

GENERALIZED LAURENT POLYNOMIAL RINGS AS QUANTUM PROJECTIVE 3-SPACES

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ABSTRACT. Given a ring R , we introduce the notion of a generalized Laurent polynomial ring over R . This class includes the generalized Weyl algebras. We show that these rings inherit many properties from the ground ring R . This construction is then used to create two new families of quadratic global dimension four Artin-Schelter regular algebras. We show that in most cases the second family has a finite point scheme and a defining automorphism of finite order. Nonetheless, a generic algebra in this family is not finite over its center.

1. INTRODUCTION

Let K be an algebraically closed field of characteristic different than 2. The primary goal of this paper is to construct two new families of quadratic Artin-Schelter regular K -algebras of global dimension 4. We recall that an \mathbb{N} -graded K -algebra A is Artin-Schelter regular if it has finite GK-dimension, finite global dimension d and satisfies the Gorenstein condition

$$\mathrm{Ext}^k(T, A) = \delta_{k,d}T,$$

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where T is the trivial module A/A_+ .

The classification of such algebras in various global dimensions has been a driving problem in noncommutative algebraic geometry for years. Classification in dimension 2 is quite easy and classification in dimension 3 is contained in the celebrated papers of Artin, Schelter, Tate and Van den Bergh ([3], [1], [2]). Lu, Palmieri, Wu and Zhang ([9]) have recently classified some Noetherian, 2-generated, global dimension 4 AS-regular algebras (having one relation of degree 3 and one of degree 4).

It is well known that an Artin-Schelter regular algebra of global dimension 4 on four generators must be a quadratic and Koszul algebra of Hilbert series $H_A(t) = (1 - t)^{-4}$ (cf. [11]), very much like a polynomial ring in four variables. For this reason we refer to such an algebra as a quantum \mathbb{P}^3 . The classification of such algebras is very far from complete. The purpose of this paper is to provide two new families of such algebras, examples which do not appear to be similar to any of the examples already known (for example graded Ore extensions, normal extensions of 3-dimensional AS-regular algebras [6], regular Clifford algebras and their “noncommutative” analogs [11], Sklyanin algebras [14] and others).

We can state our main results at once. Let x, y, d and u be indeterminates. Let $F = K\langle x, y \rangle$ and $F' = K\langle x, y, d, u \rangle$ be free algebras, graded with generators in degree 1.

Theorem 1.1. *Let $q = ax^2 + b(xy + yx) + cy^2$ be a symmetric 2-tensor in F and let σ be any graded automorphism of F . Let $B' = F'/I$ be the graded algebra where I is the homogeneous ideal of F' generated by the elements:*

$$\begin{array}{lll} xd - d\sigma(x), & yd - d\sigma(y), & ux - \sigma(x)u \\ uy - \sigma(y)u, & du - q, & ud - \sigma(q). \end{array}$$

Then the following are equivalent:

- (1) B' is a quantum \mathbb{P}^3 .
- (2) B' is Koszul.
- (3) q is nonsingular (i.e. q is not factorable in F).

Theorem 1.2. Fix $r, \alpha \in K^\times$. Let $C(r, \alpha)$ be the algebra F'/J where J is the homogeneous ideal generated by

$$\begin{aligned} &xd - dy, & yd - \alpha dx, & ux - ryu \\ &uy - r^{-1}\alpha xu, & du - (xy - r yx) & ud + r\alpha(xy - r^{-1}yx). \end{aligned}$$

Then $C(r, \alpha)$ is a quantum \mathbb{P}^3 and is Koszul.

These theorems are proved, in part, by realizing the algebras in question via a different construction, which we call a *generalized Laurent polynomial ring*. The construction is quite simple and is closely related to the notion of down-up algebras ([5]) and generalized Weyl algebras ([4]). This construction and its basic properties are discussed in section 2. Theorems 1.1 and 1.2 are then proved in sections 3 and 4 respectively.

We also include some information about the basic noncommutative algebraic geometry of the algebras in theorems 1.1 and 1.2.

Associated to any quantum \mathbb{P}^3 , say $A = F'/I$, there is a standard family of graded modules known as point modules. These are the graded, cyclic modules of Hilbert series $1/(1 - t)$. It is well known that this family of modules is parameterized by a scheme called the point scheme. The point scheme is easily described as the scheme of zeroes in $\mathbb{P}^3 \times \mathbb{P}^3$ of the quadratic component I_2 of the defining ideal of A . We denote this scheme as $\Gamma = \Gamma(A)$. The point schemes of the regular algebras B' are always at least one dimensional. The point schemes of the algebras $C(r, \alpha)$, on the other hand, are generically zero-dimensional.

It is also well known ([12]) that Γ is the graph of an automorphism. It was long thought that a quantum \mathbb{P}^3 was finite over its center if and only if the associated automorphism defining the point scheme had finite order, as is the case for Artin-Schelter regular algebras of global dimension 3 ([2]). Recently Vancliff and Stephenson [15] have found quantum \mathbb{P}^3 's with finite automorphisms which are not finite over their centers. For certain values of the parameters r and α the algebras $C(r, \alpha)$ also exhibit this phenomenon, that is, the automorphism defining $\Gamma(C(r, \alpha))$ has finite order while $C(r, \alpha)$ is not a finite module over its center. This information is detailed in section 4.

2. GENERALIZED LAURENT POLYNOMIAL RINGS

Let R be a ring with identity. Fix $\sigma \in \text{Aut}R$ and fix a normal regular element $q \in R$. Let $q^\sigma = \sigma(q)$. The element q defines an automorphism $\tau \in \text{Aut}(R)$ via the formula $rq = q\tau(r)$ for all $r \in R$. Let d be an indeterminate and consider the (right) skew Laurent polynomial ring $R[d, d^{-1}; \sigma]$ (cf. [10]). To be precise, this is the ring of Laurent polynomials in d with right R -coefficients and the multiplication rule $rd = d\sigma(r)$ for $r \in R$.

Definition 2.1. The generalized Laurent polynomial ring associated to R , σ and q is the subring $R[d, d^{-1}q]$ of $R[d, d^{-1}; \sigma]$, generated by R , d and $d^{-1}q = q^\sigma d^{-1}$. We will denote $d^{-1}q$ as u and write this ring as $R[d, u; \sigma, q]$.

The basic properties of $R[d, u; \sigma, q]$ that we will need are all straightforward.

Proposition 2.2. *Let R , σ , q be as above and let $S = R[d, u; \sigma, q]$.*

(1) *S is a free left and right R -module on the basis $\{d^i, i \geq 0\} \cup \{u^j, j > 0\}$.*

(2) *If R is a domain then S is a domain.*

(3) *If R is Noetherian then S is Noetherian.*

(4) *Suppose $f : R \rightarrow H$ is a ring homomorphism and d', u' are in H such that: $f(r)d' = d'f(\sigma(r))$, $u'f(r) = f(\sigma(\tau^{-1}(r)))u'$ for all $r \in R$, $d'u' = f(q)$ and $u'd' = f(q^\sigma)$. Then f extends uniquely to a ring homomorphism $\bar{f} : S \rightarrow H$ with $\bar{f}(d) = d'$ and $\bar{f}(u) = u'$.*

(5) *If the automorphisms σ and τ commute and the left global dimension of R is n then the left global dimension of S is either $n + 1$ or infinite.*

Proof. The first statement is obvious from the assumption that q is normal and regular. The second follows immediately. Let T be the subring of S generated by R and d . Then T is the right Ore extension $R[d; \sigma]$. Moreover, S is generated as a ring by T and u , and $T + Tu = T + uT$. The third statement follows from this in a standard way, see for example [10], Theorem 1.2.10.

Statement (4) follows from the definition of S and (1), once we have checked some formulas. We need: $rd = d\sigma(r)$ and $ur = \sigma(\tau^{-1}(r))u$ for all $r \in R$. The first of those is just the definition and the second is $ur = d^{-1}qr = d^{-1}\tau^{-1}(r)q = \sigma(\tau^{-1}(r))u$. We also need $du = q$ and $ud = q^\sigma$, which are true by definition.

Finally, assume that the automorphisms σ and τ commute and let $\zeta = \sigma\tau^{-1} = \tau^{-1}\sigma$. By direct calculation, ζ extends to an automorphism of T with $\zeta(d) = d$. Also by calculation, there is a left ζ -derivation δ of T defined by $\delta(r) = 0$ for $r \in R$ and $\delta(d) = q^\sigma - q$. Let U be another indeterminate. Then the element $dU - q$ is normal and regular in the left Ore extension $T[U; \zeta, \delta]$ and we have $S \cong T[U; \zeta, \delta]/(dU - q)$. Statement (5) follows immediately from this observation. \square

When the element q is central in R (and τ is thus the identity), our definition of generalized Laurent polynomial ring coincides with the definition of a *generalized Weyl algebra* as given by Bavula in [4]. Notationally one just takes $d = X^-$ and $u = X^+$. Generalized Weyl algebras can always be realized as quotients of iterated Ore extensions as in part 5 of the above proof, and from this it follows that the generalized Weyl algebras inherit some properties of the ground ring. For example, if R is Artin-Schelter regular then graded generalized Weyl algebras over R are again AS-Gorenstein ([7] prop 3.2).

However the method in part 5 of Proposition 2.2 does not apply to S in general. If the automorphisms σ and τ do not commute then we cannot realize S as a quotient of an iterated Ore extension.

Generalized Weyl algebras include the Noetherian subclass of the family of K -algebras known as down-up algebras. Since we will utilize several down-up algebras later, we might as well formalize this notationally as follows. Let r, s and γ be scalars. In the algebra $K\langle x, y \rangle$, for any scalar λ , let $H_\lambda = xy - \lambda yx$. Then the down-up algebra associated to r, s and γ is

$$A(r, s, \gamma) := K\langle x, y \rangle / \langle xH_r - sH_r x - \gamma x, H_r y - syH_r - \gamma y \rangle.$$

This is not the usual presentation of a down-up algebra, as given in [5], but it coincides with that definition by taking $\alpha = (r+s)$ and $\beta = -rs$.

We note that the roles of r and s in the definition of the algebra are interchangeable, due to the formula: $xH_r - sH_r x = xH_s - rH_s x$ and similarly for y .

Remark 2.3. In our notation, the algebra $A(r, s, \gamma)$ is the generalized Laurent polynomial ring $R[d, u; \sigma, q]$ where $R = K[t_1, t_2]$, $q = t_2$ and σ is defined by $\sigma(t_1) = st_1 + \gamma$ and $\sigma(t_2) = rt_2 + t_1$. It is well known that for $rs \neq 0$ the algebras $A(r, s, 0)$ are Artin-Schelter regular of global dimension 3. This observation is essentially the starting point of our endeavor, as we will use generalized Laurent polynomial rings to form Artin-Schelter regular algebras of global dimension 4.

Finally, we note that S , as in Proposition 2.2, is a \mathbb{Z} -graded ring where $\deg(d) = -1$, $\deg(u) = 1$ and $\deg(r) = 0$ for $r \in R$. This will generally be referred to as “the” \mathbb{Z} -grading, to differentiate it from natural \mathbb{N} -gradings that our later examples will have.

3. A FAMILY OF QUANTUM 3-SPACES

Let K be an algebraically closed field not of characteristic 2. Let A be the \mathbb{N} -graded K -algebra

$$A = K\langle x, y \rangle / \langle x(xy + yx) - (xy + yx)x, y(xy + yx) - (xy + yx)y \rangle$$

where the indeterminates x and y have degree 1. This is the down-up algebra $A(-1, 1, 0)$. It has a wealth of quadratic central elements. The degree two component of the center is the span of the elements x^2 , y^2 and $xy + yx$ and we will therefore identify it with the space of symmetric 2-tensors.

The graded automorphism group of A is the same as that of the graded free algebra $K\langle x, y \rangle$, which we identify with $GL_2(K)$. We fix an automorphism σ of A and let Σ be the corresponding element of $GL_2(K)$, with the formal correspondence given by $\sigma(ax + by) = (a \ b)\Sigma \begin{pmatrix} x \\ y \end{pmatrix}$. We fix a quadratic central element q and let Q be the corresponding symmetric element of $M_2(K)$, i.e. $q = (x \ y)Q \begin{pmatrix} x \\ y \end{pmatrix}$. We write q^σ for $\sigma(q) = (x \ y)Q^\sigma \begin{pmatrix} x \\ y \end{pmatrix}$ where $Q^\sigma = \Sigma^t Q \Sigma$.

Definition 3.1. In the notation of the previous section we let $B = B(\sigma, q) = A[d, u; \sigma, q]$.

We note that the algebra B has both an \mathbb{N} -grading, where the generators x, y, d and u all have degree 1, as well as the \mathbb{Z} -grading of the previous section, where x and y have degree 0, d has degree -1 and u degree $+1$. We will refer to the first of these as simply the grading and the second as the \mathbb{Z} -grading. The first thing to note is that the Hilbert series of B , with respect to the grading, is easily computed from the fact that B is a free module over A with basis $d^i, i > 0$ and $u^j, j \geq 0$. Since the Hilbert series of A is $H_A(t) = ((1-t)^2(1-t^2))^{-1}$, we get $H_B(t) = H_A(t) \cdot ((1+t)/(1-t)) = 1/(1-t)^4$.

We wish to determine the conditions under which B is Artin-Schelter regular. From the previous paragraph we know that B has GK-dimension four. The challenge now is to show that B has finite global dimension.

Throughout our discussion of B , there is a marked difference between the cases Q singular and Q non-singular. Moreover, the singular case has two distinctive subcases as distinguished in the following lemma.

Lemma 3.2. *Let A and B be as above. Assume that either Q is non-singular or that Q and Q^σ are linearly independent. Then the algebra B is isomorphic to the free algebra $K\langle x, y, d, u \rangle$ factored by the ideal generated by the six 2-tensors:*

$$\begin{aligned} xd - d\sigma(x), \quad yd - d\sigma(y), \quad ux - \sigma(x)u, \quad uy - \sigma(y)u, \\ du - q, \quad ud - q^\sigma. \end{aligned}$$

Conversely, if we assume that Q and Q^σ are singular and linearly dependent, then the graded algebra B is not quadratic.

Proof. Let B' be $K\langle x, y, d, u \rangle$ factored by the six quadratic 2-forms above and let A' be the subalgebra of B' generated by x and y . The relations imply that q and q^σ centralize x and y and are thus in the center of A' .

Now assume that Q is nonsingular or that Q and Q^σ are linearly independent. Either of these hypotheses implies that every symmetric 2-tensor in A' is in the center of A' . Thus there exists an algebra

epimorphism $\pi : A \rightarrow A'$ with $\pi(x) = x$ and $\pi(y) = y$. By universality (Proposition 2.2 part 4), π extends to an algebra epimorphism $\pi : B \rightarrow B'$ with $\pi(d) = d$ and $\pi(u) = u$. But B is also generated by x, y, d and u and satisfies (at least) the six quadratic relations of B' , so π is invertible.

Conversely, assume Q is singular and linearly dependent with Q^σ . Then we may choose coordinates x and y so that $q = x^2$ and $q^\sigma = \lambda x^2$ for some $\lambda \in K^\times$. It is clear that the six quadratic relations given above span the quadratic relations of B . Since none of the monomials y^2, xy or yx appear in those six 2-tensors, it is not possible for the relation $xy^2 - y^2x = 0$ to be generated by them. Thus B cannot be quadratic. \square

Under the hypotheses of 3.2, it is very convenient to express the relations of the algebra B in the following shorthand:

$$du = q, \quad ud = q^\sigma, \quad \begin{pmatrix} x \\ y \end{pmatrix} d = d \Sigma \begin{pmatrix} x \\ y \end{pmatrix}, \quad u \begin{pmatrix} x \\ y \end{pmatrix} = \Sigma \begin{pmatrix} x \\ y \end{pmatrix} u.$$

We can also write these in transposed form, for example $(x \ y)d = d(x \ y)\Sigma^t$.

Remark 3.3. We need one final remark before getting to our main results. Let Ω be the matrix of 2-tensors $\Omega = \begin{pmatrix} x \\ y \end{pmatrix} (x \ y) = \begin{pmatrix} x^2 & xy \\ yx & y^2 \end{pmatrix}$. We record the following interesting formula for later use

$$Q\Omega Q = qQ + \det(Q) \begin{pmatrix} -y^2 & xy \\ yx & -x^2 \end{pmatrix}.$$

A similar formula holds for q^σ and Q^σ .

Definition 3.4. Let T be the trivial graded left B -module B/B_+ . If Q is nonsingular, let $\zeta = -\det(Q^\sigma)/\det(Q)$ and if Q is singular, let ζ be an arbitrary nonzero scalar. Let P^\bullet be the following graded augmented sequence of free left B modules with indicated grading shifts:

$$\begin{array}{ccccccc} & & P^4 & & P^3 & & P^2 \\ 0 & \longrightarrow & B(-4) & \xrightarrow{\delta_4} & B(-3)^4 & \xrightarrow{\delta_3} & B(-2)^6 \\ & & & & P^1 & & P^0 \\ & & \xrightarrow{\delta_2} & B(-1)^4 & \xrightarrow{\delta_1} & B & \xrightarrow{\epsilon} T \longrightarrow 0 \end{array}$$

Here ϵ is the usual augmentation map given by the action of B on T , and the other maps are right multiplication by matrices with entries from B_1 as described below in block forms:

$$\delta_4 = \begin{pmatrix} x & y & d & u \end{pmatrix} \quad \text{and} \quad \delta_1 = \begin{pmatrix} x & y & d & u \end{pmatrix}^t,$$

$$\delta_3 = \begin{pmatrix} u\Sigma^t Q & \zeta d(\Sigma^{-1})^t Q^\sigma & \zeta Q \begin{pmatrix} x \\ y \end{pmatrix} & Q^\sigma \begin{pmatrix} x \\ y \end{pmatrix} \\ 0 & 0 & -\zeta(x \ y)Q^\sigma & -\zeta u & 0 \\ -(x \ y)Q & 0 & 0 & 0 & -d \end{pmatrix}$$

and

$$\delta_2 = \begin{pmatrix} -d\Sigma & x & 0 \\ & y & 0 \\ -u\Sigma^{-1} & 0 & x \\ & 0 & y \\ (x \ y)Q & 0 & -d \\ (x \ y)Q^\sigma & -u & 0 \end{pmatrix}.$$

Proposition 3.5. *The augmented sequence $P^\bullet \rightarrow T \rightarrow 0$ is a complex.*

Proof. We simply need to observe that $\delta_{i+1}\delta_i = 0$ in B for $1 \leq i \leq 3$.

The block form of the matrices simplifies this calculation. We have

$$\delta_2\delta_1 = \begin{pmatrix} -d\Sigma \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} d \\ -u\Sigma^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} u \\ (x \ y)Q \begin{pmatrix} x \\ y \end{pmatrix} - du \\ (x \ y)Q^\sigma \begin{pmatrix} x \\ y \end{pmatrix} - ud \end{pmatrix}$$

which is 0 in B . Similarly $\delta_4\delta_3$ is 0 in B .

Let $H = \begin{pmatrix} -y^2 & xy \\ yx & -x^2 \end{pmatrix}$ and recall our formulas 3.3 for $Q\Omega Q$ and $Q^\sigma\Omega Q^\sigma$. By direct calculation, the upper left 2 by 2 submatrix of $\delta_3\delta_2$ is:

$$\begin{aligned} & -ud\Sigma^t Q\Sigma - \zeta du(\Sigma^{-1})^t Q^\sigma \Sigma^{-1} + \zeta Q \begin{pmatrix} x \\ y \end{pmatrix} (x \ y)Q + Q^\sigma \begin{pmatrix} x \\ y \end{pmatrix} (x \ y)Q^\sigma \\ & = -q^\sigma Q^\sigma - \zeta qQ + \zeta Q\Omega Q + Q^\sigma\Omega Q^\sigma \\ & = \det(Q^\sigma)H + \zeta \det(Q)H = 0. \end{aligned}$$

Calculating the rest of $\delta_3\delta_2$ now yields

$$\begin{pmatrix} 0 & 0 & Q^\sigma(u\Sigma^{-1} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} u) & \zeta Q(d\Sigma \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} d) \\ 0 & 0 & & \\ ((x \ y)\Sigma^t u - u(x \ y))\zeta Q & 0 & & -\zeta(q^\sigma - ud) \\ ((x \ y)d - d(x \ y)\Sigma^t)Q\Sigma & -q + du & & 0 \end{pmatrix}$$

which is 0 in B . □

Theorem 3.6. *The following are equivalent:*

- (1) *The algebra B is Artin-Schelter regular of global dimension 4,*
- (2) *The algebra B is Koszul,*
- (3) *The symmetric matrix Q is nonsingular.*

Proof. Based on the fact that the Hilbert series of B is $1/(1-t)^4$, the proof that (1) implies (2) is standard and can be found, for example, in [11].

To prove (2) implies (3), let us assume that B is Koszul, but that Q is singular. Koszul algebras must be quadratic, so by 3.2, we must have Q and Q^σ linearly independent. Based on this, we may change variables in the algebra A so that $q = x^2$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $q^\sigma = y^2$, $Q^\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$ for some $\alpha, \beta \in K$ with $\alpha \neq 0$.

We consider the complex $P^\bullet \rightarrow T \rightarrow 0$. Since the entries of $\delta_2\delta_1$ span the relations of B , and B is a domain, the complex is automatically exact at P^0 , P^1 and P^4 . Moreover, since the Hilbert series of B is $1/(1-t)^4$, the kernel of δ_2 is generated in degree 3 by Koszul, where it has dimension 4. We examine the map δ_3 (with $\zeta = 1$) given by the matrix

$$\delta_3 = \begin{pmatrix} 0 & 0 & 0 & d & x & 0 \\ u & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & -y & -u & 0 \\ -x & 0 & 0 & 0 & 0 & -d \end{pmatrix}.$$

The rows of this matrix are clearly linearly independent, from which we conclude that the complex is also exact at P^2 . By the Hilbert series, the complex must now also be exact at P^3 . But this is impossible, since the linearly independent vectors (x, y, d, u) , $(0, d, 0, x)$ and $(u, 0, y, 0)$ in P^3 are all in the kernel of δ_3 . This contradiction proves that (2) implies (3).

Finally, assume that Q is nonsingular. Then B is a quadratic domain with relations given by 3.2. As above, we conclude that P^\bullet is exact at P^4 , P^1 and P^0 . It remains to prove exactness at P^3 and P^2 . Since B has Hilbert series $1/(1-t)^4$, exactness at either of these will imply

exactness at the other. It suffices to prove that the kernel of δ_3 is contained in the image of δ_4 .

Let $h = (h_1, h_2, h_3, h_4) \in P^3$ be an element of the kernel of δ_3 . This implies in particular that

$$\begin{aligned} (h_1, h_2)\Sigma^t u Q - h_4(x, y)Q &= 0, \\ \zeta(h_1, h_2)(\Sigma^{-1})^t d Q^\sigma - \zeta h_3(x, y)Q^\sigma &= 0. \end{aligned}$$

Since Q and Q^σ are nonsingular and $\zeta \neq 0$, we may cancel them from these equations and get the simpler equations

$$\begin{aligned} (h_1, h_2)\Sigma^t u &= h_4(x, y), \\ (h_1, h_2)(\Sigma^{-1})^t d &= h_3(x, y). \end{aligned}$$

We conclude, in particular, that $h_4 x \in Bu$.

We claim that $h_4 x \in Bu$ implies $h_4 \in Bu$. Recall that B has a \mathbb{Z} -grading and let $B_{(n)}$ denote the degree n component of B in that grading. Since x and u are homogeneous in the \mathbb{Z} -grading, we may assume, in the proof of the claim, that h_4 is also homogeneous, i.e. $h_4 \in B_{(n)}$ for some fixed $n \in \mathbb{Z}$. Suppose first that $n > 0$. Then $h_4 x \in B_{(n)}x \cap B_{(n-1)}u = Au^n x \cap Au^{n-1}u = A\sigma^n(x)u^n \cap Au^n = A\sigma^n(x)u^n = Au^n x \subset Bux$. Next suppose that $n \leq 0$. Then $h_4 x \in B_{(n)}x \cap B_{(n-1)}u = d^{-n}Ax \cap d^{-n+1}Au = d^{-n}Ax \cap d^{-n}Adu = d^{-n}(Ax \cap Aq)$. By assumption, Q is nonsingular, hence q is not a scalar multiple of x^2 and so $Ax \cap Aq = Axq = Aqx = dAux$. Thus $h_4 x \in d^{-n+1}Aux \subset Bux$. In either case we get $h_4 x \in Bux$ from which we can conclude $h_4 \in Bu$.

Now define $b \in B$ by $h_4 = bu$. Let $h' = (h'_1, h'_2, h'_3, 0) = h - b(x, y, d, u)$. Then h' is in the kernel of δ_3 . But the equations above, applied to h' , immediately imply that $h' = 0$. Thus $h = b(x, y, d, u)$ and h is in the image of δ_4 , as required. This completes the proof of (3) implies (1). \square

Theorem 1.1 follows immediately from 3.6 and 3.2.

Remark 3.7. Let $\Gamma(q, \sigma) \subset \mathbb{P}^3 \times \mathbb{P}^3$ be the scheme representing the point modules of $B(q, \sigma)$ (when q is nonsingular). $\Gamma(q, \sigma)$ is always at least one dimensional. Generically, $\Gamma(q, \sigma)$ has two one-dimensional components, each isomorphic to \mathbb{P}^1 , and at most two additional points.

4. A SECOND FAMILY OF QUANTUM 3-SPACES.

Throughout this section we will assume r is a nonzero scalar and we take $A(r)$ to be the graded down-up algebra $A(r, r^{-1}, 0)$. For ease of notation we will continue to use the notation s under the assumption that $rs = 1$. We write h_r and h_s for the images of H_r and H_s in $A(r)$. These are both normal and regular elements of $A(r)$.

Let α be another nonzero scalar and let σ be the graded automorphism of $A(r)$ defined by $\sigma(x) = y$ and $\sigma(y) = \alpha x$. We note that $\sigma(h_r) = -\alpha r h_s$. We also note that the automorphism τ defined by h_r is given by $\tau(x) = sx$ and $\tau(y) = ry$.

Definition 4.1. Given the notation above, we let $C = C(r, \alpha) = A(r)[d, u; h_r, \sigma]$.

We remark that as in the previous section, C is an \mathbb{N} -graded algebra with all four generators having degree one. It is also a \mathbb{Z} -graded algebra where x and y have degrees 0, d has degree -1 and u degree 1. The Hilbert series with respect to the \mathbb{N} -grading is $1/(1-t)^4$.

Lemma 4.2. *The algebra C is isomorphic to the algebra $K\langle x, y, d, u \rangle$ factored by the ideal generated by the six quadratic elements*

$$\begin{aligned} xd - dy, \quad yd - \alpha dx, \quad ux - ryu, \quad uy - \alpha sxu, \\ du - (xy - ryx), \quad ud + \alpha r(xy - syx). \end{aligned}$$

Proof. Let C' be the algebra $K\langle x, y, d, u \rangle$ factored by the six given elements and let A' be the subalgebra generated by x and y . Let $h'_r = xy - ryx = du$ in C' . Then $xh'_r = xdu = dyu = sdux = sh'_r x$ and similarly $h'_r y = syh'_r$. Thus we have an epimorphism $\beta : A(r) \rightarrow A'$ which by universality extends to an algebra epimorphism $\beta : C \rightarrow C'$. Since C is generated by x, y, d and u and satisfies at least the six given relations, this is invertible. \square

Theorem 4.3. *The algebra $C(r, \alpha)$ is Artin-Schelter regular and Koszul.*

Proof. Let ${}_C T = C/C_+$ and let $P^\bullet \rightarrow T \rightarrow 0$ be the augmented sequence of graded-projective left C modules

$$0 \rightarrow C(-4) \xrightarrow{\delta_4} C(-3)^4 \xrightarrow{\delta_3} C(-2)^6 \xrightarrow{\delta_2} C(-1)^4 \xrightarrow{\delta_1} C \xrightarrow{\epsilon} T \rightarrow 0$$

where the maps δ_i are given by right multiplication by the following matrices of degree one elements

$$\delta_4 = (x \ y \ d \ u), \quad \delta_1 = (x \ y \ d \ u)^t,$$

$$\delta_3 = \begin{pmatrix} -\alpha^{-1}u & 0 & 0 & rd & y & \alpha^{-1}ry \\ 0 & \alpha^{-2}ru & -d & 0 & -rx & -\alpha^{-1}x \\ 0 & 0 & \alpha x & -ry & -u & 0 \\ \alpha^{-2}ry & -\alpha^{-2}x & 0 & 0 & 0 & \alpha^{-2}d \end{pmatrix},$$

and

$$\delta_2 = \begin{pmatrix} 0 & -d & x & 0 \\ -\alpha d & 0 & y & 0 \\ u & 0 & 0 & -ry \\ 0 & u & 0 & -\alpha sx \\ ry & -x & 0 & d \\ -\alpha y & \alpha rx & u & 0 \end{pmatrix}.$$

The fact that $P^\bullet \rightarrow T \rightarrow 0$ is a complex is a straightforward check, similar to 3.5, which we omit. Moreover, the entries of $\delta_2\delta_1$ span the relations of C . By 4.2 and 2.2, C is a quadratic domain. Hence this complex is exact at P^4 , P^1 and P^0 . Since the Hilbert series of C is $1/(1-t)^4$, to see that the complex is exact it suffices to check it is exact at P^3 , i.e. that the kernel of δ_3 is contained in the image of δ_4 .

Let $h = (h_1, h_2, h_3, h_4) \in P^3$ be in the kernel of δ_3 . From the second column of δ_3 we obtain $rh_2u - h_4x = 0$ in C , i.e. $h_4x \in Cu$. It follows exactly as in the proof of 3.6 that $h_4 \in Cu$. The remainder of the proof of exactness is also the same as in 3.6.

The complex $P^\bullet \rightarrow T \rightarrow 0$ is now seen to be a graded projective resolution of the trivial left C -module T from which Artin-Schelter regularity and the Koszul property follow immediately. \square

Let $[X, Y, D, U]$ be homogeneous coordinates on \mathbb{P}^3 . We let $e_1 = [1, 0, 0, 0] \in \mathbb{P}^3$, $e_2 = [0, 1, 0, 0]$, etc. Recall that the point scheme Γ of any quantum \mathbb{P}^3 is the graph of a scheme automorphism $\gamma : E \rightarrow E$, where $E = \pi_1(\Gamma) \subset \mathbb{P}^3$.

Proposition 4.4. *Assume $r^4 \neq 1$. Then the point scheme of $C = C(r, \alpha)$ is zero-dimensional with four closed points: $e_1 \times e_1$, $e_2 \times e_2$, $e_3 \times e_3$ and $e_4 \times e_4$ of multiplicities 1, 1, 9 and 9 respectively. The automorphism γ is the identity on closed points and has finite order if and only if α is a root of unity.*

Proof. Let $f = \alpha(r + s)/2$. We recall that $rs = 1$ and we note that since $r^4 \neq 1$, $f \neq 0$. A point $p \times q = (X', Y', D', U') \times (X, Y, D, U)$ in $\mathbb{P}^3 \times \mathbb{P}^3$ is in Γ if and only if

$$\begin{aligned} X'D &= D'Y & Y'D &= \alpha D'X & D'U &= X'Y - rY'X \\ U'X &= rY'U & U'Y &= \alpha sX'U & U'D &= -\alpha r(X'Y - sY'X) \end{aligned}$$

The fact that the four points $e_i \times e_i$ are the only solutions to this system of equations is straightforward. We calculate the local ring over $e_3 \times e_3$ by setting $D = D' = 1$ and using affine coordinates $(x', y', u') \times (x, y, u)$. Then $x' = y$, $y' = \alpha x$, $u = x'y - rxy' = y^2 - \alpha rx^2$ and $u' = -\alpha r(x'y - sy'x) = -\alpha r(y^2 - \alpha sx^2)$. Substituting these into the remaining two equations yields $-\alpha r(y^2 - \alpha sx^2)x = \alpha rx(y^2 - \alpha rx^2)$ and $-\alpha r(y^2 - \alpha sx^2)y = \alpha sy(y^2 - \alpha rx^2)$. These simplify to the equations $fx^3 - xy^2 = 0$ and $fy^3 - \alpha^2 x^2 y = 0$. Thus we may eliminate the variables u, u', x' and y' and are left with the local ring $K[x, y]/(fx^3 - xy^2, fy^3 - \alpha^2 x^2 y)$ and the automorphism $\gamma^\#$ of that ring is given by $\gamma^\#(x) = y/\alpha$ and $\gamma^\#(y) = x$. This ring has dimension 9 and the automorphism is finite if and only if α is a root of unity.

The local ring over the point $e_4 \times e_4$ is calculated similarly and turns out to be $K[x, y]/(\alpha(r + s)x^3 - 2r^2xy^2, r^2(r + s)y^3 - 2\alpha x^2y)$ with $\gamma^\#(x) = ry$ and $\gamma^\#(y) = \alpha sx$. This ring also has dimension 9 and the automorphism has finite order if and only if α is a root of unity.

Since the total multiplicity of the scheme Γ must be 20, (see for example [13]) the remaining two points are seen to have multiplicity 1. \square

The fact that the automorphism γ will have finite order when α is a root of unity leads us to ask the question of whether or not the algebra $C(r, \alpha)$ is finite over its center in the same circumstances. Our goal in the rest of this section is to prove that $C(r, \alpha)$ is not finite over its center as long as r is not a root of unity. We must first calculate the

center of $A(r)$ in the following lemma. This result is well known, see for example [16], but we include a short proof for the convenience of the reader.

The \mathbb{Z} -grading that $A(r)$ inherits from C puts all of $A(r)$ in degree zero. However $A(r)$ has its own \mathbb{Z} -grading if we give x degree 1 and y degree -1 . This grading will be used in the proof of the Lemma 4.5 and Proposition 4.7.

Lemma 4.5. *If r is not a root of unity then the center of $A(r)$ is the polynomial ring $K[h_r h_s]$.*

Proof. Let $A(r)_{(n)}$ denote the degree n component of the \mathbb{Z} -grading on $A(r)$. Since $r \neq s$, it follows from [8] that $A(r)_{(0)}$ is the polynomial ring $K[h_r, h_s]$ and $A(r)$ is a free left or right $A(r)_{(0)}$ -module on the basis x^i , $i \geq 0$, and y^j , $j > 0$.

The center of $A(r)$, $Z(r)$, inherits both the \mathbb{N} -grading and the \mathbb{Z} -grading. Suppose that an element of the form $p(h_r, h_s)x^n$ is in $Z(r)_{(n)}$ for some $n \geq 0$ and some homogeneous polynomial p . Then $p(h_r, h_s)x^{n+1} = xp(h_r, h_s)x^n = p(sh_r, rh_s)x^{n+1}$. We conclude that $p(sh_r, rh_s) = p(h_r, h_s)$. If the degree of p is k , this can only happen if $p(h_r, h_s) = \lambda(h_r h_s)^{k/2}$ for some $\lambda \in K$ (since r is not a root of unity). So we have $p \in K[h_r h_s]$, which is clearly contained in $Z(r)$. Since p is in the center and $A(r)$ is a domain, we must now have x^n in the center.

But then $yx^n = (yx)x^{n-1} = \left(\frac{h_r - h_s}{s - r}\right)x^{n-1}$ and

$$yx^n = x^n y = x^{n-1} \left(\frac{sh_r - rh_s}{s - r}\right) = \left(\frac{s^n h_r - r^n h_s}{s - r}\right)x^{n-1}.$$

We conclude that $s^n h_r - r^n h_s = h_r - h_s$, which can only happen if $r^n = 1$. Thus $n = 0$. A similar argument works for $p(h_r, h_s)y^n$ and we conclude that the center of $A(r)$ is contained in $K[h_r h_s]$, as claimed. \square

The following statement is now clear.

Lemma 4.6. *Suppose that r is not a root of unity. If α is a primitive k^{th} root of unity then the σ -invariant elements of $Z(A(r))$ are given by*

$$Z(A(r))^\sigma = \begin{cases} K[(h_r h_s)^{k/2}] & \text{if } k \text{ is even,} \\ K[(h_r h_s)^k] & \text{if } k \text{ is odd.} \end{cases}$$

If α is not a root of unity then $Z(A(r))^\sigma = K$.

We will now calculate the center of $C(r, \alpha)$ when r is not a root of unity.

Proposition 4.7. *Assume r is not a root of unity. If α is a primitive k^{th} root of unity then the center of $C = C(r, \alpha)$ is given by*

$$Z(C) = \begin{cases} K[(h_r h_s)^{k/2}, d^{2k}, u^{2k}] & \text{if } k \text{ is even,} \\ K[(h_r h_s)^k, d^{2k}, u^{2k}] & \text{if } k \text{ is odd.} \end{cases}$$

If α is not a root of unity then the center of C is the field K .

Proof. First assume α is a primitive k^{th} root of unity. Let \bar{Z} denote the algebra on the right hand side of the equation above and let Z denote the center of C . We begin by observing that since $\alpha^k = 1$, σ^{2k} is the identity automorphism of $A(r)$. For any $a \in A(r)$ we have $ad^{2k} = d^{2k}\sigma^{2k}(a) = d^{2k}a$ and $ud^{2k} = \sigma(h_r)d^{2k-1} = d^{2k-1}\sigma^{2k}(h_r) = d^{2k-1}h_r = d^{2k}u$. Thus d^{2k} and similarly u^{2k} are in the center. Hence $\bar{Z} \subset Z$.

We also need some easy formulas, proved inductively:

$$\sigma^n(h_r) = \begin{cases} -\alpha^n r h_s & \text{if } n \text{ is odd,} \\ \alpha^n h_r & \text{if } n \text{ is even,} \end{cases}$$

and similarly with r and s interchanged.

Since Z inherits a \mathbb{Z} -grading from C , we may consider only \mathbb{Z} -homogeneous elements. Let a be in $A(r)$ and suppose that the element $d^n a$ is in Z for some $n > 0$. We will prove that $d^n a \in \bar{Z}$. The proof for central elements of the form $u^n a'$ is similar and we omit it.

For any $b \in A(r)$ we have $d^n a b = b d^n a = d^n \sigma^n(b) a$ and similarly $d^{n+1} \sigma^{-1}(a) b = d^n a (db) = (db) d^n a = d^{n+1} \sigma^n(b) a$. Since C is a domain we obtain $ab = \sigma^n(b) a$ and $\sigma^{-1}(a) b = \sigma^n(b) a = ab$ for all $b \in A(r)$. In particular a is σ -invariant as well as normal and regular in $A(r)$. We claim that n must be even. To see this, consider the \mathbb{Z} -grading on $A(r)$ (as in 4.5). The automorphism σ^n either preserves this grading or switches signs in the grading, depending as n is even or odd. But σ^n is defined by the formula $\sigma^n(b) a = ab$ and it therefore preserves the grading. So n is even.

Now we get $\alpha^n h_r a = \sigma^n(h_r) a = a h_r = h_r \tau(a)$. Canceling h_r leaves $\alpha^n a = \tau(a)$. But the eigenvalues of τ on $A(r)$ are all of the form r^j

for $j \in \mathbb{Z}$. Since r is not a root of unity, but α is a root of unity, we conclude that $\alpha^n = 1$ and $\tau(a) = a$. In particular k divides n .

Suppose first that $2k$ divides n and thus d^n is in the center Z . Since C is a domain, it follows that a is in Z as well. But then a is in $Z(A(r))^\sigma$ and so $d^n a \in \bar{Z}$ as required.

Suppose alternatively that $2k$ does not divide n . Then k must be even as we may write $n = kl$, with l odd. Since n is even, $xd^n = \alpha^{n/2} d^n x = (-1)^l d^n x \neq d^n x$ and so d^n is not in the center. However, $d^{2n} a^2 = (d^n a)^2$ is in the center and hence by the arguments above a^2 is in the center. Moreover $a^2 \in K[(h_r h_s)^{k/2}]$. But this clearly implies $a \in K[h_r h_s]$, and since a is σ -invariant we get a in Z . This contradicts d^n not being in the center and so the case where $2k$ does not divide n does not occur. This completes the proof when α is a root of unity.

Now assume that α is not a root of unity. Since we need only consider \mathbb{Z} -homogenous elements of the center, let $d^m b$ be in $Z(C)$ with $b \in A(r)$. Then $h_r h_s d^m b = d^m b h_r h_s$ and $h_r h_s d^m b = d^m \sigma^m(h_r h_s) b = \alpha^{2m} d^m h_r h_s b = \alpha^{2m} d^m b h_r h_s$. Since α is not a root of unity $m = 0$. The case for elements of the form $u^m b'$ is identical and consequently $Z(C) \subset Z(A(r))$. However elements in $Z(C)$ must be σ invariant in order to commute with d , so by Lemma 4.6 $Z(C) = K$. \square

When α is a k^{th} root of unity one can compute the formula $d^{2k} u^{2k} = (-r)^k (h_r h_s)^k$. This shows that the center of $C(r, \alpha)$, as given in the Proposition, is itself a generalized Laurent polynomial ring over the base ring $Z(A(r))^\sigma$. In particular the Hilbert series of the center is $H_{Z(C)}(t) = H_{Z(A(r))^\sigma}(t) \cdot \left(\frac{1+t^{2k}}{1-t^{2k}}\right)$. Since $Z(A(r))^\sigma$ has GK-dimension 1, we see that $Z(C)$ has GK-dimension 2. In particular:

Corollary 4.8. *If r is not a root of unity and α is a root of unity then the algebra $C(r, \alpha)$ is not a finite module over its center.*

It is of course easy to prove that $C(r, \alpha)$ is not finite over its center any time that r is not a root of unity. But it is only the case when α is a root of unity that the automorphism of the point scheme has finite order.

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