

GENERALIZING THE NOTION OF KOSZUL ALGEBRA

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ABSTRACT. We introduce a generalization of the notion of a Koszul algebra, which includes graded algebras with relations in different degrees, and we establish some of the basic properties of these algebras. This class is closed under twists, twisted tensor products, regular central extensions and Ore extensions. We explore the monomial algebras in this class and we include some well-known examples of algebras that fall into this class.

1. INTRODUCTION

Koszul algebras were originally defined by Priddy in 1970 [15] and have since revealed important applications in algebraic geometry, Lie theory, quantum groups, algebraic topology and, recently, combinatorics (cf. [9]). The rich structure and long history of Koszul algebras are clearly detailed in [13].

There exist numerous equivalent definitions of a Koszul algebra (see for example [2]). For our purposes, there are two particular equivalent definitions that motivate our discussion. Let K be a field and let A be connected graded K -algebra, finitely generated in degree 1. Let $E(A) = \bigoplus E^{n,m}(A) = \bigoplus Ext_A^{n,m}(K, K)$ be the associated bigraded Yoneda algebra of A (where n

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is the cohomology degree and $-m$ is the internal degree inherited from the grading on A). Set $E^n(A) = \bigoplus_m E^{n,m}(A)$. Then A is said to be Koszul if it satisfies either of the following (equivalent) definitions:

- (1) *Diagonal purity*: $E^{n,m}(A) = 0$ unless $n = m$.
- (2) *Low-degree generation*: $E(A)$ is generated as an algebra by $E^{1,1}(A)$.

Stated this way, these are very strong conditions and either of them immediately imply that a Koszul algebra must be quadratic.

Recently, Berger introduced the class of N -Koszul algebras [3] as a possible generalization of Koszul algebras. The definition of N -Koszul generalizes the purity definition of Koszul given above and is given as the statement: $E^{n,m}(A) = 0$ unless $m = \delta(n)$, where $\delta(n) = N(n-1)/2 + 1$ if n is odd and $\delta(n) = Nn/2$ if n is even. In particular, an N -Koszul algebra must have all of its relations in degree N and each $E^n(A)$ is *pure* in the sense that it is supported in a single internal degree. Purity then becomes a powerful homological tool, see for example [4] or [8].

In [8] it was shown that N -Koszul could also be rephrased in terms of low-degree generation. It is shown there, as we will reprove in 4.6, that A is N -Koszul if and only if:

- (1) $E(A)$ is generated as an algebra by $E^1(A)$ and $E^2(A)$, and
- (2) A has defining relations all of degree N .

Several works ([3], [8], [4], [7] among others) demonstrate similarities between N -Koszul algebras and Koszul algebras, and so provide evidence that the N -Koszul algebras should be included in a generalization of Koszulity. However the restriction to N -homogeneity is somewhat artificial and poses serious problems. In particular, the class of N -Koszul algebras, for $N > 2$, is not closed with respect to many standard operations, such as graded Ore extensions, regular normal extensions, or tensor products.

The goal of this paper is to give an alternate generalization of the notion of Koszul with four good properties: it should allow for graded algebras with relations in more than one degree; it should, at minimum, be closed under tensor products and regular central extensions; it should include the N -Koszul algebras, and it should collapse to the definition of Koszul in the case of quadratic algebras. As we will see, the following simple definition does all this, and more.

Definition 1.1. The graded algebra A is said to be \mathcal{K}_2 if $E(A)$ is generated as an algebra by $E^1(A)$ and $E^2(A)$.

It is clear that this is the next most restrictive definition one could make, following Koszul and N -Koszul, since for a non-Koszul algebra, $E(A)$ could never be generated by anything less than $E^1(A)$ and $E^2(A)$. However, this definition sacrifices homological purity. Surprisingly, many statements that one would want in a generalization of Koszulity hold even without the convenience of homological purity.

The same definition, under the name *semi-Koszul*, was used in [11] to study certain Hopf algebras over finite fields. Clearly this definition captures the Koszul and N -Koszul algebras. But there are many important examples of \mathcal{K}_2 algebras which have defining relations in more than one degree and so cannot be N -Koszul. For example, the homogeneous coordinate ring of any projective complete intersection is \mathcal{K}_2 (Corollary 9.2). Any Artin-Schelter regular algebra of global dimension four on three generators (refer to [1] or [10]) will have two quadratic relations and two cubic relations. We show in Theorem 4.7 that these algebras are all \mathcal{K}_2 .

The main results in this paper describe operations that will preserve the class of \mathcal{K}_2 algebras. For example, the class is closed under twists and twisted tensor products, as proved in section 3. Most of the results require technical hypotheses, and are therefore not stated explicitly in this introduction. Along the way we give an alternative description of \mathcal{K}_2 algebras in terms of minimal projective resolutions (Theorem 4.4) and define \mathcal{K}_2 -modules. This is analogous to the standard Koszul theory of modules with linear projective resolutions. As a corollary to this result we get Theorem 5.3, a simple combinatorial algorithm for establishing which monomial algebras are \mathcal{K}_2 .

The last five sections of the paper deal with change of rings theorems. Section 6 contains lemmas about the structure of the spectral sequences associated to a change of rings. In Section 7 we use the notion of a \mathcal{K}_2 module and prove Theorem 7.4 about algebras with \mathcal{K}_2 quotients, an analog to a standard theorem in Koszul theory. In particular, we see then that the class of \mathcal{K}_2 algebras is closed under regular central extensions. In Theorem 8.2 we analyze the inheritance of the \mathcal{K}_2 property from an algebra A to a normal regular quotient of the form A/gA where the degree of g is 1. The

case where g has degree greater than 1 is analyzed in Theorem 9.1, with a remarkably different answer. In Theorem 10.2 we show that the class of \mathcal{K}_2 algebras is closed under graded Ore extension ($A = B[z, \sigma; \delta]$), and in the process we establish an interesting and apparently previously unknown fact about maps induced on the Yoneda algebra (Lemma 10.1).

Throughout the paper we include many examples of both \mathcal{K}_2 and non- \mathcal{K}_2 algebras that we hope are both interesting and illustrative.

2. NOTATION

Let K be a field, V a finite dimensional K -vector space and $T(V)$ the usual \mathbb{N} -graded tensor algebra over V . Throughout this paper, the phrase *graded algebra* will refer to a connected graded K -algebras of the form $A = T(V)/I$, where I is an ideal of $T(V)$ that is generated by a (minimal) finite collection of homogeneous elements $\{r_1 \dots, r_s\}$ of degree at least 2. We do not assume that the relations r_i all have the same degree.

We put $I' = \bigoplus_n I'_n \subset I$ where $I'_n = \langle I_{<n} \rangle_n$. Hence I'_n is the set of element of I of degree n that are generated by elements of I of strictly smaller degree. We note that $I'_m = I_m$ for all m sufficiently large, and that I' is a proper sub-ideal of I .

Definition 2.1. An element $r \in I$ is said to be *essential* (in I) if r is not in I' , or equivalently r does not vanish in I/I' .

For any \mathbb{Z} -graded vector space W , we use $W(n)$ to denote the same vector space with degree shifted grading $W(n)_k = W_{k+n}$. It is sometimes convenient to write $W(n_1, n_2, \dots, n_k)$ for $\bigoplus_{j=1}^{j=k} W(n_j)$. We denote the projective dimension of W as an A -module by $pd_A(W)$.

Given a graded algebra A , as above, we identify the ground field K with the trivial (left or right) A -module A/A_+ and we write $\epsilon : A \rightarrow K$ for the corresponding augmentation.

Our purpose is to study certain properties of the bigraded Yoneda algebra of A which we denote $E(A)$. This is defined as $E(A) = \bigoplus_{n,k \geq 0} E^{n,k}(A)$ where $E^{n,k}(A) = Ext_A^{n,k}(K, K) = Ext_A^n(K, K)_{-k}$. In the bigrading $E^{n,k}(A)$, n is the *cohomology* degree and $-k$ is the *internal* degree (inherited from the grading on A). Note that $E(A)$ is supported in non-positive internal

degrees. We also write $E^n(A)$ for $\bigoplus_k E^{n,k}(A)$. The cup product on the ring $E(A)$ is denoted \star . The *Ext* groups involved here are calculated in the category of \mathbb{Z} -graded and locally finite A -modules with graded *Hom* spaces.

Given two graded algebras A and B and a degree zero K -algebra homomorphism $\sigma : A \rightarrow B$, we will denote the induced algebra homomorphism from $E(B)$ to $E(A)$ by $\sigma^* : E(B) \rightarrow E(A)$. In particular, if σ is a graded automorphism of A , then σ^* is a bi-graded automorphism of $E(A)$.

3. OPERATIONS ON THE CLASS \mathcal{K}_2

In this section we show that the class of \mathcal{K}_2 algebras is closed under twists by automorphisms and twisted tensor products.

Let σ be an automorphism of a graded algebra $A = T(V)/I$. Then we have the notion of the twisted algebra A^σ , as studied extensively by [18]. This algebra has the same underlying graded vector space structure as A , with the product of homogeneous elements $a \cdot b = a\sigma^{|a|}(b)$. (Here $|a|$ gives the degree of a .)

Theorem 3.1. *Let A be an \mathbb{N} -graded algebra and σ a graded automorphism of A . Then A is \mathcal{K}_2 if and only if A^σ is \mathcal{K}_2 .*

Proof. The Theorem follows immediately from the claim that $E(A^\sigma) = E(A)^{(\sigma^{-1})^*}$, where the twisting is done with respect to the internal grading. We sketch a proof of this claim. The spaces in the bar complex computing $E(A)$ and $E(A^\sigma)$ are the same, namely $\Omega_n := \text{Hom}_K(A_+^{\otimes n}, K)$. Let d be the differential on Ω_n which determines $E(A)$, and d_σ the differential which determines $E(A^\sigma)$. Then there is an explicit homotopy $\mu^* : (\Omega_n, d) \rightarrow (\Omega_n, d_\sigma)$ between the two complexes. This homotopy is the dual of a vector space isomorphism $\mu \in \text{Aut}_K(A_+^{\otimes n})$ with the following complicated formula. Let $\bar{a} = a_1 \otimes \dots \otimes a_n$ be a multihomogeneous element of $(A_+)^{\otimes n}$. Define $|\bar{a}, i| = \sum_{j=i}^n |a_j|$ for $1 \leq i \leq n$. Then we define:

$$\mu(a_1 \otimes \dots \otimes a_n) = \mu(\bar{a}) = \sigma^{|\bar{a},1|}(a_1) \otimes \sigma^{|\bar{a},2|}(a_2) \dots \otimes \sigma^{|\bar{a},n|}(a_n).$$

One checks that $d(f \circ \mu) = (d_\sigma f) \circ \mu$. This shows that μ^* induces a bigraded vector space isomorphism from $E(A)$ to $E(A^\sigma)$. Let \star be the usual cup product in $E(A)$ and \star_σ the cup product in $E(A^\sigma)$. One then calculates

explicitly that for bigraded elements f and g of $E(A)$

$$\mu^* f \star_\sigma \mu^* g = \mu^* ((\sigma^*)^{-|g|}(f) \star g)$$

where $|g|$ is the internal degree of g . This shows that $E(A^\sigma)$ is isomorphic to $E(A)^{(\sigma^{-1})^*}$ and proves the claim. \square

Let A and B be graded algebras and let σ be a graded automorphism of A . Then we can define an associative multiplication on the K -tensor product $B \otimes A$ by the rule $(b_1 \otimes a_1) \cdot (b_2 \otimes a_2) = b_1 b_2 \otimes \sigma^{|b_2|}(a_1) a_2$ whenever a_1, a_2, b_1, b_2 are homogeneous. We denote the graded algebra thus formed by $B \otimes^\sigma A$ and write $b \otimes^\sigma a$ for the element $b \otimes a$ in the algebra.

Theorem 3.2. *If A and B are graded algebras, and σ is an automorphism of A , then $B \otimes^\sigma A$ is \mathcal{K}_2 if and only if A and B are both \mathcal{K}_2 .*

Proof. Arguing in a similar fashion to the previous Theorem, one proves the formula:

$$E(B \otimes^\sigma A) = E(B) \overline{\otimes}^{(\sigma^{-1})^*} E(A).$$

(See also [13], 3.1.1.) The result follows immediately. \square

4. CONDITIONS ON MINIMAL PROJECTIVE RESOLUTIONS

It is well known that a graded algebra A is Koszul if and only if the trivial module admits a linear free resolution, see for example [13]. A simple way to express this is to say that all of the matrix entries of the maps in any minimal projective resolution of the trivial A -module K have degree 1. In this section we prove a similar, but much more complicated, version of this statement for \mathcal{K}_2 algebras, and then we use this to determine which 4 dimensional Artin-Schelter regular algebras are \mathcal{K}_2 .

We continue with the notation $A = T(V)/I$ where I is a graded ideal generated (minimally) by homogeneous elements r_1, r_2, \dots, r_s of degrees n_1, \dots, n_s respectively (all greater than 1). Let R denote both the set $\{r_1, \dots, r_s\}$ and the transpose of the $1 \times s$ matrix $(r_1 \ r_2 \ \dots \ r_s)$. We fix a minimal projective resolution (Q, d) of the trivial left A module K . It is clear that if A is a \mathcal{K}_2 algebra then every Q^n is finitely generated over A . Hence we will assume unless otherwise stated that every Q^n is finitely generated. For each $n \geq 0$, we can define integers t_n and $m_{1,n} \cdots, m_{t_n,n}$ by $Q^n = A(m_{1,n} \cdots, m_{t_n,n})$. (Note that $m_{k,n} \leq -n$ for all k .)

We choose a homogeneous A -basis for each Q^n . With respect to these bases, we let M_n be the matrix representing $d_n : Q^n \rightarrow Q^{n-1}$ and let \hat{M}_n be a lift of M_n to a matrix over $T(V)$ with homogeneous entries. The entries of \hat{M}_1 give a basis for V and we may choose our basis of Q^2 , as well as the lift \hat{M}_2 , so that $\hat{M}_2\hat{M}_1 = R$. (When multiplying matrices defined over $T(V)$ we suppress the tensor product notation.)

The following Lemma is obvious.

Lemma 4.1. *Let $\alpha_1, \dots, \alpha_s$ be homogeneous elements of $T(V)$ and suppose $r = \sum_s \alpha_s r_s$. Then r is essential in I if and only if some α_s is a unit.*

Before we state our theorem we recall the minimal projective resolution construction of the cup product on $E(A)$. Recall that $E^n(A) = \text{Hom}_A(Q^n, K)$, by minimality of the resolution. To multiply $f \in E^n(A)$ by $g \in E^k(A)$ we consider the following commutative diagram, where existence of the downward maps F_j is assured by projectivity.

$$\begin{array}{ccccccc}
 Q^{n+k} & \xrightarrow{M_{n+k}} & Q^{n+k-1} & \rightarrow \dots \rightarrow & Q^{n+1} & \xrightarrow{M_{n+1}} & Q^n \\
 F_k \downarrow & & F_{k-1} \downarrow & & F_1 \downarrow & & F_0 \downarrow & \searrow f \\
 Q^k & \xrightarrow{M_k} & Q^{k-1} & \rightarrow \dots \rightarrow & Q^1 & \xrightarrow{M_1} & Q^0 & \xrightarrow{\epsilon} K \\
 g \downarrow & & & & & & & \\
 K & & & & & & &
 \end{array}$$

Then the product of g and f in $E(A)$ is $g \star f := g \circ F_k \in \text{Hom}_A(Q^{n+k}, K)$.

It will be convenient to work with a graded basis of $E(A)$. Let $\{\varepsilon_{k,n}\}_{k=1}^{t_n}$ with $\varepsilon_{k,n} : Q^n \rightarrow K$ be the graded basis of $E^n(A)$ dual to a minimal set of homogeneous A -generators of Q^n , so that $\varepsilon_{k,n}$ has cohomology degree n and internal degree $m_{k,n}$ (since the corresponding A -generator has degree $-m_{k,n}$).

For $n \geq 1$ let $U_n = E^2(A) \star E^{n-2}(A)$ and $V_n = E^1(A) \star E^{n-1}(A)$. We note that the algebra A is \mathcal{K}_2 if and only if $E^n(A) = U_n + V_n$ for all $n \geq 3$. (This requires an inductive argument.)

We set some temporary notation to analyze the diagrams dictating the relationship of $E^n(A)$ to U_n and to V_n .

For every n and every $1 \leq k \leq t_n$, let $e_{k,n}$ be the k^{th} column of the $t_n \times t_n$ identity matrix. We can define matrices $G_{k,n}$ and $J_{k,n}$ with entries in A by

commutativity of the following diagram.

$$\begin{array}{ccccccc}
Q^{n+2} & \xrightarrow{M_{n+2}} & Q^{n+1} & \xrightarrow{M_{n+1}} & Q^n & & \\
J_{k,n} \downarrow & & G_{k,n} \downarrow & & e_{k,n} \downarrow & \searrow^{\varepsilon_{k,n}} & (*) \\
Q^2 & \xrightarrow{M_2} & Q^1 & \xrightarrow{M_1} & A & \xrightarrow{\varepsilon} & K
\end{array}$$

The entries of $J_{k,n}$ and $G_{k,n}$ are all homogeneous, but the actual degrees will not be important. Since the matrix $e_{k,n}$ is scalar and the matrix M_1 is linear, we may lift the matrix $G_{k,n}$ to a (unique) homogeneous matrix $\hat{G}_{k,n}$ over $T(V)$ which satisfies $\hat{G}_{k,n}\hat{M}_1 = \hat{M}_{n+1}e_{k,n}$. We also choose a homogeneous lift of $J_{k,n}$ to $\hat{J}_{k,n}$ in $T(V)$. Let $R_{k,n} = \hat{M}_{n+2}\hat{G}_{k,n} - \hat{J}_{k,n}\hat{M}_2$. We observe that $R_{k,n}$ is a matrix of homogeneous elements of I .

With all of this notation in hand we can state two technical lemmas, the first of which concerns U_n .

Lemma 4.2. (1) Let $1 \leq p \leq t_2$ and $1 \leq k \leq t_n$ and define scalars λ_j by $\varepsilon_{p,2}\varepsilon_{k,n} = \sum_{j=1}^{t_{n+2}} \lambda_j \varepsilon_{j,n+2}$. Then for $1 \leq i \leq t_{n+2}$, $\lambda_i \neq 0$ if and only if the entry in row i and column p of $J_{k,n}$ is a unit.

(2) Let $1 \leq i \leq t_{n+2}$. Then there exist p and k as in (1) such that $\lambda_i \neq 0$ if and only if some entry in row i of $\hat{M}_{n+2}\hat{M}_{n+1}$ is essential in I .

Proof. The first statement is clear (and the required unit is λ_i).

For the second statement, suppose first that the required p and k exist. Then the entry in row i and column p of $J_{k,n}$ is the unit λ_i . This must also be true of the matrix $\hat{J}_{k,n}$. Let f_i be row i of the $t_{n+2} \times t_{n+2}$ identity matrix. Then $f_i\hat{J}_{k,n}R := r$ is essential in I by 4.1. But we have

$$r = f_i(\hat{M}_{n+2}\hat{G}_{k,n} - R_{k,n})\hat{M}_1 = f_i\hat{M}_{n+2}\hat{M}_{n+1}e_{k,n} - f_iR_{k,n}\hat{M}_1.$$

Since $R_{k,n}$ has entries from I , the element $f_iR_{k,n}\hat{M}_1$ cannot be essential. It follows that $f_i\hat{M}_{n+2}\hat{M}_{n+1}e_{k,n}$ is essential, as required.

Conversely, suppose that an element of row i of $\hat{M}_{n+2}\hat{M}_{n+1}$ is essential, say the element in column k . Call the element r . Then

$$r = f_i\hat{M}_{n+2}\hat{M}_{n+1}e_{k,n} = f_i(\hat{J}_{k,n}\hat{M}_2 + R_{k,n})\hat{M}_1.$$

Since no entry of $f_iR_{k,n}\hat{M}_1$ is essential, we see that $f_i\hat{J}_{k,n}\hat{M}_2\hat{M}_1 = f_iJ_{k,n}R$ is essential. From Lemma 4.1, some entry of $f_iJ_{k,n}$ is thus a unit, as required. \square

Our second technical lemma concerns V_n and we leave its (easier) proof to the reader.

Lemma 4.3. (1) Let $1 \leq p \leq t_1$ and $1 \leq k \leq t_n$ and define scalars μ_j by $\varepsilon_{p,1} \star \varepsilon_{k,n} = \sum_{j=1}^{t_{n+1}} \mu_j \varepsilon_{j,n+1}$. Then for $1 \leq i \leq t_{n+1}$, $\mu_i \neq 0$ if and only if the entry in row i and column p of $E_{k,n}$ is a unit.

(2) Let $1 \leq i \leq t_{n+1}$. Then there exist p and k as in (1) such that $\mu_i \neq 0$ if and only if some entry in row i of \hat{M}_{n+1} is (nonzero) linear.

To state our theorem we need just a bit more notation. For each $n \geq 2$, let L_n be the image of \hat{M}_n modulo the ideal $T(V)_{\geq 2}$ and let E_n be the image of $\hat{M}_n \hat{M}_{n-1}$ modulo I' . We think of L_n as the linear part of \hat{M}_n and E_n as the essential part of $\hat{M}_n \hat{M}_{n-1}$. Finally, let $[L_n : E_n]$ be the $t_n \times (t_{n-1} + t_{n-2})$ matrix obtained by concatenating the rows of L_n and E_n . (Note that the entries in any given column of this matrix are all in the same vector space, either V or I/I' .)

Theorem 4.4. Let A be a graded K -algebra as above, and $Q \rightarrow K \rightarrow 0$ a minimal graded projective resolution. Then the following are equivalent:

- (1) A is \mathcal{K}_2
- (2) For $2 < n \leq \text{pd}_A(K)$, Q^n is finitely generated and the rows of the matrix $[L_n : E_n]$ are linearly independent over K .

Proof. Suppose first that for some specific $n > 2$, the rows of the matrix $[L_n : E_n]$ are linearly dependent. By changing basis (homogeneously) in the free module $Q^n = A(m_{1,n}, \dots, m_{t_n,n})$, we may assume that the first row of the matrix $[L_n : E_n]$ is zero. By lemmas 4.2 and 4.3, the coefficient of the cohomology class $\varepsilon_{i,n}$ in every element of U_n or V_n is then zero. Hence $\varepsilon_{i,n} \notin U_n + V_n$ and A is not \mathcal{K}_2 .

Conversely, suppose that A is not \mathcal{K}_2 and fix the smallest n for which $E^n(A) \neq U_n + V_n$. Choose a homogeneous basis for $U_n + V_n$, say $\varepsilon_{1,n}, \dots, \varepsilon_{k,n}$, $k < t_n$. Extend this to a full homogeneous basis of $E^n(A)$ by choosing $\varepsilon_{k+1,n}, \dots, \varepsilon_{t_n,n}$. This basis then corresponds to a homogeneous A -basis for Q^n . We may assume the matrices representing the minimal resolution are calculated with respect to this new basis. But then again by lemmas 4.2 and 4.3, the last row of $[L_n : E_n]$ must be zero. \square

An algebra will often fail the \mathcal{K}_2 hypothesis simply because of the existence of a cohomology class in the “wrong” internal degree, i.e. an internal degree that could not be generated by the internal degrees of elements of lower cohomological degree. For example, the algebra $B = K\langle x, y \rangle / \langle x^2 - xy, y^2 \rangle$ is a well-known example of a non-Koszul quadratic (and hence non- \mathcal{K}_2 algebra). Since the algebra is quadratic, the portion of $E(A)$ generated by $E^1(A)$ and $E^2(A)$ is exactly $\sum_n E^{n,n}(A)$. But $E^{3,4}(B) \neq 0$, so we have cohomology in the “wrong” internal degree.

However, an algebra can also fail to be \mathcal{K}_2 even when the errant cohomology class has an internal degree that could be generated by classes of lower cohomological degree. Theorem 4.4 can be a simple way to detect this, as exhibited by the following example.

Example 4.5. Let A be the algebra $K\langle x, y \rangle / \langle x^2 - xy, yx, y^3 \rangle$. Using the fact that A has Hilbert series $1 + 2t + 2t^2$, one can calculate the first few terms of a minimal projective resolution of ${}_A K$ to obtain:

$$A(-3, -4, -4, -4, -4) \xrightarrow{M_3} A(-2, -2, -3) \xrightarrow{M_2} A(-1, -1) \xrightarrow{M_1} A \rightarrow K \rightarrow 0$$

where

$$\hat{M}_3 = \begin{pmatrix} y & 0 & 0 \\ 0 & xy & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{pmatrix} \quad \hat{M}_2 = \begin{pmatrix} x & -x \\ y & 0 \\ 0 & y^2 \end{pmatrix} \quad \hat{M}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

Since the tensor xy^2 is in the ideal of definition of A , but is not essential ($xy^2 = -x(x^2 - xy) + (x^2 - xy)(x - y) + x(yx)$), we get

$$[L_3 : E_3] = \begin{bmatrix} y & 0 & 0 & yx & -yx \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^3 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & y & 0 & y^3 \end{bmatrix}$$

By 4.4, the algebra A is not \mathcal{K}_2 and $E^3(A) \neq U_3 + V_3$. Indeed a more careful analysis of the matrix above shows $\dim_K((U_3 + V_3)_{-4}) = 3$, whereas we know $\dim_K(E^{3,4}(A)) = 4$.

We can also reprove the following fact about N -Koszul algebras (see also [8]).

Corollary 4.6. *The algebra A is N -Koszul if and only if it is N -homogeneous and \mathcal{K}_2 .*

Proof. Suppose first that the algebra A is N -Koszul. Then the matrices \hat{M}_n will have linear entries when n is odd and entries all of degree $N - 1$ when n is even. In particular $\hat{M}_n \hat{M}_{n-1}$ will always be a matrix whose entries are all essential relations. The theorem immediately implies that A is \mathcal{K}_2 .

Conversely, suppose that A is \mathcal{K}_2 and N -homogeneous. Then a relation of A is essential if and only if it has degree N . An inductive argument, based on the condition of Theorem 4.4, then assures that the matrices \hat{M}_n must alternate between having entries of degree 1 when n is odd and degree $N - 1$ when n is even. This proves that A is N -Koszul. \square

We conclude this section by determining which 4 dimensional Artin-Schelter regular (AS-regular) algebras are \mathcal{K}_2 . See [10] for the definition of Artin-Schelter regular and for details on the degrees of the defining relations for these algebras.

Let A be an AS-regular algebra of global dimension 4. Then A has 2, 3, or 4 linear generators. If A has 4 generators then it is Koszul [16] and hence \mathcal{K}_2 . If the algebra has 2 generators then it has two relations, one of degree 3 and one of degree 4. The definition of Artin-Schelter regular assures us that ${}_A K$ has a minimal projective resolution of the form:

$$0 \rightarrow A(-7) \rightarrow A(-6)^2 \rightarrow A(-3, -4) \rightarrow A(-1)^2 \rightarrow A \rightarrow K \rightarrow 0$$

and it is easy to see that the algebra can not be \mathcal{K}_2 as there is a nonzero cohomology class in $E^{3,6}(A)$ which can not be generated by $E^2(A) = E^{2,3}(A) + E^{2,4}(A)$ and $E^1(A) = E^{1,1}(A)$.

For the remainder of this section we will assume that A is AS-regular of global dimension 4 and has 3 linear generators. Such an algebra will have 4 relations of degrees 2, 2, 3 and 3. We will write A as $K\langle x_1, x_2, x_3 \rangle / I$. In this case ${}_A K$ has a minimal projective resolution of the form:

$$0 \rightarrow A(-5) \xrightarrow{Y} A(-4)^3 \xrightarrow{N} A(-2, -2, -3, -3) \xrightarrow{M} A(-1)^3 \xrightarrow{X} A \rightarrow K \rightarrow 0 \quad (4.1)$$

where the matrices Y , N , M , X have homogeneous entries from $K\langle x_1, x_2, x_3 \rangle$. From the Gorenstein condition, we may assume:

$$Y = (x_1 \quad x_2 \quad x_3), \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

N is a 3 by 4 matrix where the entries of the first two columns are 2-tensors and those in the last two columns are 1-tensors and M is a 4 by 3 matrix with 1-tensors as entries in the first two rows and 2-tensors in the last two rows. In particular, the entries in MX and the entries in YN must each give a set of relations for A . The K -span of the two quadratic entries in MX must be the same as the K -span of the two quadratic entries in YN .

Theorem 4.7. *Let A be an AS-regular algebra of global dimension 4 with 3 linear generators. Then A is \mathcal{K}_2 .*

Proof. We see from (4.1) that the \mathcal{K}_2 matrix condition given in Theorem 4.4 can fail only at N . Let us assume that the condition fails, and find a contradiction.

Let L be the linear part of the matrix N and let E be NM modulo I' . Failure of the matrix condition is the statement that the composite matrix $[L : E]$ has linearly dependent rows. By choosing new coordinates, we may assume that the first row of this matrix is zero. Using this, let $(\sum_i x_i \otimes a_i, \sum_i x_i \otimes b_i, 0, 0)$ be the first row of N , where the terms a_i and b_i are linear.

The two quadratic entries in YN must span I_2 , so from the first row of N we see that no element of I_2 can contain a term beginning with x_1 , i.e. if $\sum_i x_i \otimes l_i \in I_2$, then $l_1 = 0$. Consequently, if $\sum_i x_i \otimes q_i \in (I')_3$, then $x_1 \otimes q_1 \in (I')_3$ and $q_1 \in I_2$ (one can make no such statement about q_2 or q_3).

Let the first two rows of M form the matrix (d_{ij}) , $i = 1, 2$ and $j = 1, 2, 3$ where the elements d_{ij} are all linear. Then the j -th entry of the first row of the matrix NM is $\sum_i x_i \otimes (a_i \otimes d_{1j} + b_i \otimes d_{2j})$. The statement that the first row of E is zero is the statement that these three 3-tensors are all in $(I')_3$. As stated above, this implies that $a_1 \otimes d_{1,j} + b_1 \otimes d_{2,j}$ is in I_2 for $1 \leq j \leq 3$, so that $(a_1 \ b_1 \ 0 \ 0)$ is in the kernel of M . Since (4.1) is exact, $(a_1 \ b_1 \ 0 \ 0)$ must

be in the image of N , which is impossible because the first two columns of N are made up of two-tensors, whereas a_1 and b_1 are linear. \square

5. MONOMIAL \mathcal{K}_2 ALGEBRAS

In this section we explore in detail the \mathcal{K}_2 property for monomial algebras and using Theorem 4.4 we present an algorithm for determining whether a given monomial algebra is \mathcal{K}_2 .

Let $\{x_1, \dots, x_n\}$ be a fixed basis of the vector space V . We use this basis to identify $T = T(V)$ with $K\langle x_1, \dots, x_n \rangle$. Throughout this section, $A = T/I$, where I is an ideal generated by a minimal finite set of monic monomials (in the variables x_i) $R = \{r_1, \dots, r_s\}$. For any element a in T , we denote its image in A by \bar{a} . Let T_{mon} be the set of monic monomials in T . We write $|a|$ for the degree of any $a \in T_{mon}$.

The combinatorics of monomial algebras is straightforward and built primarily on the finite set $LE(R)$ consisting of all $a \in T_{mon}$ for which there exists a $b \in T_{mon}$ such that $|b| > 0$ and $ab \in R$. This is the set of proper “left ends” of R .

For any pair of monomials a and b in T_{mon} such that $\bar{a}\bar{b} = 0$ we define the monic monomial $L(a, b)$ to be a' where $a = a''a'$ and a' is minimal such that $\bar{a}'\bar{b} = 0$. If $\bar{a}\bar{b} \neq 0$ we set $L(a, b) = 0$. Suppose that $\bar{a} \neq 0$ and $\bar{b} \neq 0$, but $\bar{a}\bar{b} = 0$. Let $L(a, b) = a'$. Then there exists a minimal b' so that $b = b'b''$ and $\bar{a}'\bar{b}' = 0$. The minimality of both a' and b' assures us that $a'b' \in R$, and hence that $a' \in LE(R)$.

Finally, for $b \in T_{mon}$ such that $\bar{b} \neq 0$, define

$$L(b) = \{a \in LE(R) \mid L(a, b) = a\}.$$

The following Lemma is clear from the definitions of $L(a, b)$, $L(b)$ and $LE(R)$.

Lemma 5.1. *If b is a monomial in T and $\bar{b} \neq 0$ in A then the left annihilator of \bar{b} in A is generated by $\{\bar{a} \mid a \in L(b)\}$.*

From this Lemma it is easy to give a combinatorial description of the graded vector space structure of $E(A)$. We do this by describing a minimal (monomial) projective resolution of the trivial module ${}_A K$ of the form:

$$\rightarrow A^{t_m} \xrightarrow{M_m} A^{t_{m-1}} \rightarrow \dots \rightarrow A^{t_1} \xrightarrow{M_1} A \rightarrow K \rightarrow 0.$$

To do this, it suffices to give an inductive definition of the matrices \hat{M}_m , the homogeneous lifts of the matrices M_m to matrices over T .

Let $\hat{M}_1 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Assume that $m > 1$ and that \hat{M}_{m-1} has been defined

and has the property that each row of \hat{M}_{m-1} has a unique nonzero entry and that entry is in $LE(R) \cup \{x_1, \dots, x_n\}$. Let \hat{M}_{m-1} have t_{m-1} rows. For $1 \leq i \leq t_{m-1}$, if b_i is the non-zero entry in row i of \hat{M}_{m-1} and $L(b_i) = \{a_1, \dots, a_j\}$, let $\hat{M}_m(i)$ be the j by t_{m-1} matrix with a_1, \dots, a_j arranged as column i and zeroes elsewhere. If $L(b_i)$ is empty, this is the empty matrix. Finally, take \hat{M}_m to be the matrix obtained by stacking the matrices $\hat{M}_m(i)$, $1 \leq i \leq t_{m-1}$. It is clear from construction that \hat{M}_m has the property that each row has a unique nonzero entry from $LE(R) \cup \{x_1, \dots, x_n\}$ (and in fact, for $m \geq 2$, all the nonzero entries are from $LE(R)$.)

We note that it is clear that $M_m M_{m-1} = 0$ for all $m > 1$ and hence the matrices M_m define a complex of free-graded A -modules as above. Lemma 5.1 makes the following clear.

Lemma 5.2. *The complex described above is a minimal resolution of ${}_A K$.*

Now let S be the smallest set in T such that

- (1) each x_i is in S , and
- (2) if $b \in S$ and $L(a, b) = a$ for some $a \in LE(R)$, then $a \in S$.

There is a simple inductive algorithm for calculating S . Put $S_1 = \{x_1, \dots, x_n\}$ and for $i > 1$ define $S_i = \bigcup_{b \in S_{i-1}} L(b) \setminus \bigcup_{j < i} S_j$. Then S is the disjoint union of the S_i . It should be clear that S_k is just those nonzero matrix entries of \hat{M}_k which have not appeared in \hat{M}_i for any $i < k$. Moreover, since $LE(R)$ is finite, $S_k = \emptyset$ for all k sufficiently large.

Theorem 5.3. *Let A be a monomial algebra over the field k . Then A is \mathcal{K}_2 if and only if the following condition holds for every $b \in S$: if there exists an $a \in LE(R)$ such that $L(a, b) = a$ then $ab \in R$ or $\deg(a) = 1$.*

Proof. From 4.4, we need only see that the condition given in the theorem is equivalent to the condition that for all m the rows of the matrix $[L_m : E_m]$ are linearly independent. It is clear that the condition is equivalent to the statement that every row of $[L_m : E_m]$ is non-zero. It therefore suffices to

prove that $[L_m : E_m]$ has linearly dependent rows if and only if one of its rows is zero. We begin with some simple observations about this matrix.

First we note that a monic *monomial* r in I is essential in I if and only if $r \in R$, so we may assume that E_m is a matrix with entries in the span of R (rather than in I/I'). Moreover, since every row of $\hat{M}_m \hat{M}_{m-1}$ has a single nonzero entry, the nonzero rows of E_m must be identical to the nonzero rows of $\hat{M}_m \hat{M}_{m-1}$. It is now clear that if a set of rows of E_m is linearly dependent and none of the rows are zero, then two of the rows must be identical. We may assume that these are the first two rows and that the unique nonzero entry $r \in R$ is in the first column. Now the multiplication $\hat{M}_n \hat{M}_{n-1}$ must have the following form:

$$\begin{pmatrix} a & 0 & 0 & \cdots \\ 0 & b & 0 & \cdots \\ \vdots & & & \end{pmatrix} \begin{pmatrix} c & 0 & 0 & \cdots \\ d & 0 & 0 & \cdots \\ \vdots & & & \end{pmatrix} = \begin{pmatrix} r & 0 & \cdots \\ r & 0 & \cdots \\ \vdots & & \end{pmatrix}$$

where a, b, c and d are monomials with $ac = bd = r$. In particular there is a monomial e such that $c = ed$ or $d = ec$. By symmetry we may assume $d = ec$. But then the second row of \hat{M}_{m-1} is a left multiple of the first row, which contradicts the minimality of the projective resolution. This contradiction proves that any linearly dependent set of rows of E_m must involve a row of zeroes.

Suppose now that some set of rows of $[L_m : E_m]$ is minimally linearly dependent. By the above, the E_m portion of every one of those rows must be zero. We are left with a minimally linearly dependent set of rows of L_m . But as above, the nonzero rows of L_m are identical to the corresponding rows of \hat{M}_m , which has linearly independent rows. Therefore at least one of the rows in our minimal dependence set is zero. This proves the required claim. \square

The monomial algebra A will fail to be \mathcal{K}_2 if and only if for some minimal i there exists some $b \in S_i$ and $a \in L(b)$ such that $|a| > 1$ and $ab \notin R$. In this case we will say that the \mathcal{K}_2 property fails at level i .

Example 5.4. Let A be the monomial algebra generated by $\{w, x, y, z\}$ with relations $R = \{z^2y^2, y^3x^2, x^2w, zy^3x\}$. Then $S_1 = \{w, x, y, z\}$, $S_2 = \{z^2y, y^3x, x^2, zy^3\}$, $S_3 = \{y^3\}$, $S_4 = \{z^2\}$ and $S_5 = \emptyset$. In this case \mathcal{K}_2 does

not fail at level 2, but \mathcal{K}_2 fails at level 3 because in S_3 we have y^3 which is minimally annihilated by z^2 , however z^2y^3 is not in R .

Example 5.5. Let A be generated by $\{x, y, z\}$. Let $R = \{x^4, yx^3, x^3z\}$. Then $S_1 = \{x, y, z\}$, $S_2 = \{x^3, yx^2\}$ and $S_3 = \emptyset$. We conclude that A is \mathcal{K}_2

For any monomial algebra if $b \in S_1$ and $a \in L(b)$ then $ab \in R$, so the earliest \mathcal{K}_2 can fail is at level two. In the case of N -homogeneous algebras this is also the latest \mathcal{K}_2 can fail.

Corollary 5.6. *Let A be an N -homogenous algebra. S_3 will be empty if A is \mathcal{K}_2 , and otherwise \mathcal{K}_2 will fail at level 2.*

Proof. Assume that every monomial in R is degree N . Since every quadratic monomial algebra is Koszul, and hence \mathcal{K}_2 , we will assume $N > 2$. Let $S_1 = \{x_1, \dots, x_n\}$ and let b be in S_2 . Since A is N -homogenous, $|b| = N - 1$. For any $a \in L(b)$ if $|a| > 1$ then $|a| > N$, which means $ab \notin R$ and \mathcal{K}_2 fails at level 2. If $|a| = 1$ for all $a \in L(b)$ and all $b \in S_2$, then $|ab| = N$ and $\overline{ab} = 0$, which means $ab \in R$. In this case \mathcal{K}_2 has not failed at level 2 and moreover, since $|a| = 1$ we have $a \in S_1$ so that $S_3 = \emptyset$. If $S_3 = \emptyset$ then $S_m = \emptyset$ for all $m > 2$ and \mathcal{K}_2 does not fail at any level. \square

This Corollary should be compared to the overlap condition for N -Koszul algebras given in [3] Proposition 3.8. The two conditions are logically equivalent, but the condition here requires fewer formal calculations.

6. SPECTRAL SEQUENCE LEMMAS

This section contains some spectral sequence facts which will be used in the subsequent sections. Let $\phi : A \rightarrow B$ be a (graded) ring epimorphism with kernel J . For any B module M we write ${}^\phi M$ for the corresponding A module induced via ϕ . For any graded A module N we write N^J for the graded B -module $N^J = \{n \in N \mid Jn = 0\} = \text{Hom}_A(B, N)$. The following lemma about the change of rings spectral sequence of Cartan and Eilenberg is certainly well known to experts.

Lemma 6.1. *Let $\phi : A \rightarrow B$ be an epimorphism of K algebras. For any left A -module N and any left B -modules M and ${}'M$ we have first quadrant spectral sequences E_r and ${}'E_r$, with E_2 -terms $E_2^{p,q} = \text{Ext}_B^p(M, \text{Ext}_A^q(B, N))$*

and $'E_2^{p,q} = Ext_B^p('M, Ext_A^q(B, N))$ which converge to $Ext_A(\phi M, N)$ and $Ext_A(\phi' M, N)$ respectively. The differentials d_r and $'d_r$ of these spectral sequences intertwine with the natural cup product by $Ext_B('M, M)$ from the right, that is we have the commutative diagram:

$$\begin{array}{ccc} E_r \times Ext_B('M, M) & \xrightarrow{d_r \times 1} & E_r \times Ext_B('M, M) \\ \downarrow \star & & \downarrow \star \\ 'E_r & \xrightarrow{'d_r} & 'E_r \end{array}$$

Proof. We prove the lemma only for $r = 2$, the general case being similar. We recall the construction of the spectral sequences. Let $Q \rightarrow M \rightarrow 0$ and $'Q \rightarrow 'M \rightarrow 0$ be projective resolutions (over B) of M and $'M$ respectively. Let $0 \rightarrow N \rightarrow I$ be an A -injective resolution of N . We write $J_q = Hom_A(B, I_q)$. The spectral sequence E_r is the first spectral sequence related to the double complex $F^{p,q} = Hom_B(Q^p, Hom_A(B, I_q)) = Hom_B(Q^p, J_q)$ with horizontal differential d_h and vertical differential d_v (chosen so that $d_h d_v + d_v d_h = 0$). Similarly we have $'F^{p,q} = Hom_B('Q^p, J_q)$ with differentials $'d_h$ and $'d_v$. We can use the usual trick (see for example [17]) of identifying $E_2^{p,q}$ with the set of classes of elements $(f, g) \in F^{p,q} \times F^{p+1,q-1}$ for which $d_v f = 0 = d_h f + d_v g$, modulo elements of the form $(0, g)$, $(d_v f, d_h f)$ (for $f \in F^{p,q-1}$) or $(d_h g, 0)$ (for $g \in F^{p-1,q}$). This allows us to identify $d_2 : E_2 \rightarrow E_2$ as $d_2[(f, g)] = [(d_h g, 0)]$.

Let $[\zeta] \in Ext_B^k('M, M)$ be represented by $\zeta \in Hom_B('Q^k, M)$. The following diagram then defines, via projectivity, maps ζ_j , $j \geq 0$, so that the top row of boxes are commutative and the box in the bottom row is anticommutative.

$$\begin{array}{ccccccccc} 'Q^{k+p+2} & \rightarrow & 'Q^{k+p+1} & \rightarrow & 'Q^{k+p} & \rightarrow & \dots & \rightarrow & 'Q^{k+1} & \rightarrow & 'Q^k \\ \zeta_{p+2} \downarrow & & \zeta_{p+1} \downarrow & & \zeta_p \downarrow & & \dots & & \zeta_1 \downarrow & & \zeta_0 \downarrow & \searrow \zeta \\ Q^{p+2} & \rightarrow & Q^{p+1} & \rightarrow & Q^p & \rightarrow & \dots & \rightarrow & Q^1 & \rightarrow & Q^0 & \rightarrow & M \\ & & g \downarrow & & f \downarrow & & & & & & & & \\ & & J_{q-1} & \rightarrow & J_q & & & & & & & & \end{array}$$

From this diagram it is clear that $[(f, g)] \star [\zeta] = [(f \zeta_p, g \zeta_{p+1})]$. So we may calculate $'d_2([(f, g)] \star [\zeta]) = [('d_h(g \zeta_{p+1}), 0)] = [(d_h(g) \zeta_{p+2}, 0)] = [(d_h g, 0)] \star [\zeta] = d_2[(f, g)] \star [\zeta]$, as required. \square

The following lemma should also be well-known.

Lemma 6.2. *Let $\phi : A \rightarrow B$ be an epimorphism of K algebras. For any left A -modules N and $'N$ and any left B -module M consider the first quadrant spectral sequences E_r and $'E_r$, with E_2 -terms $E_2^{p,q} = \text{Ext}_B^p(M, \text{Ext}_A^q(B, N))$ and $'E_2^{p,q} = \text{Ext}_B^p(M, \text{Ext}_A^q(B, 'N))$, which converge to $\text{Ext}_A(\phi M, N)$ and $\text{Ext}_A(\phi M, 'N)$ respectively. Then, the left cup product action of $\text{Ext}_A(N, 'N)$ from $\text{Ext}_A(B, N)$ to $\text{Ext}_A(B, 'N)$, commutes with the natural action of B . Hence the $\text{Ext}_A(N, 'N)$ -action lifts to an action (still denoted \star) from E_r to $'E_r$. Moreover, the differentials d_r and $'d_r$ of these spectral sequences intertwine with this action. That is, we have the commutative diagram:*

$$\begin{array}{ccc} \text{Ext}_A(N, 'N) \times E_r & \xrightarrow{1 \times d_r} & \text{Ext}_A(N, 'N) \times E_r \\ \downarrow \star & & \downarrow \star \\ 'E_r & \xrightarrow{'d_r} & 'E_r \end{array}$$

Proof. The first claim is clear (since the action of B is by precomposition and the action of $\text{Ext}_A(N, 'N)$ is by post composition). Let $Q \rightarrow M \rightarrow 0$ be a projective resolution (over B) of M . Let $0 \rightarrow N \rightarrow I$ and $0 \rightarrow 'N \rightarrow 'I$ be A -injective resolutions of N and $'N$. Let $F^{p,q} = \text{Hom}_B(Q^p, \text{Hom}_A(B, I_q)) = \text{Hom}_B(Q^p, I_q^J)$ with horizontal differential d_h and vertical differential d_v . Similarly let $'F^{p,q} = \text{Hom}_B(Q^p, \text{Hom}_A(B, 'I_q)) = \text{Hom}_B(Q^p, 'I_q^J)$ with differentials $'d_h$ and $'d_v$. As in the previous proof, we identify an element of $E_2^{p,q}$ with a certain class of elements $(f, g) \in F^{p,q} \times F^{p+1, q-1}$. Let $[\zeta] \in \text{Ext}_A^k(N, 'N)$ be represented by $\zeta \in \text{Hom}_A(N, 'I_k)$. The following diagram defines, via injectivity, maps ζ_j , $j \geq 0$ so that the bottom row of boxes is commutative and the top box is anticommutative. Note that we abuse notation and consider f and g as maps into I_q and I_{q-1} respectively.

$$\begin{array}{ccccccc} & & & & Q_{p+1} & \rightarrow & Q^p \\ & & & & g \downarrow & & f \downarrow \\ 0 \rightarrow N & \longrightarrow & I_0 & \rightarrow \cdots \rightarrow & I_{q-1} & \rightarrow & I_q \rightarrow I_{q+1} \\ & & \zeta \searrow \zeta_0 \downarrow & & \zeta_{q-1} \downarrow & & \zeta_q \downarrow & \downarrow \\ & & 'I_k & \rightarrow \cdots \rightarrow & 'I_{k+q-1} & \rightarrow & 'I_{k+q} \rightarrow & 'I_{k+q+1} \end{array}$$

From this diagram we see that $[\zeta] \star [(f, g)] = [(\zeta_q f, \zeta_{q-1} g)]$. (We are once again abusing notation, using the fact that image of $\zeta_q f$ must be in $'I_{r+q}^J$ and similarly for $\zeta_{q-1} g$.) Thus we have $'d_2([\zeta] \star [(f, g)]) = [('d_h(\zeta_{q-1} g), 0)] =$

$[(\zeta_{q-1}d_h(g), 0)] = [\zeta] \star [(d_h(g), 0)] = [\zeta] \star d_2[(f, g)]$. This proves the lemma for $r = 2$, and the general proof is essentially the same. \square

For the remainder of this section let A be a connected graded K -algebra and $g \in A$ a homogeneous element of degree d that is normal and regular in A . Set $B = A/gA$ and $\phi : A \twoheadrightarrow B$ the associated epimorphism. For any graded A -module N , let N^g be the B -module $\{n \in N \mid gn = 0\}$. Then the Cartan-Eilenberg spectral sequence for $Ext_A(\phi M, N)$, $E_2^{p,q} = Ext_B^p(M, Ext_A^q(B, N))$, satisfies $E_2^{p,0} = Ext_B^p(M, N^g)$, $E_2^{p,1} = Ext_B^p(M, N/gN)$ and $E_2^{p,q} = 0$ for $q > 1$. The edge homomorphism $E_2^{p,0} \rightarrow Ext_A^p(\phi M, N)$ of this spectral sequence is the map induced by ϕ^* and we denote it by ϕ^* as well. Since the spectral sequence has only two nonzero rows, there is an associated long exact sequence:

$$\begin{aligned} \cdots \rightarrow Ext_A^n(\phi M, N) \xrightarrow{\gamma} Ext_B^{n-1}(M, N^g) \xrightarrow{d_2} Ext_B^{n+1}(M, N/gN) \\ \xrightarrow{\phi^*} Ext_A^{n+1}(\phi M, N) \xrightarrow{\gamma} Ext_B^n(M, N^g) \rightarrow \cdots \end{aligned}$$

We are primarily interested in applying this long exact sequence to the case when M and N are both the trivial module, K . But we need to know that the map γ , like d_2 , intertwines with the right cup product from $Ext_B('M, M)$. This theorem should also be well-known and we omit the proof.

Theorem 6.3. *Let A , g , B and ϕ be as above. For any left A -module N and any left B -modules M and $'M$ let $E_2^{p,q}$ and $'E_2^{p,q}$ be as in Lemma 6.1. There is a well defined cup product:*

$$Ext_A^n(\phi M, N) \times Ext_B^m('M, M) \xrightarrow{\star} Ext_A^{n+m}(\phi 'M, N).$$

Moreover, the maps $\gamma : Ext_A^{p+1}(\phi M, N) \rightarrow E_2^{p,1}$ and $'\gamma : Ext_A^{p+1}(\phi 'M, N) \rightarrow 'E_2^{p,1}$ intertwine with the right action of $Ext_B('M, M)$. That is we have a commutative diagram

$$\begin{array}{ccc} Ext_A^p(\phi M, N) \times Ext_B^k('M, M) & \xrightarrow{\gamma \times 1} & E_2^{p,1} \times Ext_B^k('M, M) \\ \downarrow \star & & \downarrow \star \\ Ext_A^{p+k}(\phi 'M, N) & \xrightarrow{'\gamma} & 'E_2^{p+k,1} \end{array}$$

Consider now the case $N = M = K$, the trivial module for either A or B (from now on we will suppress the notation ϕK). The spectral sequence above then becomes $E_2^{p,q} = 0$ for $q > 1$, $E_2^{p,1} = Ext_B^p(K, K(d)) = E^p(B)(d)$

and $E_2^{p,0} = \text{Ext}_B^p(K, K) = E^p(B)$. The associated long exact sequence is therefore

$$\cdots \rightarrow E^{p-2}(B)(d) \xrightarrow{d_2} E^p(B) \xrightarrow{\phi^*} E^p(A) \xrightarrow{\gamma} E^{p-1}(B)(d) \rightarrow \cdots \quad (6.1)$$

The lemmas show that this can be interpreted as an exact triangle of *right* $E(B)$ modules.

There is a second Cartan-Eilenberg spectral sequence converging to $E(A)$ given by $\bar{E}_2^{p,q} = \text{Ext}_q^p(\text{Tor}_q^A(B, K), K)$. This spectral sequence is also supported on $p \geq 0$ and $q = 0, 1$ with nonzero terms $\bar{E}_2^{p,1} = \text{Ext}_B^p(K(-d), K) = E^p(B)(d)$ and $\bar{E}_2^{p,0} = \text{Ext}_B^p(K, K) = E^p(B)$. The edge homomorphism of this spectral sequence is also $\phi^* : E(B) \rightarrow E(A)$ and we have a long exact sequence:

$$\cdots \rightarrow E^{p-2}(B)(d) \xrightarrow{\bar{d}_2} E^p(B) \xrightarrow{\phi^*} E^p(A) \xrightarrow{\bar{\gamma}} E^{p-1}(B)(d) \rightarrow \cdots \quad (6.2)$$

Exactly as above, we see that the differential \bar{d}_2 , as well as the induced map $\bar{\gamma}$ intertwine with the *left* cup product by elements of $E(B)$. We therefore interpret (6.2) as an exact triangle of *left* $E(B)$ -modules.

Finally, we note that although the terms of the two exact triangles above are all the same, and the map ϕ^* is common to them, the sequences are not, in general, the same. In particular, they are not generally triangles of $E(B)$ -bimodules.

7. \mathcal{K}_2 MODULES AND A CHANGE OF RINGS THEOREM

For Koszul algebras, Positselskii [14] establishes a powerful change of rings theorem, which states that if A is a quadratic algebra and B is a quadratic factor algebra of A which admits a linear free resolution as an A -module, then the Koszul property for B lifts to the Koszul property for A , (see also [12]). In this section we prove the \mathcal{K}_2 analog of that theorem using the definition of a \mathcal{K}_2 module given below.

Let $A = T(V)/I$ be a graded algebra and W a graded left A -module. Let $D(A)$ be the subalgebra of $E(A)$ generated by $E^1(A)$ and $E^2(A)$. Let $Q^* \rightarrow W \rightarrow 0$ be a projective resolution of W as a left A module, where the maps $Q^n \rightarrow Q^{n-1}$ are given by matrices M_n with entries in $T(V)$. Let

L_n be the linear part of M_n , i.e. M_n modulo $T(V)_{\geq 2}$. For $n \geq 2$, let $E_n = M_n M_{n-1}$ modulo I' and let $E_1 = 0$.

Definition 7.1. We say W is a \mathcal{K}_2 A -module if for all $1 \leq n \leq pd_A(W)$, the rows of the matrix $[L_n : E_n]$ are linearly independent.

Theorem 4.4 can be interpreted as the statement that the algebra A is \mathcal{K}_2 if and only if the trivial module is a \mathcal{K}_2 -module. The following lemma is proved in exactly the same way as that theorem.

Lemma 7.2. *The module W is \mathcal{K}_2 if and only if $Ext_A(W, K)$ is generated as a left $D(A)$ module by $Ext_A^0(W, K)$.*

Before we can state and prove the analog of Positselskii's theorem for \mathcal{K}_2 algebras, we need a technical lemma. Let $A = T(V)/I$ be a graded algebra and let B be a graded factor algebra of A . Let $\phi : A \rightarrow B$ be the associated epimorphism.

Lemma 7.3. *Suppose that B , as a left A -module, is a \mathcal{K}_2 -module. Then the natural action of B on $Ext_A(B, K)$ is trivial.*

Proof. Assume that the rows of the matrix $[L_n : E_n]$ are linearly independent and fix $b \in B_k$ for some $k > 0$. The left action of b on $Ext_A(B, K)$ is induced from the right action of b on B . We choose matrices N_i with homogeneous entries in $T(V)$ to produce a commutative diagram of A -modules:

$$\begin{array}{ccccccc} \rightarrow & Q^i & \xrightarrow{M_i} & Q^{i-1} & \rightarrow & \dots & \rightarrow & Q^0 & \rightarrow & B & \rightarrow & 0 \\ & N_i \downarrow & & N_{i-1} \downarrow & & & & N_0 \downarrow & & \cdot b \downarrow & & \\ \rightarrow & Q^i & \xrightarrow{M_i} & Q^{i-1} & \rightarrow & \dots & \rightarrow & Q^0 & \rightarrow & B & \rightarrow & 0 \end{array}$$

To prove the lemma, it suffices to prove the claim that there are no non-zero entries in N_i of degree 0. We do this by induction on i . Write $N_i = N_i^0 + N_i^+$, where the entries of N_i^0 are of degree 0 and the entries of N_i^+ are of positive degree. We must prove $N_i^0 = 0$ for all $i > 0$.

For $i = 0$, N_0 is a diagonal matrix with b on the diagonal, hence $N_0^0 = 0$. The commutativity of the above diagram implies that $N_1 M_1 = M_1 b \text{ mod } I$. Since I has no elements of degree 1 the linear part of $N_1 M_1$ is the linear part of $M_1 b$, which is 0. $N_1^0 L_1$ is the linear part of $N_1 M_1$, hence $N_1^0 L_1 = 0$. Since the rows of $[L_1 : E_1]$ are linearly independent, $N_1^0 = 0$. Assume

now that $i > 1$ and the claim is true for all $k < i$. Consider the matrix $N_i^0[L_i : E_i] = [N_i^0L_i : N_i^0E_i]$. First we note that $N_i^0L_i$ is the linear component of the matrix N_iM_i . Since I contains no elements of degree 1 this is also the linear component of M_iN_{i-1} . By the inductive hypothesis, every entry of this matrix has degree at least 2. Thus $N_i^0L_i = 0$. Similarly, $N_i^0E_i = N_i^0(M_iM_{i-1} \bmod I') = (N_i^0M_iM_{i-1} \bmod I')$. By definition, $(N_i^+M_iM_{i-1} \bmod I') = 0$, since every entry of M_iM_{i-1} is already in I . Now $N_iM_i = M_iN_{i-1} \bmod I$, so $N_iM_iM_{i-1} = M_iN_{i-1}M_{i-1} \bmod I'$, and likewise $M_iN_{i-1}M_{i-1} = M_iM_{i-1}N_{i-2} \bmod I'$. Thus, again by induction, $N_i^0E_i = (N_iM_iM_{i-1} \bmod I') = (M_iM_{i-1}N_{i-2} \bmod I') = 0$. Hence N_i^0 annihilates the matrix $[L_i : E_i]$. Since the rows of $[L_i : E_i]$ are linearly independent it follows that $N_i^0 = 0$. This completes the induction and proves the lemma. \square

Theorem 7.4. *Let A be a graded algebra and B a graded factor algebra of A . Assume that B is \mathcal{K}_2 as an A -module and also \mathcal{K}_2 as an algebra. Then A is \mathcal{K}_2 .*

Proof. Let $\phi : A \rightarrow B$ be the algebra epimorphism. As usual, we have a change of rings spectral sequence $E_r^{p,q}$ converging to $E(A)$ with E_2 term $E_2^{p,q} = \text{Ext}_B^p(K, \text{Ext}_A^q(B, K))$. By Lemmas 6.1 and 6.2 this is a spectral sequence of right $E(B)$ modules and left $E(A)$ modules. By the previous lemma, the B -action on $\text{Ext}_A(B, K)$ is trivial, and thus we have $E_2^{p,q} = \text{Ext}_A^q(B, k) \otimes_K E(B)$ as an $E(A)$ - $E(B)$ bimodule. Since ${}_A B$ is a \mathcal{K}_2 -module, this bimodule is generated by $E_2^{0,0} = \text{Ext}_B^0(B, K) \otimes E^0(B)$ as a $D(A)$ - $E(B)$ bimodule. Since d_2 is a bimodule homomorphism and $d_2(E_2^{0,0}) = 0$, we conclude that $d_2 = 0$. Similarly, for all $r \geq 2$, $d_r = 0$ and $E_r^{p,q} = E_2^{p,q}$.

Since our spectral sequence has collapsed, there is a filtration $F^r E(A)$ on $E(A)$ for which $(F^q E^{p+q}(A) / F^{q-1} E^{p+q}(A)) = E_2^{p,q}$. $E(A)$ is a $D(A)$ - $E(B)$ -bimodule, where $E(B)$ acts on the right through ϕ^* . The convergence of the spectral sequence is compatible with the two bimodule actions. Hence we conclude that $E(A)$ is generated by $E^0(A)$ as an $D(A)$ - $E(B)$ bimodule. By hypothesis $\phi^*(E(B)) = \phi^*(D(B)) \subset D(A)$, and thus $E(A)$ is generated by $E^0(A)$ over $D(A)$, i.e. $E(A) = D(A)$, as required. \square

Example 7.5. Let A and B be defined by

$$A = \frac{K\langle x, y, z \rangle}{\langle xz - zx, yz - zy, x^3z, y^4 + xz^3 \rangle}, \quad B = A/\langle z \rangle = \frac{K\langle x, y \rangle}{\langle y^4 \rangle}$$

Note that A is not a regular central extension of B . Clearly B is a \mathcal{K}_2 algebra, and we claim B is also a \mathcal{K}_2 A -module. To see this it suffices to show that the following is a projective resolution of B as an A -module. We omit any details.

$$\cdots \rightarrow A(-8) \begin{pmatrix} x^3 \\ \longrightarrow \end{pmatrix} A(-5) \begin{pmatrix} z \\ \longrightarrow \end{pmatrix} A(-4) \begin{pmatrix} x^3 \\ \longrightarrow \end{pmatrix} A(-1) \begin{pmatrix} z \\ \longrightarrow \end{pmatrix} A \rightarrow B \rightarrow 0.$$

We conclude that A is also a \mathcal{K}_2 algebra.

8. NORMAL, REGULAR FACTOR RINGS I, THE DEGREE 1 CASE

Let $A = T(V)/I$ be a graded K -algebra and g a homogeneous element of degree 1 that is both normal and regular in A . We set $B = A/gA$ and let $\phi : A \rightarrow B$ the natural graded algebra epimorphism. The purpose of this section is to investigate the extent to which A and B inherit the \mathcal{K}_2 property from one another. We remind the reader that the definition of \mathcal{K}_2 was specifically motivated by the desire that the class of algebras be closed when passing from B to A . Fortunately, this holds as a consequence of Theorem 7.4.

Theorem 8.1. *If the algebra B is \mathcal{K}_2 then the algebra A is \mathcal{K}_2 .*

The converse of the previous theorem, as we will see, is not true. However, one extra hypothesis will give us a converse.

Theorem 8.2. *Let A be \mathcal{K}_2 and assume that $E^3(B)$ is generated by $E^1(B)$ and $E^2(B)$. Then B is \mathcal{K}_2 .*

Proof. Assume A is \mathcal{K}_2 . Consider the exact triangle of right $E(B)$ -modules (6.1), with $d = 1$. Since $E^0(B)(1)$ is supported in degree -1 and $E^{2,1}(B) = 0$, the map d_2 must be zero, the ring homomorphism $\phi^* : E(B) \rightarrow E(A)$ is injective and γ is surjective. Choose any $\hat{g} \in E^{1,1}(A)$ for which $\gamma(\hat{g}) = 1 \in E^0(B)$. Since the image of γ is a projective (bigraded-free) right $E(B)$ -module, we can split what is now a short exact sequence to get $E(A) = E(B) \oplus \hat{g} \star E(B)$ (we suppress the ϕ^* -notation). Now consider the long exact sequence (6.2). Since $E^1(A) = E^1(B) \oplus K\hat{g}$ and $E^1(B)$ is in the kernel of γ' , we see $\gamma'(\hat{g}) \neq 0$. Hence, arguing as above, we get $E(A) = E(B) \oplus E(B) \star \hat{g}$.

Let $D = \sum_{j,k} D^{j,k}$ be the bigraded subalgebra of $E(B)$ generated by $E^1(B)$ and $E^2(B)$. We have $E^1(A) = D^1 + K\star\hat{g}$ and $E^2(A) = D^2 \oplus D^1\star\hat{g} = D^2 \oplus \hat{g}\star D^1$. Thus D and \hat{g} generate $E(A)$. Since $E^3(B) = D^3$, we have $D^2\star\hat{g} \subset D^3 \oplus \hat{g}\star D^2$. Therefore $D\star\hat{g} \subset D \oplus \hat{g}\star D$. Since D and \hat{g} generate $E(A)$, we conclude that $E(A) = D \oplus \hat{g}\star D$. This proves that $D = E(B)$, as required. \square

Remark 8.3. The hypothesis that $E^3(B)$ should be generated by $E^1(B)$ and $E^2(B)$ is easily translated into the hypothesis that the matrix $[L_3 : E_3]$, from Theorem 4.4, has linearly independent rows.

The following example shows that the $E^3(B)$ hypothesis of the theorem cannot be avoided. The example also highlights an interesting connection between Theorem 8.2 and the main theorem of [6], where it is shown that the regularity of a central extending variable is controlled by information encoded in $E^3(B)$.

Example 8.4. Let $A = K\langle x, y, z, w, g \rangle / I$ where I is generated by

$$\{y^2z, zx^2 + gw^2, y^2w^2, xg - gx, yg - gy, wg - gw, zg - gz\}.$$

Let $B = A/gA$. We make the following three non-obvious claims:

- (1) B is not \mathcal{K}_2 .
- (2) The element g is central and regular in A .
- (3) A is \mathcal{K}_2

Proof. Let $T = K\langle x, y, z, w, \rangle$.

To prove the first claim, we apply the algorithm of Section 5 to get a minimal projective resolution of the trivial B -module. This has the form

$$0 \rightarrow B(-5) \xrightarrow{M_3} B(-3, -3, -4) \xrightarrow{M_3} B(-1, -1, -1, -1) \xrightarrow{M_1} B \rightarrow K \rightarrow 0$$

where

$$M_3 = \begin{pmatrix} 0 & y^2 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & y^2 & 0 \\ zx & 0 & 0 & 0 \\ 0 & 0 & 0 & y^2w \end{pmatrix}, \quad M_1 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

We calculate $[L_3 : E_3]$ for this, as in theorem 4.4, and see that its first row is zero. This proves (1).

To prove (2) we use the algorithm from [6] for checking the regularity of a central extending variable. We work over the algebra $T[g]$. Define:

$$f_2 = \begin{pmatrix} 0 \\ w^2 \\ 0 \end{pmatrix} \quad \text{and} \quad f_3 = (0 \ 0 \ 0 \ 0)$$

so that $M_2M_1 + f_2g$ and $M_3M_2 - f_3g$ are both identically 0 in A . Put

$$\hat{M}_n = \begin{pmatrix} M_n & f_n \\ (-1)^{(n-1)}gI & M_{n-1} \end{pmatrix}$$

for all $n \geq 1$. The principal theorem of [6] states that g is regular in A if and only if the matrix $\hat{M}_3\hat{M}_2$ is zero in A . Furthermore, if g is regular in A , then the matrices \hat{M}_n provide the maps in a minimal projective resolution of the trivial A -module. We have

$$\hat{M}_3 = \begin{pmatrix} 0 & y^2 & 0 & 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 & 0 & y^2 & 0 \\ 0 & g & 0 & zx & 0 & 0 & 0 \\ 0 & 0 & g & 0 & 0 & 0 & y^2w \end{pmatrix}, \quad \hat{M}_2 = \begin{pmatrix} 0 & 0 & y^2 & 0 & 0 \\ zx & 0 & 0 & 0 & w^2 \\ 0 & 0 & 0 & y^2w & 0 \\ -g & 0 & 0 & 0 & x \\ 0 & -g & 0 & 0 & y \\ 0 & 0 & -g & 0 & z \\ 0 & 0 & 0 & -g & w \end{pmatrix}.$$

It follows immediately that $\hat{M}_3\hat{M}_2$ is 0 in A and that g is regular.

To prove the last claim, we consider the matrix condition 4.4. We will refrain from including the calculation. The salient point is that the term y^2w^2 , which is an essential relation of A , appears in the first row of $[L_3 : E_3]$ and that $-g$ appears in the first row of $[L_4 : E_4]$. \square

Remark 8.5. The example above becomes substantially more complex if we make the rings A and B commutative (that is, factor A and B by the commutativity relations). Nonetheless, using Theorem 7.4 it can be shown that the interesting features of the example remain the same, i.e. A is \mathcal{K}_2 , g is regular, and B is not \mathcal{K}_2 .

9. NORMAL, REGULAR FACTOR RINGS II, THE DEGREE $d > 1$ CASE

Throughout this section, let $A = T(V)/I$ be a graded algebra, g a normal and regular homogeneous element of A of degree $d > 1$ and $B = A/gA$. As before, $\phi : A \rightarrow B$ is the associated algebra epimorphism. We examine the conditions under which the \mathcal{K}_2 property will descend from A to B .

Since the map $d_2 : E(B)(d) \rightarrow E(B)$ associated to the long exact sequence (6.1) is a right $E(B)$ module homomorphism, there is a distinguished element $\hat{g} \in E^{2,d}(B)$ such that $d_2(a) = \hat{g} \star a$.

Theorem 9.1. *Assume A is \mathcal{K}_2 and the map $\gamma : Ext_A^2(K, K) \rightarrow Ext_B^1(K, K(d))$ of (6.1) is 0. Then B is \mathcal{K}_2 , \hat{g} is a normal and regular element of $E(B)$ and $E(A) = E(B)/\hat{g}E(B)$.*

Proof. Since $d > 1$, $\phi^* : E^1(B) \rightarrow E^1(A)$ is an isomorphism (i.e. A and B have the same set of generators). From (6.1) the hypothesis on γ tells us that $\phi^* : E^2(B) \rightarrow E^2(A)$ is surjective.

Let $D = \bigoplus_n D^n$ be the graded subalgebra of $E(B)$ generated by $E^1(B)$ and $E^2(B)$.

We use induction on n to prove the following two claims: $D^n = E^n(B)$ and $\phi^* : D^n \rightarrow E^n(A)$ is surjective. We may assume the claim for $n-1$ and $n-2$. Our long exact sequence (6.1) then looks like:

$$0 \rightarrow D^{n-2}(d) \xrightarrow{d_2} E^n(B) \xrightarrow{\phi^*} E^n(A) \xrightarrow{\gamma} D^{n-1} \rightarrow \dots$$

Since A is \mathcal{K}_2 , we have by induction,

$$\begin{aligned} E^n(A) &= E^1(A)E^{n-1}(A) + E^2(A)E^{n-2}(A) \\ &= \phi^*(D^1)\phi^*(D^{n-1}) + \phi^*(D^2)\phi^*(D^{n-2}) \\ &= \phi^*(D^n) \end{aligned}$$

This proves the second claim and also shows that $\gamma(E^n(A)) = 0$. Since d_2 is multiplication by $\hat{g} \in E^2(B) = D^2$, we get $E^n(B) = D^n + \hat{g} \cdot D^{n-2} = D^n$, completing the induction.

We now see that ϕ^* is surjective, $\gamma = 0$ and d_2 is injective. In particular, \hat{g} is left regular in $E(B)$. Since ϕ^* is surjective we also see, from (6.2) that \bar{d}_2 is injective and $\bar{\gamma} = 0$. There exists $\hat{g}' \in Ext^{2,d}(B)$ for which $\bar{d}_2(a) = a \star \hat{g}'$. Then $\hat{g} \star E(B) = E(B) \star \hat{g}' = \ker(\phi^*)$. It follows that $\hat{g} = \mu \hat{g}'$ for some scalar $\mu \neq 0$ and hence \hat{g} is normal and regular in $E(B)$, as required. \square

There are several ways to assure the hypothesis that the image of $\gamma : Ext_A^2(K, K) \rightarrow Ext_B^1(K, K(d))$ is 0. In particular, the hypothesis can only fail if there are relations of A of degree $d+1$ (since $Ext_B^1(K, K(d))$ is supported in degree $d+1$.) More specifically, let σ be the automorphism of A defined by $ag = g\sigma(a)$. Then σ defines an automorphism of $T(V)$ as well. Choose $g' \in T(V)_d$, a preimage of $g \in A$. It is not difficult to see that the

map fails to be zero if and only if there exists an *essential* relation of the form $x \otimes g' - g' \otimes \sigma(x)$ for some $x \in T(V)_1$.

Corollary 9.2. *If A is a commutative or graded-commutative \mathcal{K}_2 -algebra and $g \in A_d$ is regular, then A/gA is \mathcal{K}_2 . In particular, any graded complete intersection is \mathcal{K}_2 .*

The hypothesis $\gamma(\text{Ext}_A^2(K, K)) = 0$ is certainly not necessary in Theorem 9.1. For example, in the \mathcal{K}_2 -algebra $A = K\langle x, y \rangle / \langle xy^2 - y^2x \rangle$, y^2 is normal and regular and $A/y^2A = K\langle x, y \rangle / \langle y^2 \rangle$ is also \mathcal{K}_2 , but $\gamma(\text{Ext}_A^2(K, K)) \neq 0$. However, the theorem is false without the γ hypothesis as shown by the following important example.

Example 9.3. Let $A = K\langle x, y \rangle / \langle x^2y - yx^2, xy^3 - y^3x \rangle$ and let $B = A/y^3A = K\langle x, y \rangle / \langle x^2y - yx^2, y^3 \rangle$. It is evident that y^3 is central in A , and by a straightforward application of Bergman's diamond lemma [5] we see that y^3 is regular in A . Moreover A is \mathcal{K}_2 . This is proved by applying theorem 4.4 to the following minimal projective resolution of ${}_A K$:

$$0 \rightarrow A(-5) \begin{pmatrix} y^2 & -x \\ & \end{pmatrix} \rightarrow A(-3, -4) \begin{pmatrix} yx & -x^2 \\ y^3 & -xy^2 \end{pmatrix} \rightarrow A(-1, -1) \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow A \rightarrow K \rightarrow 0.$$

However, B is not \mathcal{K}_2 . The minimal projective resolution of B is

$$\dots \rightarrow B(-5, -4) \begin{pmatrix} y^2 & x^2 \\ 0 & y \end{pmatrix} \rightarrow B(-3, -3) \begin{pmatrix} yx & -x^2 \\ 0 & y^2 \end{pmatrix} \rightarrow B(-1, -1) \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow B \rightarrow K \rightarrow 0,$$

which fails the criteria of 4.4. Alternatively, simply note that $E^1(B)$ and $E^2(B)$ are supported in internal degrees -1 and -3 respectively, and thus they can not possibly generate the nonzero cohomology class in $E^{3,5}(B)$.

10. GRADED ORE EXTENSIONS

Fix a graded algebra B , a graded automorphism σ of B and a degree $+1$ graded σ -derivation δ of B (i.e. $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$). We let A be the associated Ore extension $B[z; \sigma, \delta]$. The extending variable z is assumed to

have degree 1 in A . In this section we investigate when the \mathcal{K}_2 property can pass between A and B .

Let J be the two sided ideal $J = AB_+ = B_+A \subset A$ and put $C = A/J$, with factor morphism $\phi : A \rightarrow C$. We note that C is just the polynomial ring $C = K[\bar{z}]$, $\bar{z} = \phi(z)$. Let $\alpha : B \rightarrow A$ be the inclusion homomorphism.

Let $\zeta : A \rightarrow A$ be the left A -module homomorphism given by *right* multiplication by z . Let ζ also denote the induced map $\zeta : C \rightarrow C$ (still given by multiplication by z). We have a short exact sequence of left A -modules:

$$0 \rightarrow C(-1) \xrightarrow{\zeta} C \rightarrow K \rightarrow 0$$

This induces a long exact sequence

$$\cdots \rightarrow Ext_A^{q-1}(C, K)(1) \rightarrow E^q(A) \rightarrow Ext_A^q(C, K) \xrightarrow{\zeta^*} Ext_A^q(C, K)(1) \rightarrow \cdots$$

Since A is free as a right (or left) B -module and ${}_A C = A \otimes_B K$, $Ext_A(C, K) = Ext_B(K, K) = E(B)$, in particular we may consider ζ^* as a map on $E(B)$. As usual, let σ^* be the automorphism of $E(B)$ induced from the automorphism σ .

The following Lemma does not seem to be widely known.

Lemma 10.1. *The map $\zeta^* : E(B) \rightarrow E(B)$ is a σ^* -derivation (with respect to the cup product \star) of homological degree 0 and internal degree +1. Moreover, ζ^* vanishes on $E^1(B)$ and $E^2(B)$.*

Proof. Let $Bar(B) = \bigoplus_n B \otimes B_+^{\otimes n}$ be the usual bar resolution of ${}_B K$, with differential

$$\partial(b_0 \otimes \cdots \otimes b_n) = \sum_{i=1}^n (-1)^i b_0 \otimes \cdots \otimes b_{i-1} b_i \otimes \cdots \otimes b_n.$$

Since A_B is free and $A \otimes_B K = C$, we may tensor $Bar(B)$ by A to get an A -projective resolution $A \otimes_B Bar(B) = \bigoplus_n A \otimes B_+^{\otimes n}$ of ${}_A C$ with boundary map ∂_A .

We define maps $\check{\delta}_n$ and ζ_n as follows:

$$\check{\delta}_n : B_+^{\otimes n} \rightarrow B_+^{\otimes n} \quad \check{\delta}_n = \sum_{i+j=n-1} 1^{\otimes i} \otimes \delta \otimes \sigma^{\otimes j},$$

and

$$\zeta_n : A \otimes B_+^{\otimes n} \rightarrow A \otimes B_+^{\otimes n} \quad \zeta_n = \zeta \otimes \sigma^n + 1 \otimes \check{\delta}_n, \quad \zeta_0 = \zeta.$$

It is straightforward to check $\zeta_n \circ \partial_A = \partial_A \circ \zeta_{n+1}$. Therefore ζ^* on $Ext_A^m(C, K)$ is given by $[f] \mapsto [\zeta^* f]$, for $f \in Hom_A(A \otimes B_+^{\otimes m}, K)$ (a cocycle). However, $(\zeta \otimes \sigma^{\otimes m})^* f = 0$, simply because z has positive degree, and thus $\zeta_m^* f = (1 \otimes \check{\delta}_m)^* f$. In particular, under the identification $Hom_A(A \otimes B_+^{\otimes m}, K) = Hom_K(B_+^{\otimes m}, K)$, ζ_m^* becomes $\check{\delta}_m^*$.

The definition of $\check{\delta}_{n+m}$ makes it clear that for $f \in Hom(B_+^{\otimes n}, K)$ and $g \in Hom(B_+^{\otimes m}, K)$, $\check{\delta}_{n+m}^*(f \otimes g) = f \otimes \check{\delta}_m^*(g) + \check{\delta}_n^*(f) \otimes (\sigma^{\otimes m})^*(g)$. This proves that ζ^* is a σ^* -derivation, as claimed.

Since ζ^* is a σ^* -derivation of internal degree $+1$, It is now clear that ζ^* vanishes on $E^1(B) = E^{1,1}(B)$ and on $E^{2,2}(B)$ (since $E^{2,1}(B) = 0$). Suppose now that $[f] \in E^{2,q}(B)$ for some $q > 2$ and consider $\zeta^*[f] \in E^{2,q-1}(B)$. Define $g \in Hom(B_+, K)$ as follows: $g(B_1) = 0$ and for any $b_i, c_i \in B_+$, $g(\sum_i b_i c_i) = f(\sum_i (\delta(b_i) \otimes \sigma(c_i) + b_i \otimes \delta(c_i)))$. From the formula $\partial(\sum_i (\delta(b_i) \otimes \sigma(c_i) + b_i \otimes \delta(c_i))) = \delta(\sum_i b_i c_i)$ and the fact that f is a cocycle, it follows that g is well-defined. But then $\partial^* g = \check{\delta}_2^* f$, showing that $\zeta^*[f] = 0$, as required. \square

Theorem 10.2. *If B is a \mathcal{K}_2 algebra, then the graded Ore extension $A = B[z; \sigma, \delta]$ is also \mathcal{K}_2 .*

Proof. By the lemma, since B is \mathcal{K}_2 , the map $\zeta^* : E(B) \rightarrow E(B)$ is zero. The long exact sequence above becomes the short exact sequences

$$0 \rightarrow E^{q-1}(B)(1) \rightarrow E^q(A) \rightarrow E^q(B) \rightarrow 0$$

It is easy to see that the second map in this sequence is just the ring homomorphism α^* . Let $\check{z} \in E^{1,1}(A)$ span the kernel of $\alpha^* : E^1(A) \rightarrow E^1(B)$. Then the fact that α^* is surjective and has a kernel isomorphic to $E(B)(1)$ implies that \check{z} generates the kernel of α^* , i.e. our short exact sequence is encoding:

$$0 \rightarrow \check{z} \star E(A) \rightarrow E(A) \rightarrow E(B) \rightarrow 0.$$

It follows at once that A is \mathcal{K}_2 . \square

Remark 10.3. If we hypothesize that the map ζ^* of Lemma 10.1 is known to be zero, then $\alpha^* : E(A) \rightarrow E(B)$ is surjective. It follows then that B inherits the \mathcal{K}_2 property from A . However, the following example shows that the map ζ^* need not be zero. In this example, neither the algebra B

nor its Ore extension $B[z; \delta]$ are \mathcal{K}_2 . We do not know if the converse to 10.2 is true or false.

Example 10.4. Let B be the monomial algebra $K\langle x, y \rangle / \langle xyx, xy^2x, y^3 \rangle$. Let σ be the identity automorphism of B and let δ be the derivation of B extended from the formula $\delta(x) = 0$ and $\delta(y) = y^2$. The algebra B fails the \mathcal{K}_2 criterion of Section 5. Since B is a monomial algebra, each graded component B_n has a canonical monomial basis, and these determine monomial bases for the graded components of $B_+^{\otimes m}$. We define $f \in \text{Hom}(B_+^{\otimes 3}, K)_{-6}$ by:

$$f(xy^2 \otimes xy \otimes x) = f(xy \otimes yxy \otimes x) = f(x \otimes y^2xy \otimes x) = f(xy^2 \otimes x \otimes yx) = 1$$

and by insisting that f vanishes on all other monomials of degree 6. One checks easily that $\partial^* f = 0$, so that $[f]$ is a nonzero cohomology class in $E^{3,6}(B)$. By the lemma, $\zeta^*[f] = [\check{\delta}_3^* f] \in E^{3,5}(B)$. We calculate: $\check{\delta}_3^* f(xy \otimes xy \otimes x) = f(xy^2 \otimes xy \otimes x + xy \otimes xy^2 \otimes x) = 1$ and similarly $\check{\delta}_3^* f(x \otimes yxy \otimes x) = f(x \otimes y^2xy \otimes x + x \otimes yxy^2 \otimes x) = 1$ and $\check{\delta}_3^* f(xy \otimes x \otimes yx) = f(xy^2 \otimes x \otimes yx + xy \otimes x \otimes y^2x) = 1$. We see that $\check{\delta}_3^* f$ vanishes on all other monomials of degree 5. This assures that $[\check{\delta}_3^* f] \neq 0$ and shows that $\zeta^* : E^3(B) \rightarrow E^3(B)$ is not zero.

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