{u} \cdot \vec{v}$
   b.) $||\vec{u}||$
   c.) $2\vec{v} - 5\vec{u}$

5.3.3. Consider the vectors $\vec{u} = 3\vec{i} + 5\vec{j}$ and $\vec{v} = -6\vec{i} + \vec{j}$.
   a.) Compute $\vec{u} \cdot \vec{v}$.
   b.) Find the angle between $\vec{u}$ and $\vec{v}$. Round your answer to two decimal places.

5.3.4. Let $\vec{u} = 3\vec{i} - 4\vec{j}$, let $\vec{v} = -2\vec{i} + 7\vec{j}$, and let $\vec{w} = 3\vec{i} - \vec{j}$. Leave your answers in exact form.
   a.) Calculate $\vec{u} + \vec{v}$.
   b.) Calculate $5\vec{w} - 2\vec{v}$.
   c.) Calculate $\vec{u} - 3(\vec{v} + 2\vec{w})$.
   d.) Calculate $\vec{v} \cdot \vec{w}$.
   e.) Calculate $||\vec{v}|| (\vec{u} \cdot \vec{w})$
   f.) Calculate $||\vec{v}|| (\vec{u} + \vec{w})$

5.3.5. Find the angle between the vectors $\vec{u} = -6\vec{i} + 7\vec{j}$ and $\vec{v} = 3\vec{i} - 3\vec{j}$. Round to two decimal places.

5.3.6. Find the angle between the vectors $\vec{u} = -6\vec{i} - 13\vec{j}$ and $\vec{v} = -4\vec{i} + 7\vec{j}$. Round to two decimal places.

5.3.7. Find the angle between the vectors $\vec{u} = 6\vec{i} + 3\vec{j}$ and $\vec{v} = -6\vec{i} + 2\vec{j}$. Round to two decimal places.

5.3.8. Let $\vec{u} = (t - 6)\vec{i} + t^2\vec{j}$ and $\vec{v} = \vec{i} + \vec{j}$. For what values of $t$ are $\vec{u}$ and $\vec{v}$ perpendicular?

5.3.9. Find all possible values of $t$ (if any) such that $\vec{u} = 4t\vec{i} - 6\vec{j}$ and $\vec{v} = -t\vec{i} + 10t\vec{j}$ are perpendicular.

5.3.10. Let $\vec{u} = (t + 4)\vec{i} + \vec{j}$ and $\vec{v} = -2\vec{i} + t^2\vec{j}$. For what values of $t$ (if any) are $\vec{u}$ and $\vec{v}$ perpendicular?

5.3.11. Consider the vectors $\vec{u} = 2\vec{i} + 4\vec{j}$ and $\vec{v} = -3\vec{i} + t\vec{j}$. If the angle between $\vec{u}$ and $\vec{v}$ is $\pi/3$ and $||\vec{v}|| \geq 5$, find $\vec{v}$. Round to two decimal places.

5.3.12. Find any vector which is perpendicular to $-2\vec{i} + \vec{j}$. Justify your answer.

5.3.13. Find $\theta$ below. Round to two decimal places.

5.3.14. Consider the triangle drawn below:

Assuming that $P$, $Q$, and $R$ are points with integer components, find a unit vector decomposition for $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RP}$. What is the angle $\theta$? Round to two decimal places.
5.3.15. Two boats leave port at the same time. The first boat travels along the vector \( \vec{v} = 3\hat{i} + \hat{j} \) and the second boat travels along the vector \( \vec{u} = -\hat{i} + 5\hat{j} \). Find the angle between their paths. Round your answer to two decimal places.

5.3.16. John and Amy leave their house by bike at the same time. John is traveling along a vector of \( \vec{u} = 2\hat{i} + 5\hat{j} \) and Amy is traveling along a vector of \( \vec{v} = -\hat{i} + 6\hat{j} \). Find the angle between their paths. Round to two decimal places.

5.3.17. A small box is sitting on a hill at an incline of 60°. The force of gravity on the box is 150 N. This force slides the box down the hill 20 m before the box comes to rest. Calculate the work imparted by the force of gravity on the box and round to two decimal places.

5.3.18. A barrel is rolling down a hill which is at an angle of 10° with the horizontal. The only force acting on the barrel is the force of gravity which has a magnitude of 500 N and acts perpendicular to the horizontal. If the barrel rolls a total of 100 m, find the work done on the barrel by the force of gravity. Round to two decimal places.

5.3.19. A force \( \vec{F} \) with a magnitude of 1000 N acts on an object as shown below. The object subsequently moves 200 m to the right (parallel to the ground). Find the work done by \( \vec{F} \) in moving the object. Round your answer to two decimal places.

5.3.20. In the force diagram below, \( \vec{G} \) is the force of gravity and \( \vec{P} \) is another force pushing on the box. You are given that \( \|\vec{G}\| = 322 \, \text{N} \) and that \( \|\vec{P}\| = 480 \, \text{N} \). Under the influence of these two forces, the box moves up the hill (i.e. up and to the right) a total of 50 m. Note: Vectors are not necessarily drawn to scale.

5.3.21. Consider the three forces below, \( \vec{F}_t \), \( \vec{F}_n \), and \( \vec{F}_g \), acting on the circular object in the center of the diagram where \( \|\vec{F}_t\| = 750 \), \( \|\vec{F}_n\| = 500 \), and \( \|\vec{F}_g\| = 200 \) (all forces in Newtons).

A.) Find the resultant force, \( \vec{F} \), that acts on the above object. Round to two decimal places when appropriate.

B.) Suppose that the object in part (a) is moved a distance of 30 meters straight up in the air (that is, in the \( \hat{j} \) direction). What is the work done by the resultant force (from part (a))? Round your answer to two decimal places.
(5.3.c1) Challenge Problem. Suppose \( \vec{u} \) and \( \vec{v} \) are two vectors such that \( \|\vec{u} - \vec{v}\| = \|\vec{u} + \vec{v}\| \). Find \( \vec{u} \cdot \vec{v} \) and justify your answer. Note: You can assume that these are both two dimensional vectors, but the answer would be the same for vectors of any dimension.


5.4 APPLICATIONS OF VECTORS

This section corresponds to section 12.3 in Functions Modeling Change [4].

This section covers additional applications of the vectors that we’ve seen so far. The material from this section comes from parts of sections 12.3 and 12.4 in Functions Modeling Change [4].

Motivation

We’ve already seen a few important applications of vectors. In this section we would like to look at some additional applications. There are applications of vectors everywhere; we will look at the following:

• Gravitation
• Computer Graphics
• Vector-Valued Functions

Theory

Before we can get to the applications there are a couple mathematical topics that need covered.

Remember from Section 5.2 that \|c\vec{v}\| = c\|\vec{v}\| whenever c > 0. It follows that when \vec{v} \neq \vec{0},

\[
\left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \frac{1}{\|\vec{v}\|} \|\vec{v}\| = 1.
\]

This motivates Lemma 5.4.1 below.

\begin{quote}
(5.4.1) Lemma. If \vec{v} is a non-zero vector then the unit vector in the direction of \vec{v} is

\[
\frac{1}{\|\vec{v}\|} \vec{v}.
\]

Recall that a unit vector is a vector with a magnitude of 1.
\end{quote}

Oftentimes it can be useful to have a vector in the same direction as a known vector but with a different length. Example 5.4.2 is an example of this.
(5.4.2) Example. Consider the vector \( \vec{v} = 48\vec{i} - 55\vec{j} \).

a. Find a vector \( \vec{u} \) which is in the same direction as \( \vec{v} \) but with a magnitude of 1.

b. Find a vector \( \vec{w} \) which is in the same direction as \( \vec{v} \) but with a magnitude of 50.

Solution:

a. We can use Lemma 5.4.1. We first need the magnitude of \( \vec{v} \).

\[ ||\vec{v}|| = \sqrt{(48)^2 + (-55)^2} = 73. \]

By Lemma 5.4.1 we have

\[ \vec{u} = \frac{1}{||\vec{v}||} \vec{v} = \frac{1}{73} (48\vec{i} - 55\vec{j}) = \frac{48}{73}\vec{i} - \frac{55}{73}\vec{j}. \]

Answer: \( \vec{u} = \frac{1}{73} (48\vec{i} - 55\vec{j}) = \frac{48}{73}\vec{i} - \frac{55}{73}\vec{j} \)

b. We already know a vector of length 1 which is in the same direction as \( \vec{v} \); that was the vector \( \vec{u} \) found in the previous part. In order to find a vector of length 50 in that direction we need only multiply \( \vec{u} \) by 50.

\[ \vec{w} = 50\vec{u} = 50 \left( \frac{48}{73}\vec{i} - \frac{55}{73}\vec{j} \right) = \frac{2400}{73}\vec{i} - \frac{2750}{73}\vec{j} \]

Answer: \( \vec{w} = \frac{2400}{73}\vec{i} - \frac{2750}{73}\vec{j} \)

In Section 5.3 we talked about things traveling along a vector. We said that if a boat was traveling along the vector \( \vec{v} = \vec{i} + 3\vec{j} \) then the boat is traveling in the direction of \( \vec{v} \) but for some undetermined distance. Suppose this boat actually travels a distance of 50 km and assume that \( \vec{i} \) and \( \vec{j} \) are both a length of 1 km. Using the strategies in Example 5.4.2 we can find the boat’s actual displacement vector, \( \vec{d} \). We want a vector in the direction of \( \vec{v} \) but with a magnitude of \( ||\vec{d}|| = 50 \).

\[ \vec{d} = \frac{||\vec{d}||}{||\vec{v}||} \vec{v} \]

\[ = \frac{50}{\sqrt{10}} (\vec{i} + 3\vec{j}) \]

\[ = 5\sqrt{10}\vec{i} + 15\sqrt{10}\vec{j} \]

We also want to be able to find the angle that a vector makes with the horizontal. Let \( \vec{v} = v_1\vec{i} + v_2\vec{j} \) be a vector with \( v_1 \neq 0 \) and let \( \theta \) be
the angle that it makes with the horizontal. Then, from Lemma 5.2.12, we know
\[
v_1 = \|\vec{v}\| \cos(\theta) \quad \text{and} \quad v_2 = \|\vec{v}\| \sin(\theta).
\]

By dividing these two equations we get
\[
\frac{v_2}{v_1} = \frac{\|\vec{v}\| \sin(\theta)}{\|\vec{v}\| \cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta).
\]

We would like to use the inverse tangent function to find \(\theta\) but we don’t quite know where \(\theta\) lives. Of course, if \(\tan^{-1}(\frac{v_2}{v_1})\) is on the right side of the unit circle. If the vector is pointing to the right - that is, if \(v_1 > 0\) - then \(\theta = \tan^{-1}(\frac{v_2}{v_1})\). On the other hand, if \(v_1 < 0\) then \(\theta = \pi + \tan^{-1}(\frac{v_2}{v_1})\). These things are summarized in Lemma 5.4.3 and can be verified with Lemma 4.4.14.

\[
(5.4.3) \text{ Lemma.} \quad \text{Let} \ \vec{v} = v_1 \vec{i} + v_2 \vec{j} \text{ with } v_1 \neq 0 \text{ and let } \theta \text{ be the angle that } \vec{v} \text{ makes with the horizontal. Then } \tan(\theta) = \frac{v_2}{v_1} \text{ and}
\]

- if \(v_1 > 0\) then \(\theta = \tan^{-1}(\frac{v_2}{v_1})\).
- if \(v_1 < 0\) then \(\theta = \pi + \tan^{-1}(\frac{v_2}{v_1})\).

In each case \(\theta\) is defined up to \(2\pi n\) for some integer, \(n\).

We’ve left out the case where \(v_1 = 0\). This case is easy, though. If \(\vec{v} = v_2 \vec{j}\) then \(\vec{v}\) points either straight up or straight down. If \(v_2 > 0\) then \(\theta = \frac{\pi}{2}\) and if \(v_2 < 0\) then \(\theta = -\frac{\pi}{2}\). Using Lemma 5.4.3 we can rotate vectors as in Example 5.4.4.

\[
(5.4.4) \text{ Example.} \quad \text{Consider the vector } \vec{p} = 4\vec{i} + \vec{j}. \text{ Let } \vec{q} \text{ be the vector which results from rotating } \vec{p} \text{ counterclockwise by an angle of } \frac{2\pi}{3}. \text{ Find a unit vector decomposition for } \vec{q}.
\]

\[
\text{Solution:}
\]

We know that \(\vec{q}\) has the same magnitude as \(\vec{p}\) so
\[
\|\vec{q}\| = \|\vec{p}\| = \sqrt{4^2 + 1^2} = \sqrt{17}.
\]

If we can find the angle, \(\phi\), that \(\vec{q}\) makes with the horizontal then we will be able to find \(\vec{q}\) using Lemma 5.2.12. We know that \(\vec{q}\) is a rotation of \(\vec{p}\) counterclockwise by \(\frac{2\pi}{3}\) so if we can
find the angle, \( \theta \), that \( \vec{p} \) makes with the horizontal then \( \phi = \theta + \frac{2\pi}{3} \).

We can find \( \theta \) using Lemma 5.4.3 with \( v_1 = 4 \) and \( v_2 = 1 \). Since \( v_1 > 0 \) we have that

\[
\theta = \tan^{-1} \left( \frac{1}{4} \right) = 0.24.
\]

Then \( \phi = 0.24 + \frac{2\pi}{3} = 2.34 \) and, from Lemma 5.2.12,

\[
\vec{q} = (\|\vec{q}\| \cos(\phi))\hat{i} + (\|\vec{q}\| \sin(\phi))\hat{j}
= (\sqrt{26} \cos(2.34))\hat{i} + (\sqrt{26} \sin(2.34))\hat{j}
= -3.54\hat{i} + 3.67\hat{j}.
\]

**Answer:** \( \vec{q} = -3.54\hat{i} + 3.67\hat{j} \)

For the purpose of demonstration, the vectors \( \vec{p} \) and \( \vec{q} \) are shown below with their starting points at the origin.

We have been talking about *the* angle that a vector makes with the horizontal as if it is unique when, of course, it is not unique. If you start with a vector, \( \vec{v} \), and you rotated it counterclockwise by \( 2\pi \) then the vector obtained after the rotation is just the same as \( \vec{v} \). In Lemma 5.2.12 and Lemma 5.4.3 we are not concerned with which of the many angles the vector in question makes with the horizontal. In Lemma 5.2.12 it is irrelevant as they all share the same cosine and the result in Lemma 5.4.3 simply picks one of the many choices.

The last thing to mention before the applications is scientific notation. Depending on your background in other areas you may or may not have seen scientific notation before. Suppose that \( N \) is the number of atoms in a person who weighs 70 kg. Well, \( N \) is very, very large. In fact, it is approximately

\[ N \approx 7,000,000,000,000,000,000,000,000,000,000,000. \]

That is an absurdly large number and it is inconvenient to write out all 27 zeros every time we want to use \( N \). Scientific notation is a con-
venient way to write extremely large numbers. In scientific notation we would write

\[ N \approx 7 \times 10^{27}. \]

There is no trick to this. When we write \( 7 \times 10^{27} \) we literally mean the number 7 times \( 10^{27} \). This allows us to write extremely large numbers in a compact format. We can write very small numbers similarly. For example, the wavelength of light in X-rays is approximately \( \epsilon \) where

\[ \epsilon \approx 0.0000000001. \]

Instead, we can write this as

\[ \epsilon \approx 1 \times 10^{-10}. \]

When you calculator is asked to compute very large numbers it will likely give you answers in scientific notation by default. Some calculators have a button dedicated to scientific notation but any calculator that can handle exponents will be able to input scientific notation in some form.

Rounding becomes a little more complicated in scientific notation. You will be asked to round to two decimal places in scientific notation. Suppose we perform a calculation and find that the mass of a very large object is

\[ 485,748,920,109,289,384,830,294,887.928475932 \text{ kg}. \]

This, in some sense, is a ludicrous number. Just because a calculator outputs that many decimal places does not mean that our calculation can provide nearly that much accuracy. However, rounding to two decimal places also seems just as ludicrous. The quantity

\[ 485,748,920,109,289,384,830,294,887.93 \text{ kg} \]

does not seem any better. However, when we round to two decimal places in scientific notation this quantity becomes

\[ 4.86 \times 10^{26} \text{ kg}, \]

which is a more reasonable approximation.

**Applications**

We have already seen the application of vectors to force diagrams. This is an important application that you will see again if you take an introductory physics class. Forces are always vectors and there are, of course, other uses of force outside of force diagrams. Here we introduce Newton’s law of Gravitation which describes how objects impart gravitational forces on each other.
(5.4.5) Theorem (Newton’s Law of Universal Gravitation). Consider two objects in space with masses of $m_1$ and $m_2$. Let $r$ be the distance between these two objects and let $F$ be the magnitude of the force of gravity between them. Then

$$F = G \frac{m_1 m_2}{r^2}$$

where $G$ is a constant called the gravitational constant. If the masses are in kg and the distance is in km then

$$G = 6.6738 \times 10^{-17} \text{ N km}^2/\text{kg}^2$$

and the units on $F$ are Newtons.

The way that gravity works is that every mass imparts a force on every other mass. That is, every two masses are being pulled toward each other by gravity. This force pulls the centers of the objects together. The mass of Earth pulls you toward the center of Earth, but your mass also pulls Earth toward your center.

Consider the two objects of mass $m_1$ and $m_2$ shown in Figure 5.4.6. Gravity pulls them toward each other so that there is a force pulling on each object which points toward the center of the other object. These forces are shown as $\vec{F}_1$ and $\vec{F}_2$. Both forces have the same magnitude (even if the masses are different) and they point in opposite direction (that is, they are negatives of each other). Theorem 5.4.5 tells us that

$$||\vec{F}_1|| = ||\vec{F}_2|| = G \frac{m_1 m_2}{r^2}.$$ 

The number $G$ is just a constant whose value depends on the units of $m_1$, $m_2$, and $r$. In this section we will only use kg for mass and km and Theorem 5.4.5 tells us the value of $G$ in that case.

In many cases it is simply enough to know the magnitude of the gravitational force. However, if we know the position of the two objects then we can do better. Let $\vec{d}_1$ be the displacement vector from
m_1 \text{ to } m_2. \text{ We know that } \vec{F}_1 \text{ is in the same direction as } \vec{d} \text{ but with a magnitude of } ||\vec{F}_1||. \text{ Using Lemma 5.4.1 we then have }

\[ \vec{F}_1 = \frac{||\vec{F}_1||}{||\vec{d}_1||} \vec{d}_1. \]

Multiplying \( \vec{d}_1 \) by \( \frac{1}{||\vec{d}_1||} \) gives it a length of 1 and then multiplying by \( ||\vec{F}_1|| \) gives it a length of \( ||\vec{F}_1|| \). Then, of course, \( \vec{F}_2 = -\vec{F}_1 \) as we see in Figure 5.4.6.

(5.4.7) Example. The diagram below shows the position of Earth, the moon, and a satellite at a particular moment.

We know the following things about these objects:

- The mass of the satellite is \( m_S = 100,000 \text{ kg} \).
- The mass of Earth is \( m_E = 5.98 \times 10^{24} \text{ kg} \).
- The mass of the moon is \( m_L = 7.34 \times 10^{22} \text{ kg} \).
- In the diagram, \( \vec{r}_E = -280000\vec{i} - 90000\vec{j} \) (with units in km).
- In the diagram, \( \vec{d} = 384000\vec{i} \) (with units in km).

Let \( \vec{F}_E \) be the gravitational force exerted on the satellite by Earth and let \( \vec{F}_L \) be the gravitational force exerted on the satellite by the moon.

A. Find \( \vec{F}_E \).

B. Find \( \vec{F}_L \).

C. Which object exerts a larger gravitational force on the satellite?

Solution:
A. We can use Theorem 5.4.5 to find \( \| \vec{F}_E \| \). Of course, the distance between Earth and the satellite is \( \| \vec{r}_E \| \) and

\[
\| \vec{r}_E \| = \sqrt{(-280000)^2 + (-90000)^2} = 294108.82.
\]

Then we have

\[
\| \vec{F}_E \| = G \frac{m_S m_E}{\| \vec{r}_E \|^2} = \frac{(6.6738 \times 10^{-11})(100000)(5.98 \times 10^{24})}{294108.82^2} = 461.38.
\]

Hence \( \| \vec{F}_E \| = 461.38 \) N.

From the diagram we know that the force should point in the direction of \( \vec{r}_E \) since it should point from the satellite toward Earth. Then

\[
\vec{F}_E = \frac{\| \vec{F}_E \|}{\| \vec{r}_E \|} \vec{r}_E = \frac{461.38}{294108.82} (-280000\vec{i} - 90000\vec{j}) = -439.25\vec{i} - 141.17\vec{j}.
\]

**Answer:** \( \vec{F}_E = -439.25\vec{i} - 141.17\vec{j} \)

B. We will also use Theorem 5.4.5 to find \( \| \vec{r}_L \| \) but first we need to find the distance between the satellite and the moon. The vector \( \vec{r}_L \) is added below:

From the diagram we see that \( \vec{r}_L = \vec{r}_E + \vec{d} \) so

\[
\vec{r}_L = \vec{r}_E + \vec{d} = (-280000\vec{i} - 90000\vec{j}) + (384000\vec{i}) = 104000\vec{i} - 90000\vec{j}.
\]

Of course, the distance between the satellite and the moon is \( \| \vec{r}_L \| \) and

\[
\| \vec{r}_L \| = \sqrt{104000^2 + (-90000)^2} = 137535.45.
\]
By Theorem 5.4.5,

\[
\|\vec{F}_L\| = G \frac{m_S m_L}{\|\vec{r}_L\|} = \frac{(6.6738 \times 10^{-11})(100000)(7.34 \times 10^{22})}{137535.45^2} = 25.90.
\]

Hence \(\|\vec{F}_L\| = 25.90\) N.

From the diagram we see that the force should be in the direction of \(\vec{r}_L\) so

\[
\vec{F}_L = \frac{\|\vec{F}_L\|}{\|\vec{r}_L\|} \vec{r}_L = \frac{25.90}{137535.45}(104000\hat{i} - 90000\hat{j}) = 19.58\hat{i} - 16.95\hat{j}.
\]

\textbf{Answer:} \(\vec{F}_L = 19.58\hat{i} - 16.95\hat{j}\)

c. Over the course of the previous parts we found that \(\|\vec{F}_E\| = 461.38\) N and \(\|\vec{F}_L\| = 25.90\) N. Of course, since \(\|\vec{F}_E\| > \|\vec{F}_L\|\), Earth exerts a stronger gravitational force on the satellite than the moon. The moon is significantly closer but Earth’s gravity overwhelms the moon’s gravity because Earth is more massive.

The influence of distance and mass on the force of gravity is interesting. Since its magnitude is inversely proportional to the square of the distance between the two objects, the gravitational force tapers very quickly as the distance increases. The size of gravitational forces can be surprising at times.

\textbf{Answer:} Earth

Our next application is that of computer graphics. When a computer screen displays images on the screen it does so by displaying single points of color (called \textit{pixels}). A programmer treats the screen like an \(xy\)-plane with the origin in the bottom left corner. It is convenient to represent the position of these pixels by a vector which describes their displacement from the origin.

This use of vectors is useful when programming first-person video games. When the character moves or turns, the perspective of the things on screen change. When the character just moves a little bit most of the things on the screen remain the same, they just shift by a little bit. In order to optimize computing power it is important
that the video processor doesn’t need to render the scene completely every time the character moves, but instead uses previous frames and small shifts to make new frames.

The types of shifts which correspond to these small movements are called linear transformations. In general these transformations are complicated but we already understand two of them. We know how to rotate a pixel (using Lemma 5.4.3) and we know how to translate a pixel (by adding a vector). This is explored in Example 5.4.8.

(5.4.8) Example. On a computer screen with a resolution of $1280 \times 800$ the pixels can be at points which correspond to vectors $\vec{p} = p_1 \vec{i} + p_2 \vec{j}$ where $p_1$ is an integer with $0 \leq p_1 \leq 1280$ and $p_2$ is an integer with $0 \leq p_2 \leq 800$. Consider a pixel which is at $\vec{p} = 500\vec{i} + 150\vec{j}$.

a. After a certain movement, the pixel at $\vec{p}$ gets rotated by an angle of $0.4$ around the origin. Find the position of the pixel after the rotation and round two the nearest whole numbers.

b. After another movement, this same pixel is then translated up 200 pixels and to the right by 600 pixels (that is, after the rotation described in the previous part). Find the position of the pixel after the rotation.

Solution:

a. Let $\vec{q}$ be the vector which describes the pixel’s position after the rotation. The pixel is originally at a vector of $\vec{p} = 500\vec{i} + 150\vec{j}$. Since this describes the position from the origin and we want to rotate the pixel’s position by $0.4$ around the origin then we simply need to rotate $\vec{p}$ by $0.4$. This is shown below:
We can do this as we did in Example 5.4.4. Then, by using Lemma 5.4.3, we know that $\vec{p}$ makes an angle of $\theta$ with the horizontal where

$$\theta = \tan^{-1}\left(\frac{150}{500}\right) = 0.29.$$

It follows that if $\vec{q}$ makes an angle of $\phi$ with the horizontal then $\phi = 0.29 + 0.4 = 0.69$. Since $\vec{q}$ has the same magnitude as $\vec{p}$ we have

$$\|\vec{q}\| = \|\vec{p}\| = \sqrt{500^2 + 150^2} = 522.02.$$

We can then calculate

$$\vec{q} = (\|\vec{q}\| \cos(\phi))\vec{i} + (\|\vec{q}\| \sin(\phi))\vec{j} = 402.12\vec{i} + 333.87\vec{j}.$$

**Answer:** $\vec{q} = 402\vec{i} + 333\vec{j}$

b. Let $\vec{r}$ be the vector which describes the pixel’s final position after this translation. The translation up by 200 pixels corresponds to the vector $200\vec{j}$ and the translation to the right by 600 pixels corresponds to the vector $600\vec{i}$. To translate this pixel in this manner we simply add $600\vec{i} + 200\vec{j}$ to $\vec{q}$. This is shown below:

It follows that

$$\vec{r} = \vec{q} + 600\vec{i} + 200\vec{j} = 402\vec{i} + 333\vec{j} + 600\vec{i} + 200\vec{j} = 802\vec{i} + 533\vec{j}.$$

**Answer:** $\vec{r} = 802\vec{i} + 533\vec{j}$
In Section 5.1 we discussed how force diagrams represent objects by a single point. This is not just a matter of making the diagram easier to draw; there is a reason for this assumption. If an object has volume then forces will cause it to rotate which can result in additional complications. If a physicist wants to neglect all rotation then he or she will assume that an object is a single particle with the appropriate mass. This seems like an unrealistic oversimplification but it is still extremely useful at times. In elementary physics there are three primary quantities involved in describing the motion of a particle. These are position, velocity, and acceleration; each of which are vector quantities.

Position is straightforward as vectors are naturally used to describe displacement. We can describe a particle’s position as its displacement from a fixed point.

(5.4.9) Example. A ball is launched in the air. Then $t$ seconds after the ball is thrown its displacement from a fixed point on the ground is

$$(4t)\vec{i} + (-5t^2 + 20t + 3)\vec{j}$$

where $\vec{i}$ has a length of 1 m in a direction parallel to the ground and $\vec{j}$ has a length of 1 m pointing directly upward.

A. Find the ball’s height off of the ground at the moment it is thrown.

B. How long does it take for the ball to reach its highest point?

C. When the ball hits the ground, how far away is it from the point from the fixed point?

Solution: Before we handle the questions, let’s sketch a picture of this ball’s path:
At each value of $t$ we simply draw the vector described starting at the given fixed point. For example, when $t = 0$ we should have

$$\vec{r} = (4(0))\vec{i} + (-5(0)^2 + 20(0) + 3)\vec{j} = 3\vec{j}$$

which is drawn above.

**A.** We know that $t$ is the number of seconds after the ball is thrown so the moment the ball is thrown corresponds to $t = 0$. We just calculated that the ball’s position vector is $3\vec{j}$ when $t = 0$. Since the position vector measure’s the ball’s position from the fixed point which is on the ground, the $\vec{j}$-component of the position vector should be the ball’s height. Hence when $t = 0$ the ball’s height is $3$ m.

**B.** The $\vec{j}$-component of the position vector is the height of the ball so that finding the value of $t$ which maximizes the $\vec{j}$-component will be the value we want. As a function of time, the $\vec{j}$-component is $f(t)$ where

$$f(t) = -5t^2 + 20t + 3.$$  

The vertex of $f$ happens when $t = -\frac{b}{2a} = \frac{-20}{2(-5)} = 2$ which means that the $\vec{j}$-component of the position vector is maximized 2 seconds after the ball is thrown.

**Answer:** 2 seconds

**C.** The ball hits the ground when its height is zero. The height is the $\vec{j}$-component of the position vector, so we first need to know when

$$-5t^2 + 20t + 3 = 0.$$  

Using the quadratic formula we see that this happens when $t = -0.14$ and when $t = 4.14$. Of course it doesn’t
make sense to talk about values of $t$ which are less than zero as this is before the ball is thrown. Hence the ball hits the ground when $t = 4.14$. When $t = 4.14$ the position vector is


The position vector is the ball’s displacement from the fixed point, so at this point the ball is 16.58 m from the fixed point.

**Answer:** 16.58 m

Example 5.4.9 gives us a glimpse into what mathematician call **vector valued functions**. We are accustomed to functions which input and output real numbers. That is, if $f$ is a function then we expect $f(t)$ to be a number. However, functions can input and output pretty much anything. A vector valued function outputs vectors. In Example 5.4.9 we could have defined a function $\vec{s}$ such that

$$\vec{s}(t) = (4t)i + (-5t^2 + 20t + 3)j.$$

We use vector notation on this function to indicate that its outputs are vectors. These functions are useful but we won’t discuss them any further here.

The other two quantities which describe a particle’s movement are velocity and acceleration. We have already been discussing acceleration in secret. Newton’s second law of motion tells us that if a particle has mass $m$, has an acceleration vector of $\vec{a}$, and is being influenced by a resultant force $\vec{F}$, then

$$\vec{F} = m\vec{a}.$$ 

Because of this relationship, studying acceleration is the same as studying force which is something we’ve already done at length.

A particle’s velocity describes how it is moving (while the acceleration vector describes how its movement is changing). The magnitude of the velocity vector is the particle’s speed and the direction is the direction of the particle’s motion. A particle can be moving in any sort of crazy path so it might seem like an oversimplification to assume that a particle is moving in a particular “direction.” However, thanks to calculus, if our particle is moving in a sufficiently smooth path then at any given moment it looks like it’s moving in roughly a straight line. In a calculus class you will learn that a particle’s velocity is the “derivative” (whatever that means) of the particle’s position, but we will not study velocity here.
5.4.1. Consider the vector $\vec{u} = 8\hat{i} - 15\hat{j}$.

A.) Find a vector $\vec{v}$ which is in the same direction as $\vec{u}$ and has a magnitude of 1. Leave your answer in exact form.

B.) Find a vector $\vec{w}$ which is in the same direction as $\vec{u}$ and has a magnitude of 100. Leave your answer in exact form.

C.) Find a vector $\vec{x}$ which results from rotating the vector $\vec{u}$ counterclockwise by an angle of $\frac{\pi}{3}$. Round to two decimal places.

5.4.2. Two tugboats are pulling a larger boat according to the diagram below where the filled circles are the tugboats and the rectangle is the larger boat.

The force with which tugboat A is pulling has a magnitude of 1,200,000 N. What is the magnitude of the force with which tugboat B must pull in order to keep the larger boat moving parallel to the dashed line? Round your answer to two decimal places. Hint: The large boat travels along the resultant force vector.

5.4.3. The following information may be helpful:

- The mass of the sun is $1.9891 \times 10^{30}$ kg.
- The mass of the earth is $5.972 \times 10^{24}$ kg.
- The mass of jupiter is $1.8981 \times 10^{27}$ kg.

5.4.4. Suppose that the earth is at point E, the sun is at point S, and jupiter is at point J. Also suppose that $\vec{SE} = 1496000000\hat{i}$ and that $\vec{EJ} = -538850000\hat{i} + 674200777\hat{j}$ (all of these units are in kilometers). Compute the values below and round your answer to two decimal places (two decimal places in scientific notation):

A.) Find the magnitude of the gravitational force between the sun and the earth.

B.) Find the magnitude of the gravitational force between the sun and jupiter.

C.) Find the magnitude of the gravitational force between the earth and jupiter.

5.4.4. A.) The mass of the earth is $5.972 \times 10^{24}$ kg. A person is standing on the surface of the earth and her position vector with respect to the center of the earth is given by $5400\hat{i} + 1500\hat{j} + 3000\hat{k}$. Find a function $f$ which describes the magnitude of the gravitational force imparted on her by the earth; that is if her mass is $m$ (in kilograms) then $f(m)$ should be the magnitude of the gravitational force between her and the earth. Round to two decimal places when necessary.

B.) Newton’s second law of motion says that the relationship between a force $\vec{F}$ and the acceleration $\vec{a}$ that it imparts on an object is given by $||\vec{F}|| = m||\vec{a}||$ where $m$ is the mass of the object in kilograms and the forces are in Newtons. Find the magnitude of the acceleration due to gravity imparted on the person in part (a) with mass $m$. 
c.) Suppose the person in part (a) has a mass of 70 kg. First find the magnitude of the gravitational force between her and the earth. Next find the force vector imparted on the center of the earth by gravity. Round to two decimal places when necessary. 

*Hint: The force vector has the same direction as the person’s position vector.*

5.4.5. A small military ship is parked in a river which flows directly north and south. A coordinate system is implemented such that \( \hat{i} \) points directly east and has a length of 1 mi while the vector \( \hat{j} \) points directly north and has a length of 1 mi. The ship’s radar detects a helicopter as it flies near the ship. The ship then tracks the helicopter’s movements and finds that \( t \) hours after it passes the ship its displacement vector from the ship is given by 

\[
(10t^2 - 70t)\hat{i} + (t^2 - 4)\hat{j}.
\]

a.) How far from the ship is the helicopter after five hours? Round to two decimal places.

b.) How long is it after the ship starts tracking the helicopter that the helicopter passes the river for the second time?

c.) Find a vector \( \vec{d} \) which describes the helicopter’s displacement from its position when \( t = 4 \) to its position when \( t = 8 \).

5.4.6. A triangle is displayed on a computer screen. The pixels at its corners are described by vectors \( \vec{p}, \vec{q}, \) and \( \vec{r} \) where 

\[
\vec{p} = 50\hat{i} + 100\hat{j}, \\
\vec{q} = 50\hat{i} + 50\hat{j}, \\
\vec{r} = 200\hat{i} + 50\hat{j}.
\]

Recall that these position vectors describe each pixel’s displacement from the bottom left corner of the screen (in pixels).

a.) It is easy to see that this triangle is a right triangle. Find the length of its hypotenuse (in pixels). Round to the nearest pixel (no decimal places).

b.) This triangle is shifted up by 100 pixels. Find vectors \( \vec{p}_1, \vec{q}_1, \) and \( \vec{r}_1 \) which describe the corners of the triangle after it is shifted.

c.) After the shift in the previous part, the entire triangle is rotated about the origin by an angle of \( \frac{\pi}{6} \). Find vectors \( \vec{p}_2, \vec{q}_2, \) and \( \vec{r}_2 \) which describe the corners of the triangle after this rotation. Round to the nearest pixel.

*Hint: Do this by rotating each of the individual vectors.*

---

**Challenge Problems**

**(5.4.c1) Challenge Problem.** During this section we learned how to rotate a vector about the origin. We wish to extrapolate on that idea here.

a. Let \( \vec{p} = x\hat{i} + y\hat{j} \) be some vector (think of a point on a computer screen if you’d like). Come up with a general formula for the vector \( \vec{q} \) which is obtained after rotating \( \vec{p} \) by an angle of \( \theta \) (clockwise). That is, if \( \vec{q} = a\hat{i} + b\hat{j} \) then find a formula for both \( a \) and \( b \) written in terms of \( x, y, \) and \( \theta \). Try to find a formula which doesn’t involve any inverse trig functions (although, depending
on how you do it, you may need to use inverse trig functions along the way).

b. Let $p = (4, 2)$ be a point in the plane (it may be helpful to think of it as the vector $\vec{p} = 4\vec{i} + 2\vec{j}$).

(i) Find the point $q_1$ obtained after rotating $p$ by an angle of $\frac{\pi}{3}$ about the origin.

(ii) Find the point $q_2$ obtained after rotating $p$ by an angle of $\frac{\pi}{3}$ about the point $(1, 3)$.

Hint: You already know how to find $q_1$ after translating these points into the language of vectors. On the other hand, you may not know how to find $q_2$ immediately since it is a rotation about a point that is not the origin. However, there is a clever way to use vectors and the skills that you already know to find $q_2$. 
5.5 **ABSTRACT VECTORS**

*This section corresponds to section 12.3 in Functions Modeling Change [4]*

In theoretical mathematics there is a notion of a vector space and mathematicians consider vectors to be things which fall in such a space. In this section we get closer to understanding such spaces and some of their applications, though we won’t use that terminology.

**Motivation**

To this point we have only defined two dimensional and three dimensional vectors, and for seemingly good reasons. Vectors include a “direction,” which needs to point somewhere. The space in which we live is three dimensional so it seems like the idea of direction shouldn’t need more than three dimensions. The idea of direction doesn’t always transfer, but higher dimensional vectors have their uses.

For example, the space in which we live isn’t actually three dimensional; it is four dimensional. There are three dimensions of space and one dimension of time. Using three dimensions you can describe any place in the universe. Adding a fourth dimension, however, allows us to describe any time and place in the universe.

We defined vectors by defining their magnitude and their direction. We then developed the notion of a unit vector decomposition by defining special vectors called \( \vec{i} \), \( \vec{j} \), and \( \vec{k} \). We can’t really define the direction of a four dimensional vector, but we can add a vector \( \vec{\ell} \) and define four dimensional vectors to be things which look like

\[
x \vec{i} + y \vec{j} + z \vec{k} + w \vec{\ell}.
\]

We think of \( \vec{\ell} \) as a vector of length 1 in the positive time direction and we can add, multiply, and dot vectors in the same way as we did before.

**Theory**

This is a very convenient notation for four dimensional vectors. Here we show the operations on them:

\[
(x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k} + w_1 \vec{\ell}) + (x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k} + w_2 \vec{\ell}) = (x_1 + x_2) \vec{i} + (y_1 + y_2) \vec{j} + (z_1 + z_2) \vec{k} + (w_1 + w_2) \vec{\ell}
\]

\[
c(x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k} + w_1 \vec{\ell}) = (cx_1) \vec{i} + (cy_1) \vec{j} + (cz_1) \vec{k} + (cw_1) \vec{\ell}
\]

\[
(x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k} + w_1 \vec{\ell}) \cdot (x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k} + w_2 \vec{\ell}) = x_1 x_2 + y_1 y_2 + z_1 z_2 + w_1 w_2
\]
Observe that everything happens on the components. To add two vectors we add their \( \vec{i} \)-components to find the \( \vec{i} \)-component of the sum. To multiply a vector by a constant we multiply each of its components by that constant. Dotting two vectors together is a matter of multiplying each pair of components together and then adding up those products.

We can easily define five dimensional vectors by defining a vector \( \vec{\mathbf{i}} \) and letting them be vectors of the form

\[
x\vec{i} + y\vec{j} + z\vec{k} + w\vec{l} + v\vec{m}.
\]

You can see how this gets unruly very quickly. If you ever wanted a 78 dimensional vector you would run out of letters.

Notice that we don’t really need the vectors \( \vec{i}, \vec{j}, \) and etc. for anything; they are just placeholders. There is an alternate notation which can be more convenient at times. We can write vectors as follows:

\[
x\vec{i} + y\vec{j} = (x, y) \\
x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z) \\
x\vec{i} + y\vec{j} + z\vec{k} + w\vec{l} = (x, y, z, w)
\]

The danger in this is that it makes vectors look like points which is misleading. This notation can be easier when there is no confusion. With the new notation we perform all operations on the components. For example,

\[
(x_1, y_1, z_1, w_1) + (x_2, y_2, z_2, w_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2)
\]

One of the advantages of writing vectors in this manner is that they can be generalized easily. You can imagine how to write a 78 dimensional vector if you ever needed to write such an awful thing. This notation is summarized in Definition 5.5.1. Unfortunately it is not really possible two write down a vector whose length is unknown so we write \( \vec{v} = (v_1, v_2, \ldots, v_n) \) to mean a vector with \( n \) dimensions. What this means is that there are \( n \) terms in the list. For example,

\[
\vec{v} = (v_1, v_2) \quad \text{when } n = 2, \\
\vec{v} = (v_1, v_2, v_3) \quad \text{when } n = 3, \\
\vec{v} = (v_1, v_2, v_3, v_4) \quad \text{when } n = 4, \\
\]

and so on.
(5.5.1) **Definition.** Let \( n \) be any positive integer. An \( n \)-dimensional vector \( \vec{v} \) is written as

\[ \vec{v} = (v_1, v_2, \ldots, v_n). \]

For each integer \( i \) with \( 1 \leq i \leq n \), \( v_i \) is a real number called the \( i \)-**th component** of \( \vec{v} \). These vectors are added by adding their components. A vector is multiplied by a constant by multiplying each of its components by that constant. The dot product of two \( n \)-dimensional vectors is computed according to

\[ (v_1, v_2, \ldots, v_n) \cdot (w_1, w_2, \ldots, w_n) = v_1w_1 + v_2w_2 + \ldots + v_nw_n. \]

The magnitude of an \( n \)-dimensional vector, \( \vec{v} \) is computed according to

\[ ||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}. \]

It is straightforward to work with vectors in this form using the rules outlined in **Definition 5.5.1**. You will likely find that computations are easier in this form once you get used to them.

(5.5.2) **Example.** Let \( \vec{v} = (3, 2, -5, 1) \) and \( \vec{w} = (7, 0, 1, -3) \).

a. Find \( \vec{v} + \vec{w} \).

b. Find \( -3\vec{v} \).

c. Find \( \vec{v} \cdot \vec{w} \).

d. Find \( ||\vec{v}|| \).

**Solution:**

a. We have

\[ \vec{v} + \vec{w} = (3, 2, -5, 1) + (7, 0, 1, -3) \]
\[ = (3 + 7, 2 + 0, -5 + 1, 1 - 3) \]
\[ = (10, 2, -4, -2). \]

**Answer:** \( \vec{v} + \vec{w} = (10, 2, -4, -2) \)

b. We have

\[ -3\vec{v} = -3(3, 2, -5, 1) = (-9, -6, 15, -3). \]
Answer: \(-3\vec{v} = (-9, -6, 15, -3)\)

c. We have
\[
\vec{v} \cdot \vec{w} = (3, 2, -5, 1) \cdot (7, 0, 1, -3)
= (3)(7) + (2)(0) + (-5)(1) + (1)(-3)
= 13.
\]

Answer: \(\vec{v} \cdot \vec{w} = 13\)

d. We have
\[
\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{3^2 + 2^2 + (-5)^2 + (-3)^2} = \sqrt{39}.
\]

Answer: \(\|\vec{v}\| = \sqrt{39}\)

Definition 5.5.1 defines the notion of magnitude of a vector but that is hard to correlate to “length.” If you had the ability to draw and measure a line in four dimensional space then magnitude would correspond to length. All of the properties that we learned about vectors hold true for \(n\)-dimensional vectors except for Theorem 5.3.7 which tells us that
\[
\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos(\theta)
\]
where \(\theta\) is the angle between \(\vec{u}\) and \(\vec{v}\). The only reason this doesn’t hold is because we don’t have a notion of “angle between two vectors” in larger dimensions. Since we can’t draw two vectors it’s hard to see angles between them. Some mathematicians, though, define angles in higher dimensions such that the angle, \(\theta\), between vectors \(\vec{v}\) and \(\vec{u}\) is
\[
\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right).
\]

We won’t deal with those sorts of things here, though.

Applications

One of the most accessible applications of vectors is to use them as lists of data. Imagine a boy named Timmy who is running a lemonade stand. Timmy sells four beverages: lemonade, iced tea, root beer, and water. The chart below shows Timmy’s economics. That is, it shows what it costs Timmy to buy one of each of these items (remember, this is called cost) and it shows how much Timmy charges his customers for each of them (remember, this is called revenue).
We can list these two pieces of data in a vector. That is, define vectors $\vec{C}$ and $\vec{R}$ such that

\[
\vec{C} = (0.06, 0.04, 0.45, 0.00)
\]

\[
\vec{R} = (0.50, 0.50, 1.00, 0.25).
\]

Recall from Section 2.1 that profit is the revenue minus the cost. If I want to know the profit that Timmy makes from the sale of one lemonade I simply subtract the cost from the revenue. Timmy makes a profit of $0.50 - 0.06 = 0.44$. We can do the same thing with each of the other beverages, but that corresponds to subtracting the components of the vectors $\vec{R}$ and $\vec{C}$ which, of course, corresponds to the difference of the vectors.

The table shows the profit from each beverage. We can also calculate

\[
\vec{P} = \vec{R} - \vec{C} = (0.44, 0.46, 0.55, 0.25).
\]

The vectors $\vec{R}$ and $\vec{C}$ expressed the revenue of the four beverages and the vector $\vec{P}$ expresses the profit of each of the four beverages. We were able to find the profit by the vector operation $\vec{R} - \vec{C}$ because profit is calculated at each component and that's what vector subtraction does.

At the end of the day Timmy is tallying his sales. The table below shows the number of each beverage that he purchased at the beginning of the day and the number of each beverage that he sold by the end of the day.

We can write these as vectors, as well. Let $\vec{P}$ be a vector containing the purchase information and $\vec{S}$ be a vector containing the sales information so that

\[
\vec{P} = (30, 30, 30, 30)
\]

\[
\vec{S} = (28, 19, 24, 12)
\]

Observe that in each of the vectors here the 1st component holds information about lemonade, the 2nd component holds information about
iced tea, and so on. Let $\vec{L}$ be a vector which stores the number of each beverage that Timmy had leftover at the end of the day. The way that we calculate the amount of lemonade that was leftover is to subtract the number that he sold from the number that he purchased, which is $30 - 28 = 2$. Of course, we can do the same on each component, so

$$\vec{L} = \vec{P} - \vec{S} = (2, 11, 6, 18).$$

What if Timmy wants to know his total revenue for the day? His revenue from selling lemonade is the number of glasses of lemonade he sold times the price of each glass, or $28 \cdot 0.50 = 14$. That is, he made $14 in revenue from selling lemonade. We can calculate the revenue from each of the other products in the same way. If $r$ is the total revenue then we have

$$r = (28)(0.50) + (19)(0.50) + (24)(1) + (12)(0.25) = 50.5$$

so that his total revenue is $50.50. Look at this last calculation, though. We took the vectors $\vec{S}$ and $\vec{R}$, multiplied pairs of their components together, and then added them up, which is precisely the dot product. Hence $r = \vec{S} \cdot \vec{R}$. Using a similar method, try to convince yourself that $\vec{P} \cdot \vec{C}$ is Timmy’s total costs.

Would it make a difference if the vectors were longer? Suppose that Timmy’s Lemonade Stand was actually a store selling 100 different flavors of ice cream. Suppose $\vec{R}$ and $\vec{C}$ were still the revenue and cost vectors and if $\vec{P}$ and $\vec{S}$ are the vectors containing purchase and sales information for a day. These vectors are now 100-dimensional vectors, but it is still the case that $\vec{R} - \vec{C}$ contains the profit of each item and $\vec{S} \cdot \vec{R}$ is still the total revenue.

The key to this process is that in each of the vectors involved the $i$-th component of the vector always corresponds to the same flavor of ice cream (or type of beverage, or whatever the case may be). This process of using vector operations to perform these computations may seem like overkill, but when computers are using large lists of things it is convenient to store them as vectors. The remainder of this section is dedicated to more examples of using vectors in this manner.

(5.5.3) Example. A small car dealership uses vectors to store its sales information. In each of the vectors, the $i$-th vector corresponds to information about a particular model of car. This car dealership sells exactly five models of cars so each of these vectors is 5-dimensional. You are given several vectors of information about a particular month:
• The inventory vector, \( \vec{P} \), stores the number of each car that the dealership purchased at the beginning of the month.
\[
\vec{P} = (30, 30, 20, 20, 10)
\]

• The sales vector, \( \vec{S} \), stores the number of each car that the dealership sold during the month.
\[
\vec{S} = (28, 29, 17, 19, 8)
\]

• The revenue vector, \( \vec{R} \), stores the revenue that the dealership makes from selling one of each model of car. All of the units are in dollars.
\[
\vec{R} = (21000, 27000, 39000, 41000, 90000)
\]

• The cost vector, \( \vec{C} \), stores the dealership’s costs from buying one of each model of car. All of the units are in dollars.
\[
\vec{C} = (11000, 13000, 19000, 25000, 37000)
\]

Use these vectors to perform the following computations:

A. Find a vector, \( \vec{L} \), which describes the number of each type of car left at the end of the month.

B. Find a vector, \( \vec{P} \), which describes the profit from selling each type of car.

C. Find the dealership’s total revenue, total cost, and total profit at the end of the month.

Solution:
This example is intentionally designed to work a lot like the example of Timmy’s lemonade stand.

A. The way that we compute the number of cars that are left over at the end of the month for each particular type of car is to subtract the number sold from the number purchased at the beginning of the month. Since this is done the same way at each component, we have
\[
\vec{L} = \vec{P} - \vec{S} = (2, 1, 3, 1, 2).
\]

[Answer:] \( \vec{L} = (2, 1, 3, 1, 2) \).
b. To compute the profit from selling a particular car we subtract that car’s cost from its revenue. Since this is done the same way at each component, we have

\[ \vec{P} = \vec{R} - \vec{C} = (10000, 14000, 20000, 16000, 53000). \]

**Answer:** \( \vec{P} = (10000, 14000, 20000, 16000, 53000) \)

c. The revenue is computed for each car by multiplying the number sold by the revenue from selling one car. To compute the total revenue we add all of those computations. As vectors, this is the dot product, \( \vec{S} \cdot \vec{R} \).

\[ \vec{S} \cdot \vec{R} = 3,533,000 \]

Hence the total revenue is \$3,533,000. Cost is computed by multiplying the number of cars purchased by the cost of purchasing one car at each component and then adding those. As vectors, total cost is \( \vec{P} \cdot \vec{C} \).

\[ \vec{P} \cdot \vec{C} = 1,970,000 \]

Hence the total cost is \$1,970,000.

To calculate total profit we can simply subtract the total cost from the total revenue.

\[ 3533000 - 1970000 = 1563000 \]

Hence the total profit is \$1,563,000.

**Answer:** The total revenue is \$2,084,000, the total cost is \$1,970,000, and the total profit is \$1,291,000.

(5.5.4) **Example.** A researcher is studying the populations of six states in 1995 and 2005. The vector \( \vec{P} \) stores the population of these six states in 1995 and

\[ \vec{P} = (3.28, 1.24, 6.07, 1.15, 0.99, 0.59) \]

where each component is in millions of people. The vector \( \vec{Q} \) stores the population of these six states in 2005 and

\[ \vec{Q} = (3.51, 1.32, 6.40, 1.31, 1.08, 0.62) \]

where each component is in millions of people.
A. Find a vector, \( \vec{R} \), which describes the amount (in millions of people) by which the population of each state increased between 1995 and 2005.

B. The researcher predicts that in 2015 these six states will have populations which are 8% higher than their population in 2005. Find a vector, \( \vec{S} \), which describes this prediction.

C. A different researcher predicts that the population of these six states in 2015 will be described by \( \vec{S} = \vec{Q} + 2\vec{R} \). Describe what this prediction means conceptually.

**Solution:**

A. To find the population change of a single state we subtract the 1995 population from the 2005 population. By doing this at each component we have \( \vec{R} = \vec{Q} - \vec{P} \).

\[
\vec{R} = \vec{Q} - \vec{P} = (0.23, 0.08, 0.33, 0.16, 0.09, 0.03)
\]

**Answer:** \( \vec{R} = (0.23, 0.08, 0.33, 0.16, 0.09, 0.03) \)

B. To increase a number by 8% we multiply it by 1.08. To increase each of the populations in the vector \( \vec{Q} \) (which holds the population information for 2005) by 8% we multiply \( \vec{Q} \) by 1.08.

\[
\vec{S} = 1.08\vec{Q} = (3.79, 1.43, 6.91, 1.41, 1.17, 0.67)
\]

**Answer:** \( \vec{S} = (3.79, 1.43, 6.91, 1.41, 1.17, 0.67) \)

C. The vector \( \vec{R} \) stores the change from 1995 to 2005 and the vector \( \vec{Q} \) stores the population in 2005. Adding 2\( \vec{R} \) to \( \vec{Q} \) is adding twice the change between 1995 and 2005 to the population in 2005.

**Answer:** The researcher predicts that between 2005 and 2015 the population will increase by twice as much as it increased between 1995 and 2005.

Example 5.5.3 and Example 5.5.4 give concrete vectors with given values. However, we can perform the same calculations in theory even if we don’t know the values of the vectors. Even if the vectors \( \vec{R} \) and \( \vec{C} \) were not given in Timmy’s Lemonade Stand we would still know that the profit vector is \( \vec{P} = \vec{R} - \vec{C} \).
Example. A class has 36 students. This class uses 36-dimensional vectors to store information. The class had four tests throughout the term and the students scores are stored in vectors $\vec{T}_1, \vec{T}_2, \vec{T}_3,$ and $\vec{T}_4.$ That is, the vector $\vec{T}_1$ stores each students score (out of 100) on the first test. Find a vector $\vec{A}$ which stores each students’ average test score.

Solution: Imagine a single student with four test scores. To find the average of those four test scores we would add them up and divide them by four. To do this on each component of the four test vectors we can add the four vectors and then multiply that sum by $\frac{1}{4}.$ Hence

$$\vec{A} = \frac{1}{4}(\vec{T}_1 + \vec{T}_2 + \vec{T}_3 + \vec{T}_4).$$

Answer: $\vec{A} = \frac{1}{4}(\vec{T}_1 + \vec{T}_2 + \vec{T}_3 + \vec{T}_4)$

Example. A truck driver has been tracking his fuel purchases on his trips this year. He has made 73 stops for gas during the year and he uses 73-dimensional vectors to store data about these purchases. The vector $\vec{G}$ stores the number of gallons of gas he bought at each stop and the vector $\vec{P}$ stores the price per gallon of gas at each stop. Find an expression which calculates the total amount of money that he spent on fuel this year.

Solution: To calculate the amount he spent at each stop we multiply the number of gallons he bought by the price per gallon. To figure out the total amount of money he spent over the course of all of those stops we add up the amount he spent at each stop. This is the dot product of $\vec{G}$ and $\vec{P}.$

Answer: $\vec{G} \cdot \vec{P}$

Example. A wholesale mattress supply store sells to retailers. They offer 30 different mattress and they keep their sales information in 30-dimensional vectors. The normal price of each mattress is stored in the vector $\vec{P}.$
A. A 5% discount is given to local retailers. Find a vector which stores the prices for local retailers.

B. A local retailer is looking to purchase several different types of mattresses. The vector $\vec{Q}$ stores how many of each type of mattress they wish to purchase. Find an expression which calculates the total amount that this local retailer will be spending on these mattresses.

**Solution:**

A. To reduce a number by 5% we multiply it by 0.95. To reduce each of the components of $\vec{P}$ by 5% we multiply the whole vector by 0.95. Hence the price vector for local retailers is $0.95\vec{P}$.

**Answer:** $0.95\vec{P}$

B. Each component of these vectors represent one type of mattress. To figure out what the retailer spends on each type of mattress we multiply the price of that mattress by the number of that type of mattress that they buy. We then add up all of those values for all of the mattress types to find the total amount that the retailer needs to pay. The vector $0.95\vec{P}$ stores the prices and the vector $\vec{Q}$ stores the number of each type that they want to buy. It follows that we calculate the total cost of the purchase by

$$(0.95\vec{P}) \cdot \vec{Q} = 0.95(\vec{P} \cdot \vec{Q}).$$

**Answer:** $0.95(\vec{P} \cdot \vec{Q})$
5.5.1. Let \( \vec{v} \) and \( \vec{w} \) be 5-dimensional vectors where
\[
\vec{v} = (3, 0, -5, 2, 11) \quad \text{and} \quad \vec{w} = (4, 19, 9, -1, 0).
\]
\begin{enumerate}
\item Find \( 3\vec{w} - 2\vec{v} \).
\item Find \( \|\vec{v}\| \). Leave your answer in exact form.
\item Find \( \vec{v} \cdot \vec{w} \).
\end{enumerate}

5.5.2. A small rental car company has five types of cars that they rent to customers. They use 5-dimensional vectors to hold information about these cars. A vector \( \vec{M} \) holds the gas mileage of each of these cars in miles per gallon and
\[
\vec{M} = (23, 25, 29, 19, 26).
\]
The vector \( \vec{G} \) holds the number of gallons of gas used by each type of car in the past month where
\[
\vec{G} = (900, 1150, 2640, 580, 1500).
\]
\begin{enumerate}
\item A visitor from Europe wants to know the fuel mileage in kilometers per liter. To convert a mileage measurement in miles per gallon to kilometers per liter, simply multiply by 0.425. That is
\[
(\text{miles per gallon}) \cdot 0.425 = (\text{kilometers per liter}).
\]
Find a vector \( \vec{D} \) which holds the gas mileage of each of the cars in kilometers per liter.
\item Find the total number of miles driven by all of the company’s cars in the last month.
\item The company has put an additive in their fuel which should increase the fuel mileage of each car by 4%. Find a vector \( \vec{N} \) which holds the predicted gas mileage of each of these cars in miles per gallon.
\end{enumerate}

5.5.3. A baseball team uses vectors to store its team’s stats over the last five seasons where each component of the vectors correspond to one of these five seasons. The vector \( \vec{G} \) holds the number of games played in each season, the vector \( \vec{W} \) holds the number of wins in each season, and the vector \( \vec{R} \) holds the number of runs scored in each season (by the entire team).
\begin{enumerate}
\item Find a vector \( \vec{L} \) which holds the number of losses in each season.
\item If there are 18 hitters on the roster (assuming that the hitters are the only players which can score runs), find a vector which describes the average number of runs scored by a hitter in each season.
\end{enumerate}

5.5.4. An airline uses vectors to store information about the fuel levels in its airplanes. These vectors have 382 components; each of which corresponds to a particular aircraft. The vector \( \vec{M} \) stores the maximum amount of fuel (in gallons) that each airplane can hold in its tank. The vector \( \vec{f} \) stores the current amount of fuel (in gallons) that each airplane has in its tank. The vector \( \vec{C} \) holds the cost per gallon of the fuel that each aircraft uses.
\begin{enumerate}
\item Find a vector which describes the amount of fuel needed to refill the tanks of the airplanes.
\item How much would it cost the airline to fill the tanks of all of the airplanes (from their current fuel level)?
\item If the tank of every single airplane were empty, how much would it cost the airline to refuel all of the airplanes?
\end{enumerate}
5.5. A grocery store uses vectors to store data about the products that they carry. These vectors have 1047 components; each of which corresponds to a unique product. Some of the vectors that they use are as follows:

- $\vec{I}$ stores the number of each product purchased in a particular week.
- $\vec{S}$ stores the number of each product sold in a particular week.
- $\vec{P}$ stores the price at which the store sells each item.
- $\vec{W}$ stores the price for which the store purchases each item from their supplier.

Because the food spoils, the store must dispose of all of its leftover inventory at the end of each week.

A.) Find a vector which describes the amount above wholesale that the store charges for each item.

B.) Find a vector for the number of each product that is leftover at the end of each week.

C.) Find the amount of money that the store spends on inventory that is wasted.

D.) What is the store’s costs for the week? Assume that the only costs incurred are from purchasing their inventory.

E.) What is the most amount of revenue that the store could possibly make in a week? That is, what is their revenue if they sell their entire inventory?

F.) What is the maximum profit that the store could possibly make in a week?

G.) What is the actual revenue that the store makes in a week?

H.) What is the actual profit that the store makes in a week?

---

**Challenge Problems**

(5.5.1) **Challenge Problem.** Consider the collection of all polynomials whose degree is at most 2. Such things look like

$$f(x) = ax^2 + bx + c$$

where $a$, $b$, and $c$ can be any real number (including zero). Explain how these polynomials act like three dimensional vectors (except for the notion of magnitude) by answering the following questions.

A. How do two of these polynomials add? Relate that to addition of vectors.

B. How do we multiply these polynomials by a constant? Relate that to addition of vectors.

C. What is the polynomial equivalent to the zero vector? That is, what is the polynomial which acts like zero should act? Hint: What else could it possibly be?

D. What are the polynomial equivalents to $\vec{i}$, $\vec{j}$, and $\vec{k}$? There are a lot of correct answers to this question, but there is one “good” choice.
Mathematicians have an abstract notion of a **vector space** which is essentially any sort of collection of things which act like vectors. Polynomials are one example of a vector space. In general, mathematicians do not consider it a requirement that a vector space should have a notion of magnitude. In fact, there are several different ways to add an additional pieces of structure to a vector space in order to yield new and useful things. The set of vectors that we know are a vector space plus the notion of magnitude which mathematicians call a **normed vector space**. The set of degree three polynomials is a vector space plus the notion of multiplication which mathematicians call an **algebra**.
Practice Exercise 2.1.a:

(i) \((f - g)(x) = \frac{2}{x^2 + 4x + 3}\)

(ii) \((f/h)(x) = x - 1\)

Practice Exercise 2.1.b:

(i) \(A(t) = 21t\)

(ii) 27 min

(iii) 21 ft in length by 27 ft in width

Practice Exercise 2.2.a:

(i) \((g \circ f)(x) = 9x^2 - 12x + 5\)

(ii) \((h \circ g)(x) = \frac{x^2 + 2}{x^2}\)

Practice Exercise 2.2.b:

\[S(p) = \frac{1175p - 2890}{p - 2}\]

Practice Exercise 2.3.a:

\(f^{-1}(y) = \sqrt{y + 8}\)

Practice Exercise 2.3.b:

Yes

Practice Exercise 2.3.c:

\(F(k) = 0.0169492k - 4.32835\)

Practice Exercise 2.4.a:
(i) $Q(5) = 77.88$
(ii) $Q(-50) = 36.79$
(iii) $Q(25) = 77.88$
(iv) There are no values of $t$ such that $Q(t) = 10$.

Practice Exercise 2.4.b:
The period is 4, the amplitude is 3, and the midline is $y = 1$.

Practice Exercise 3.1.a:
$10 \text{ in}^2$

Practice Exercise 3.1.b:
$(0.93, -0.36)$ and $(-0.53, -0.84)$

Practice Exercise 3.2.a:
$\sin(\theta) = 0.94$

Practice Exercise 3.2.b:
$x = 3.18$

Practice Exercise 3.2.c:
$590.88 \text{ ft}$

Practice Exercise 3.3.a:
(i) $\sin(315^\circ) = -\frac{\sqrt{2}}{2}$ and $\cos(315^\circ) = \frac{\sqrt{2}}{2}$
(ii) $\sin(120^\circ) = \frac{\sqrt{3}}{2}$ and $\cos(120^\circ) = -\frac{1}{2}$
(iii) $\sin(240^\circ) = -\frac{\sqrt{3}}{2}$ and $\cos(240^\circ) = -\frac{1}{2}$

Practice Exercise 3.3.b:
(i) $\sin(-135^\circ) = -\frac{\sqrt{2}}{2}$ and $\cos(-135^\circ) = -\frac{\sqrt{2}}{2}$
(ii) $\sin(930^\circ) = -\frac{1}{2}$ and $\cos(930^\circ) = -\frac{\sqrt{3}}{2}$

Practice Exercise 3.4.a:
$(-5.93, 4.30)$

Practice Exercise 3.4.b:
$f(\theta) = 14 \sin(\theta) + 40$

Practice Exercise 3.5.a:
(i) \( \tan(45^\circ) = 1 \)
(ii) \( \tan(120^\circ) = -\sqrt{3} \)
(iii) \( \tan(900^\circ) = 0 \)
(iv) \( \tan(-570^\circ) = -\frac{\sqrt{3}}{3} \)
(v) \( \tan(342^\circ) = -0.32 \)

**Practice Exercise 3.5.b:**
\[ x = \frac{50}{\sqrt{3}} \]

**Practice Exercise 3.6.a:**
\[ \theta = 30.96^\circ \]

**Practice Exercise 3.6.b:**
\[ \theta = 30.96^\circ \]

**Practice Exercise 4.1.a:**
\[ x = 4 \text{ and } \theta = 0.64 \]

**Practice Exercise 4.1.b:**
3997.47 mi  
*Note: This is surprisingly close (the average radius is around 3960 mi) and actually provides a practical way to approximate the radius of the earth if you have the right tools and enough patience.*

**Practice Exercise 4.2.a:**
\[ \ell = 17.81 \]

**Practice Exercise 4.2.b:**
\[ \theta = 0.65 \]

**Practice Exercise 4.2.c:**
\[ 4764.38 \text{ m} \]

**Practice Exercise 4.3.a:**
\[ f(x) = \frac{5}{2} \sin\left(\frac{2\pi}{3}(x - \frac{1}{4})\right) - \frac{7}{2} \]

**Practice Exercise 4.3.b:**
\[ H(t) = 100 \sin\left(\frac{\pi}{2}(t - \frac{1}{2})\right) + 125 \]

**Practice Exercise 4.4.a:**
\[ \theta = 0.36 + 2\pi n \text{ and } \theta = 2.78 + 2\pi n \text{ for integers, } n. \]

**Practice Exercise 4.4.b:**
\[ t = -0.0106 + \frac{n}{60} \text{ and } t = -0.006 = \frac{n}{60} \text{ for integers, } n. \]
Practice Exercise 4.5.a:
\[
\sin(37.5^\circ) = \frac{\sqrt{2 - \sqrt{2 - \sqrt{3}}}}{2}
\]

Practice Exercise 5.1.a:
(i) \(|\vec{u} + \vec{v}| = 2\sqrt{7}\) with \(\vec{u} + \vec{v}\) shown below
(ii) \(|\vec{u} - \vec{v}| = 2\sqrt{3}\) with \(\vec{u} - \vec{v}\) shown below

Practice Exercise 5.1.b:
If the resultant force is \(\vec{F}\) then \(|\vec{F}| = 50.99\) N and \(\vec{F}\) is shown below:

Practice Exercise 5.2.a:
(i) \(-4\vec{u} = -8\vec{i} + 12\vec{j}\)
(ii) \(|\vec{v}| = 7.62\)
(iii) \(3\vec{u} - 2\vec{v} = 12\vec{i} + 5\vec{j}\)

Practice Exercise 5.2.b:
\(\vec{u} = -2\sqrt{2}\vec{i} + 2\sqrt{2}\vec{j}\) and \(\vec{v} = -7\vec{i} - 3\vec{j}\)

Practice Exercise 5.2.c:
854.03 N

Practice Exercise 5.3.a:
(i) \(\vec{u} \cdot \vec{v} = 32\)
(ii) \(\theta = 0.6\)
Practice Exercise 5.3.b:

(i) $W_1 = -300 \text{ N}$

(ii) $W_2 = 169.71 \text{ N}$

(iii) $W_3 = 311.77 \text{ N}$
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