KOSZUL ALGEBRAS FROM GRAPHS AND HYPERPLANE ARRANGEMENTS

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1. *Introduction*

This work was started as an attempt to apply theory from noncommutative graded algebra to questions about the holonomy algebra of a hyperplane arrangement. We soon realized that these algebras and their deformations, which form a class of quadratic graded algebras, have not been studied much and yet are interesting to algebra, arrangement theory and combinatorics.

Let X be a topological space having homotopy type of a finite cell complex. Let $H_*(X)$ be the homology coalgebra with coefficients in a field and comultiplication dual to the cup product. Then the holonomy Lie algebra G_X of X is the quotient of the free Lie algebra on $H_1(X)$ over the ideal generated by the image of the comultiplication $H_2(X) \to \Lambda^2(H_1(X))$. The universal enveloping algebra $U(X)$ of G_X is called the holonomy algebra of X.

Holonomy algebras were introduced to arrangement theory by T. Kohno in [**14, 15**]. Let $\mathscr{A}$ be an arrangement over $\mathbb{C}$, that is, a set $\{H_1, \ldots, H_n\}$ of linear hyperplanes in a linear space $\mathbb{C}^l$. Let X be the complement of $\bigcup_{i=1}^n H_i$ in $\mathbb{C}^l$ and let $U(\mathscr{A}) = U(X)$. In [**14**], $U(\mathscr{A})$ is defined explicitly by generators and relations that can be obtained from the combinatorics of $\mathscr{A}$, see Section 4. Recall that there is another graded algebra defined by the combinatorics of $\mathscr{A}$, the Orlik–Solomon algebra $A(\mathscr{A})$ [**19**]. A well-known theorem of Brieskorn–Orlik–Solomon says $A(\mathscr{A})$ is isomorphic to $H^*(X, \mathbb{C})$.

In his papers, Kohno studied a complex, $\tilde{K}$ (the Aomoto–Kohno complex), of free modules over $U(\mathscr{A})$, defined by $K_p = \mathrm{Hom}_{\mathbb{C}}(A(\mathscr{A})_p, U(\mathscr{A}))$ for $p = 0, 1, \ldots$, and $K_{-1} = \mathbb{C}$. He proved the acyclicity of this complex for certain classes of reflection arrangements. He also proved that if this complex is acyclic then the Lower Central Series (LCS) formula holds:

$$P(-t) = \prod_{n \geqslant 1} (1 - t^n)^{\phi_n},$$

where $P(t)$ is the Poincare polynomial of X and the ϕ_n are the ranks of successive quotients in the lower central series of its fundamental group. The LCS formula was later extended to all supersolvable arrangements (equivalently fiber-type) by Falk and Randell [**8**] (see also [**13**]). The acyclicity of the complex for arbitrary supersolvable arrangements remained open in spite of attempts to prove it (for example, see [**12**] and the correction in [**13**]).

We begin our work with the simple but crucial observation that the algebra $U(\mathscr{A})$ is dual (in the sense of Koszul algebra theory) to the quadratic closure $\bar{A}(\mathscr{A})$ of $A(\mathscr{A})$.

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Notice that the algebras $U(\mathscr{A})$ and $A(\mathscr{A})$ are defined over an arbitrary field F and we substitute it for $\mathbb{C}$. Let T be the free graded F-algebra on a set of degree one generators $x_1, \ldots, x_n$. All of our graded F-algebras will be graded quotients of such a T. The Koszul dual of a quadratic graded F-algebra B is the quadratic graded F-algebra whose generating relations form an orthogonal complement in T_2 to the quadratic relations of B. This algebra is denoted by $B^!$. The algebra B is said to be Koszul if $B^!$ is isomorphic to the cohomology ring of the trivial graded B-module $F = B/B_{>0}$. An immediate implication of our observation above is that the Aomoto-Kohno complex $\tilde{K}$ is never exact if $A(\mathscr{A})$ is not quadratic. Moreover, in the quadratic case the exactness of $\tilde{K}$ is equivalent to $U(\mathscr{A})$ (equivalently $A(\mathscr{A})$) being a Koszul algebra.

To analyze the class of algebras $U(\mathscr{A})$ we use the idea of deformation theory, attempting to deform $U(\mathscr{A})$ into a simpler quadratic algebra. In particular, if there exists a monomial basis of $\bar{A}(\mathscr{A})$ such that the complementary set of monomials in the free algebra T forms an ideal generated in degree 2, then there is a nice deformation of $U(\mathscr{A})$ into an algebra in a class of quadratic algebras we call graph algebras.

A *graph algebra* is an algebra given by a collection of relations of the form $x_i x_j - q_{i,j} x_j x_i = 0$ for some pairs (i,j) where $q_{i,j} \in F^*$. Graph algebras seem, by themselves, to form an interesting class of quadratic algebras. Fröberg studied the Hilbert series of these algebras in [**11**] and proved that the algebras are Koszul. Although we need only this result for deformations of $U(\mathscr{A})$, we analyze the structure of these algebras more carefully. One outcome of this analysis is a simple combinatorial formula for the Hilbert series of graph algebras. It also provides another proof of Fröberg's theorem.

Now the success of the deformation method depends on the existence of a good monomial basis for the Orlik–Solomon algebras $A(\mathscr{A})$, as mentioned above. These bases were considered by Bjorner and Ziegler in [**5**]. As was shown there, every supersolvable arrangement has such a basis. Moreover, it is known that for supersolvable arrangements $\bar{A}(\mathscr{A}) = A(\mathscr{A})$. Using this, we give a deformation U_t of U($\mathscr{A}$) with the properties that $U_t \cong U(\mathscr{A})$ for $t \neq 0$ and U_0 is a graph algebra. Now an application of a theorem of Drinfeld [**6**] gives the Koszul property for $U(\mathscr{A})$, whenever $\mathscr{A}$ is a supersolvable arrangement.

Our presentation is outlined as follows. In Section 2 we recall several equivalent definitions of the Koszul property and some basic results about Koszul algebras. In Section 3 we study the structure of graph algebras and compute their Hilbert series giving algebraic meaning to some interesting combinatorial facts previously known from [**9, 10**], see Corollary 3.4. In Section 4 we return to the study of arrangements. Here we recall the basic definitions of the algebras associated to an arrangement and prove the two main results: $A(\mathscr{A})$ must be quadratic for the complex $\tilde{K}$ to be exact; and $U(\mathscr{A})$ is Koszul for all supersolvable arrangements. We conclude, in Section 5, with three examples of arrangements that are not supersolvable and which we analyze by more *ad hoc* arguments. The last of these examples provides an open question.

2. *Koszul algebras: preliminaries*

We collect in this section some of the basic results about Koszul Algebras, cf. [**4, 3, 18**]. Let F be a field and V an n-dimensional vector space over F. We let $T = T(V)$ denote the full F-tensor algebra over V. Choosing a basis $x_1, \ldots, x_n$ for V we can write $T \cong F\langle x_1, \ldots, x_n \rangle$, the free F-algebra on $x_1, \ldots, x_n$. We use the usual grading on T where $T_1 = V$ and $T_0 = F$. Fix an F-inner product on the space of 2-tensors $V \otimes V = T_2$. To ease the notation we shall usually assume that this is the standard inner product induced by the basis $x_1, \ldots, x_n$.

Fix a homogeneous ideal I of T and let $U = U(I)$ be the graded algebra T/I. We may assume that I contains no non-zero elements of degree 1 and we say that U is quadratic if I is generated, as an ideal, by its elements of degree 2. Since $T_1 \to U_1$ is an isomorphism, we identify these spaces and use $x_1, \ldots, x_n$ to denote a basis of the space.

Definition 2.1. Let $U = U(I)$ be a quadratic algebra. Let $I^!_2$ be the orthogonal complement to I_2 in $V \otimes V$ and $I^!$ the ideal of T generated by $I^!_2$. The quadratic algebra $U^! = U(I^!) = T/I^!$ is called the *Koszul dual* of U.

We observe at once that $(U^!)^! = U$.

Definition 2.2. Let $U = U(I)$ be a quadratic algebra and let ${}_UF$ be the trivial graded left U-module $U/U_{>0}$. The algebra U is said to be *Koszul* if ${}_UF$ admits a free graded resolution

$$\longrightarrow \cdots P^i \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow {}_UF \longrightarrow 0$$

such that P^i is generated by its component of degree i.

There are many equivalent ways of expressing this definition. The following theorem collects some of these variations. We denote by $E(U)$ the graded cohomology algebra $\mathrm{Ext}^*_U({}_UF, {}_UF)$. For any graded F-vector space M we denote the Hilbert Series of M by $H(M,t) := \sum_n \dim_F(M_n)t^n$. The Koszul complex of U is the sequence

$$\cdots \longrightarrow K_i \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 \longrightarrow {}_UF \longrightarrow 0$$

of free (except ${}_UF$) left U-modules and their homomorphisms, where $K_i = \mathrm{Hom}_F(U^!_i, U)$, and $d_i : K_i \to K_{i-1}$ is defined by $d_i f(a) = \sum_{k=1}^n f(x_k\, a)\, x_k$ for every $a \in U_{i-1}$.

Theorem 2.3. *Let $U = U(I)$ be a quadratic algebra. The following statements are all equivalent:*

(a) *U is Koszul,*
(b) *$U^!$ is Koszul,*
(c) *$E(U)$ is a quadratic F-algebra generated as an algebra in degree* 1,
(d) *$E(U) \cong U^!$,*
(e) *the Koszul complex of U is acyclic,*
(f) *$H(U,t) \cdot H(E(U), -t) = 1$.*

The various equivalences of the theorem can be found in [**3**, **4**, **18**]. Several more equivalent versions of the Koszul condition can also be found in these references.

Corollary 2.4. *If U is a Koszul algebra, then $H(U,t) \cdot H(U^!, -t) = 1$.*

Quite recently, Positselski [**24**] and Roos [**21**] have independently constructed examples showing that the converse of this corollary is false.

3. *Graph algebras*

Let F be a field. Let Γ be an edge-labelled graph (without loops or multiple edges) on a set of n vertices $V = \{1, 2, \ldots, n\}$ with a set E of edges. Each edge $\{i, j\}$ in E is labelled by a non-zero field element q_{ij}. We associate two F-algebras to Γ. Recall that $T = \bigoplus_{d \geqslant 0} T_d$, the free F-algebra on n generators $x_1, \ldots, x_n$, naturally graded. Define the ideal $I(\Gamma)$ of T as

$$I(\Gamma) = (x_i x_j - q_{i,j} x_j x_i \,|\, \{i, j\} \in E \text{ with } i < j).$$

One checks that

$$I(\Gamma)^! = I^!(\Gamma) = (x_i x_j, q_{k,l} x_k x_l + x_l x_k \,|\, \{i, j\} \notin E, \{k, l\} \in E \text{ with } k < l).$$

Notice that $x_i^2 \in I^!(\Gamma)$ for every i. Define

$$U(\Gamma) = T/I(\Gamma), \quad A(\Gamma) = U^!(\Gamma) = T/I^!(\Gamma).$$

Clearly both algebras are quadratic and dual to each other. Let $\Lambda(V)$ denote the exterior algebra on V over F. We note that $A(\Gamma)$ is a deformation of the factor algebra $\Lambda(V)/(x_i x_j \,|\, \{i, j\} \notin E \text{ with } i \neq j)$ and the two algebras have the same Hilbert series. In particular it follows, as for the exterior algebra, that $\dim_F A(\Gamma) \leqslant 2^n$.

EXAMPLES 3.1. 1. Let Γ be discrete, that is, $E = \varnothing$. Then $U(\Gamma) = T$ and $A(\Gamma) = F \oplus V$ with zero multiplication on V.

2. Let Γ be the complete graph K_n with all labels $q_{i,j} = 1$. Then $U(\Gamma) = F[x_1, \ldots, x_n]$ and $A(\Gamma) = \Lambda(V)$.

3. Let Γ be the complete bipartite graph $K_{k,l}$ (with $k + l = n$) and all the labels $q_{i,j} = 1$. Then $U(\Gamma) = T' \otimes T''$, where T' and T'' are free algebras on k and l generators, respectively. Also $A(\Gamma)$ is the exterior algebra on n generators with extra relations of products of generators for a fixed k-subset of generators and its complement. Equivalently, $A(\Gamma) = A(\Gamma') \underline{\otimes} A(\Gamma'')$, where Γ' and Γ'' are discrete graphs on k and l vertices respectively, and $\underline{\otimes}$ is the operation Koszul dual to tensor multiplication (*see* [**18**]).

4. Let $n = 2$ and assume that $\{1, 2\} \in E$. Set $q = q_{1,2}$. Then the algebra $U(\Gamma)$ is usually denoted by $F_q[x_1, x_2]$. This ring is often referred to as the quantum line as it can be realized as the twisted homogeneous coordinate ring of projective one-space. Alternatively one may think of this ring as the set of polynomials $F[x_1, x_2]$ with a new multiplicative structure, $\odot$, defined by $f(x_1, x_2) \odot g(x_1, x_2) = f(x_1, q^{-1}x_2)\, g(x_1, x_2)$ for all $f, g \in F[x_1, x_2]$.

We begin with a theorem due to Fröberg.

THEOREM 3.2. [**11**]. *The algebras $U(\Gamma)$ and $A(\Gamma)$ are Koszul.*

Our purpose in the remainder of this section is to make a more careful analysis of the internal structure of the algebras $U(\Gamma)$ and to give a combinatorial formula for the Hilbert series of the algebra $A(\Gamma)$ (and hence of $U(\Gamma)$). We must first set some notation.

For any subset J of the vertex set V, let Γ_J be the full subgraph of Γ on the set of vertices in J. Also, if v is a fixed vertex of Γ, then we set $C(v) = \{i \in V \,|\, \{i, v\} \in E\}$. We shall write Γ_v for the full subgraph of Γ on the set of vertices $V \backslash \{v\}$.

Finally, let $X = X(\Gamma)$ be the simplicial complex on $\{1, \ldots, n\}$ defined as follows: the set $K = \{i_1, \ldots, i_p\}$ is a $(p-1)$-simplex in X if and only if the unlabelled subgraph Γ_K of Γ is a complete graph (X is called the flag-complex of Γ, cf. [**22**]). Notice that Γ itself is the 1-skeleton of X. For any vertex $v \in V$ we have $X_v = X(\Gamma_v)$, the subcomplex in the vertices $V \backslash \{v\}$, and $X_{C(v)} = X(\Gamma_{C(v)})$, the link of the vertex v in X. For any simplicial complex X, let $F_p(X)$ be the number of $(p-1)$-simplices of X (with the convention that $F_0(X) = 1$). We denote the Euler characteristic polynomial of X by $E(X, t)$, that is, $E(X, t) = \sum_i (-1)^i F_i(X)\, t^i$.

THEOREM 3.3. *Let Γ be an edge labelled graph as above and let J be any subset of the vertices of Γ. Then*

1. *the canonical ring homomorphism $U(\Gamma_J) \to U(\Gamma)$ is injective, that is, the set of generators x_i for $i \in J$ generate a quadratic subalgebra of $U(\Gamma)$ isomorphic to $U(\Gamma_J)$;*
2. *$U(\Gamma)$ is free as a left or right $U(\Gamma_J)$-module;*
3. *for any vertex $v \in V$, we have a canonical short exact sequence of $U(\Gamma)$-modules*

$$0 \longrightarrow \left(U(\Gamma) \underset{U(\Gamma_{C(v)})}{\otimes} F \right)[-1] \longrightarrow U(\Gamma) \underset{U(\Gamma_v)}{\otimes} F \longrightarrow F \longrightarrow 0,$$

where F denotes the one-dimensional graded trivial module concentrated in degree zero (over the appropriate ring).

Fröberg's Theorem tells us that $H(U(\Gamma), t)\, H(A(\Gamma), -t) = 1$ for any edge labelled graph Γ, including Γ_v and $\Gamma_{C(v)}$. Using this, Parts (2) and (3) of the structure theorem above yield the functional equation

$$H(A(\Gamma), -t) = H(A(\Gamma_v), -t) - tH(A(\Gamma_{C(v)}), -t).$$

But this is exactly the same as the functional equation satisfied by the polynomials $E(X(\Gamma), t)$. By induction on the number of vertices of Γ we get the following corollary.

COROLLARY 3.4. *$H(A(\Gamma), -t) = E(X, t)$ and $H(U(\Gamma), t) = 1/E(X, t)$.*

In particular, Corollary 3.4 claims that all the coefficients of the series $1/E(X, t)$ are non-negative. This fact has been known to combinatorialists for some time [**9, 10, 22**]; it has beautiful combinatorial corollaries like Turán's theorem for triangles [**22**]. Corollary 3.4 can be viewed as an algebraic justification of this fact.

Also, applying the algebraic mapping cone from [**17**] to the exact sequence from Theorem 3.3(3) one can obtain another proof of Fröberg's theorem by induction on n.

We now turn to the proof of Theorem 3.3. To simplify the exposition we assume for the remainder of this section that all of the labels $q_{i,j}$ are 1, that is, we consider only relations in $U(\gamma)$ of the form $x_i x_j - x_j x_i$. We leave it to the reader to make the minor technical adjustments needed to extend to the more general case.

We shall apply to the free algebra T the usual terminology from polynomial rings. For instance, each element a of T is the uniquely defined sum (with nonzero coefficients) of monomials. These monomials form the support $S(a)$ of a. If $S(a)$ has only two elements, then we say that a is a binomial.

The fact that graph algebras form a manageable class is based on the following simple observation.

LEMMA 3.5. *Let $I = I(\Gamma)$. If $a \in I$ then $a = \sum a_i$, where each a_i is a binomial from I and $S(a) = \bigcup_i S(a_i)$. In particular, if $\mathscr{B}$ is any set of monomials of T, composed of at most one element from each nontrivial coset of I, then the image of $\mathscr{B}$ in $U = T/I$ under the standard projection is F-linearly independent.*

Proof. To write a given $a \in I$ as $a = \sum a_i$, where each a_i is a binomial in I, is a triviality, since I is generated by binomials. Moreover, the containment $S(a) \subset \bigcup S(a_i)$ is clear. It is the opposite containment that is not automatically true. Among all possible representations $a = \sum a_i$, choose one with a minimal number of terms, say k.

Let us prove that $S(a_k) \subset S(a)$. Suppose not. Choose a monomial $\mu \in S(a_k)$ with $\mu \notin S(a)$. For each i with $1 \leqslant i \leqslant k$, let r_i be the coefficient of μ in the binomial a_i (of course $r_i = 0$ unless μ is one of the two elements of $S(a_i)$). Since $\mu \notin S(a)$ we must have $\sum_i r_i = 0$. Define $b_i = a_i - r_i(r_k)^{-1}a_k$ for $1 \leqslant i \leqslant k-1$. By construction, each b_i is a binomial in I and $a = \sum_{i=1}^{k-1} b_i$. This contradicts the minimality of the representation $a = \sum_i a_i$. This proves the first claim of the lemma. The second claim is then clear.

Now we need to fix a specific monomial basis of $U = U(\Gamma)$. To do this we impose a total order on the monomials of T as follows. Let π be the standard projection $T \to F[x_1, \ldots, x_n]$. We order the monomials of $F[x_1, \ldots, x_n]$ first by degree and then by the inverse lexicographic order. Now if $\pi(\mu_1) \neq \pi(\mu_2)$ for monomials μ_1 and μ_2 from T, we say that $\mu_1 < \mu_2$ whenever $\pi(\mu_1) < \pi(\mu_2)$. If, on the other hand, $\pi(\mu_1) = \pi(\mu_2)$, we use the inverse lexicographic order. Now we say that a monomial μ of T is *standard* if μ is minimal among all the monomials in the coset $\mu + I$. We shall identify standard monomials with their images in U. By Corollary 3.6, standard monomials form an F-basis of U that we call the *standard basis*.

We can easily check whether two monomials of T are in the same I-coset. Fix a pair of indices (i,j) with $1 \leqslant i < j \leqslant n$. If $\{i,j\} \notin E$, then define the ring epimorphism $\pi_{i,j}: T \to F\langle x_i, x_j\rangle$ by evaluation of x_k at 1 for every $k \neq i$ or j. If $\{i,j\} \in E$ then define $\pi_{ij}: T \to F[x_i, x_j]$ similarly. Either way, the map $\pi_{i,j}$ factors through $U(\Gamma) = T/I$.

LEMMA 3.6. *Let m_1 and m_2 be monomials in T. Then $m_1 - m_2 \in I(\Gamma)$ if and only if $\pi_{ij}(m_1 - m_2) = 0$ for every pair (i,j) with $1 \leqslant i < j \leqslant n$.*

Proof. Suppose that $\pi_{ij}(m_1 - m_2) = 0$ for every pair (i,j) but $m_1 - m_2$ is not in $I(\Gamma)$. Changing each monomial in its I coset, one can assume that they are both standard monomials. Write the monomials as $m_1 = a_1 x_i b$ and $m_2 = a_2 x_j b$, where a_1, a_2, and b are some monomials of T and $i \neq j$. One can assume that $i > j$. Then applying the condition for (j,i) one can write $m_1 = a x_j x_{i_1} \cdots x_{i_k} x_i b$ for some monomial a and some integers k, i_1, ..., i_k. Applying the condition for the pairs j, i_r (with $r = 1, \ldots, k$) and (i,j) one sees that all these pairs belong to E, whence m_1 is in the same I-coset as the monomial $m_3 = a x_{i_1} \cdots x_{i_k} x_i x_j b$. Since $m_3 < m_1$ this contradicts the assumption that m_1 is standard.

Now we can prove the first claim of Theorem 3.3. Fix a subset J of the vertices of Γ and let $U_J(\Gamma)$ be the subalgebra of $U(\Gamma)$ generated by the x_i for $i \in J$. Let T_J be the subalgebra of T generated by the x_i for $i \in J$. The image of $U(\Gamma_J)$ in $U(\Gamma)$ is just $T_J/(I(\Gamma) \cap T_J)$, so it suffices to see that $T_J \cap I(\Gamma) = I(\Gamma_J)$. By Lemma 3.5, it suffices to consider binomials in $T_J \cap I(\Gamma)$. But then by Lemma 3.6, applied to $I(\Gamma_J)$, it is clear that any such binomial must be in $I(\Gamma_J)$.

We shall now identify $U(\Gamma_J)$ with its image in $U(\Gamma)$. To prove the second claim of Theorem 3.3 we fix a vertex v of Γ. By transitivity and induction, it suffices to prove that $U(\Gamma)$ is free as a right $U(\Gamma_v)$-module.

Reorder the vertices of Γ so that $v = n$, the last vertex. Notice that the ordering we have defined on the monomials in $T_n = T_{V\setminus\{n\}}$ is the same as the ordering inherited from the ordering of monomials in T. Let $\mathscr{B}$ be the set $\{1\}$ union with the set of all standard monomials in T of the form ax_n, where a is some monomial. If ax_n is in $\mathscr{B}$ and b is a monomial in T_n, then $ax_n b$ is a standard monomial in T if and only if b is a standard monomial in T_n. Applying Lemma 3.5 to both Γ_n and Γ, it follows that the image of $\mathscr{B}$ in $U(\Gamma)$ is a basis for $U(\Gamma)$ as a right $U(\Gamma_n)$-module.

Finally, we consider the short exact sequence in Part 3 of Theorem 3.3. We may still assume that $v = n$. Let K be the kernel of the canonical left module homomorphism $U(\Gamma)\otimes_{U(\Gamma_n)} F \to F$. The module K is graded and cyclic and generated in degree one by the tensor $x_n \otimes 1_F$. We need only to compute the annihilator of this tensor.

Let m be any standard monomial in T. Then there are two exclusive possibilities for mx_n: either mx_n is also a standard monomial, or mx_n is not a standard monomial, in which case $mx_n \in m'x_j + I(\Gamma)$ for some monomial $m'x_j$ where $j \neq n$. In the latter case, by Lemma 3.6, $\pi_{j,n}(mx_n - m'x_j) = 0$ which can only happen if $j \in C(n)$. Thus, either mx_n is standard or m is in $\sum_{j\in C(n)} Tx_j + I(\Gamma)$.

Let $a \in U(\Gamma)$ be in the annihilator of $x_n \otimes 1_F$. Write $a = \sum_\alpha a_\alpha \overline{m}_\alpha$, where $a_\alpha \in F$ and $\overline{m}_\alpha$ is the image in $U(\Gamma)$ of a standard monomial m_α in T. Then $ax_n = \sum_\alpha a_\alpha \overline{m}_\alpha x_n$ in $U(\Gamma_n)^+$. Since the images of the standard monomials form an F-basis, this can only happen if $a_\alpha = 0$ for every α such that $m_\alpha x_n$ is a standard monomial. Thus $a \in \sum_{j\in C(n)} U(\Gamma)x_j$ and the annihilator of $x_n \otimes 1_F$ is therefore exactly $\sum_{j\in C(n)} U(\Gamma)x_j$. This proves that K is isomorphic to $(U(\Gamma)\otimes_{U(\Gamma_{C(v)})} F)\,[-1]$ and thus completes the proof of Theorem 3.3.

4. *Holonomy algebras of arrangements*

In this section we are concerned with certain quadratic algebras related to an arrangement of hyperplanes.

Let $\mathscr{A} = \{H_1 \ldots, H_n\}$ be a set of $(l-1)$-dimensional linear subspaces of an l-dimensional linear space V over a field F. We fix linear functions α_i such that $\ker \alpha_i = H_i$ and call a subset of $\mathscr{A}$ independent if the respective set of functionals is linearly independent. The collection of minimal dependent subsets of $\mathscr{A}$ (*circuits*) forms a matroid $\mathscr{M}$. From the point of view of matroid theory $\mathscr{A}$ is a representation of $\mathscr{M}$ over F. In fact most constructions in this section depend only on $\mathscr{M}$ and not on its representation $\mathscr{A}$.

Associated with $\mathscr{A}$, is the well-known Orlik-Solomon algebra $A(\mathscr{A}) = A$ [**19**]. A slightly unusual definition of A is as follows. Recall from the previous sections that T is the free F-algebra on generators $\{x_1, \ldots, x_n\}$ and denote by $J(\mathscr{A}) = J$ the ideal of T generated by $x_i^2, x_i x_j + x_j x_i$ for every $1 \leqslant i < j \leqslant n$, and

$$\sum_{j=1}^{k} (-1)^{j-1} x_{i_1} \cdots x_{i_{j-1}} x_{i_{j+1}} \cdots x_{i_k}$$

for all dependent subsets $\{H_{i_1}, \ldots, H_{i_k}\}$ of $\mathscr{A}$. Then $A = T/J$. One can easily see that $A_p = 0$ for $p > l$.

The algebra A is not necessarily quadratic. First of all, for it to be quadratic $\mathscr{A}$ must be formal, that is, all the linear relations among the functionals of H_i should be linearly generated by relations among triples of them. Indeed if $\mathscr{A}$ is not formal then there exists a formal arrangement $\bar{\mathscr{A}}$ such that $J(\bar{\mathscr{A}}) \subset J(\mathscr{A})$ and $J(\bar{\mathscr{A}})_2 = J(\mathscr{A})_2$ but $J(\bar{\mathscr{A}}) \neq J(\mathscr{A})$ (cf. [**23**]). But even formal arrangements do not in general produce a quadratic algebra A. A necessary condition for a formal arrangement to have a quadratic Orlik-Solomon algebra is contained in [**7**]. It is proved there, in particular, that for all the reflection arrangements of type D_k with $k \geqslant 3$, the algebras A are not quadratic.

It is easy to construct a quadratic algebra that is, in a way, the quadratic closure of A. This is the algebra $\bar{A} = \bar{A}(\mathscr{A}) = T/\bar{J}$, where $\bar{J}$ is the ideal of T generated by J_2. Notice that $\bar{A}$ is a finite-dimensional algebra since it is a factor of the exterior algebra on n generators. It is also graded since $\bar{J}$ is homogeneous but, unlike A, it can have graded components of degree higher than l. Another algebra associated with $\mathscr{A}$ is the algebra $U = U(\mathscr{A})$ studied by Aomoto and Kohno [**1, 14, 15**] that is the universal enveloping algebra of the holonomy Lie algebra of the complement of $\bigcup H_i$ in V (for $F = \mathbb{C}$). The explicit description of U (over an arbitrary field F) is as follows. Let $I(\mathscr{A}) = I$ be the ideal of T generated by $[x_i, \sum_{j\in X} x_j]$ for every i and every maximal $X \subset \{1, \dots, n\}$ such that $i \in X$ and $\bigcap_{j\in X} H_j$ has codimension 2 in V. Here we put $[a, b] = ab - ba$ for $a, b \in T$. Then $U = T/I$. The following simple observation has initiated this work.

LEMMA 4.1. *For every arrangement $\mathscr{A}$ we have $U(\mathscr{A}) = (\bar{A}(\mathscr{A}))^!$.*

Now we define a complex K_* of free left U-modules (the Aomoto-Kohno complex). For every $p \geqslant 0$ put $K_p = \operatorname{Hom}_F(A_p, U)$ and define $d_p\colon K_p \to K_{p-1}$(for $p = 1, \dots, l$) via

$$d_p f(a) = \sum_{i=1}^{n} f(x_i\, a)\, x_i$$

for every $f \in \operatorname{Hom}_F(A_p, U)$ and $a \in A_{p-1}$. Clearly $\operatorname{Im} d_1 = U_+$, whence K_* can be augmented on the right by the canonical map $d_0\colon U \to {}_U F$. In exactly the same manner one can construct a complex $\bar{K}_*$ using $\bar{A}$ instead of A.

The natural question about K_* is whether this complex is exact. Kohno proved the exactness for the reflection arrangements of types A_k in [**15**] and claimed it for types C_k, D_k, G_2, and $I_2(p)$ in his unpublished but often cited paper [**16**]. Since the arrangements of the first two types are supersolvable, the result for them also follows from the main theorem of this section. The following proposition shows that it cannot be true for D_k with $k > 3$.

PROPOSITION 4.2. *If K_* is exact then A is quadratic.*

Proof. Suppose that A is not quadratic and $\bar{A}_i = A_i$ for $i = 0, 1, \dots, p-1$, while $\bar{A}_p \neq A_p$. If K_* is not exact in some dimension less than $p-1$ then the result is proved. Suppose that K_* is exact in all dimensions less than $p-1$. It suffices to prove that K_* is not exact in K_{p-1}.

Notice that $\bar{K}_* = K_*$ up to dimension $p-1$. Suppose that K_* is exact in dimension $p-1$, that is, $\operatorname{Im} d_p = \ker d_{p-1}$. Denote by $\bar{d}$ the differential $\bar{K}_p \to \bar{K}_{p-1}$ of $\bar{K}_*$. Since

$\bar{J} \subset J$ we have a surjective graded homomorphism $\bar{A} \to A$ that allows us to view K_p as a subspace of $\bar{K}_p$. Besides, $d|_{K_p} = d_p$. Thus the exactness assumption implies that

$$\operatorname{Im} d = \operatorname{Im} d_p. \tag{1}$$

On the other hand, since $\bar{A}_p \neq A_p$ there exists a nonzero map $f: \bar{A}_p \to F = U_0$ such that $f \notin K_p$. Notice that $\deg f = 0$ and (1) implies that there exists $g \in K_p$ such that $f - g \in \ker d_p$. But it is easy to see from definition of d_p that $\ker d_p$ cannot have nonzero elements of degree 0. Thus $f = g$ which contradicts the choice of f. This contradiction completes the proof.

Now we focus our attention on the complex $\bar{K}_*$. It is clear from definition and Lemma 4.1 that this complex is the usual Koszul complex for the algebra U, whence this complex is exact if and only if U is Koszul. In the rest of the section we prove that U is Koszul for supersolvable arrangements.

There are many different ways to characterize supersolvable arrangements (cf. [**19**]). The best suitable definition for our goal is the one given by Bjorner and Ziegler in [**5**]. First let us recall that for any (ordered) arrangement $\mathscr{A}$ one can exhibit a specific monomial F-basis of $A = A(\mathscr{A})$ called the broken circuit basis. A circuit is a sequence of hyperplanes such that their functionals form a minimal dependent set. A broken circuit is a sequence $(H_{i_1}, \ldots, H_{i_p})$ such that $i_1 < \cdots < i_p$ and $(H_{i_1}, \ldots, H_{i_p}, H_j)$ is a circuit for some $j > i_p$. Now the broken circuit basis is formed by the set of monomials $x_{i_1} \cdots x_{i_p}$ such that $i_1 < \cdots < i_p$ and the respective sequence of hyperplanes does not contain any broken circuit. Finally an arrangement $\mathscr{A}$ is supersolvable if every minimal broken circuit consists of two hyperplanes.

It follows from [**7, 8**] that for a supersolvable arrangement the algebra A is quadratic, that is, $\bar{A} = A$. Since the proof there involves rational homotopy theory, we give a direct elementary proof below.

LEMMA 4.3. *If $\mathscr{A}$ is supersolvable then $A = A(\mathscr{A})$ is quadratic.*

Proof. Define the F-linear map $d: T \to T$ via

$$d(x_{i_1} \cdots x_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_1} \cdots x_{i_{j-1}} x_{i_{j+1}} \cdots x_{i_p}.$$

Clearly $d^2 = 0$ and $d(ab) = (da)b + (-1)^p a db$ for $a \in T_p$ and $b \in T$. Since all generators of $\bar{J}$ are annihilated by d, the ideal $\bar{J}$ is invariant with respect to d. Recall from the beginning of this section that only generators of J of degree different from 2 have form $d(\mu_S)$, where $\mu_S = x_{i_1} \cdots x_{i_p}$ for a dependent sequence $S = (H_{i_1}, \ldots, H_{i_p})$ with $p > 3$.

Suppose now that A is not quadratic, that is, $\bar{J} \neq J$. Then the previous paragraph implies that there exists a monomial $\mu = \mu_S \notin \bar{J}$, where the sequence S is dependent. Without any loss of generality, we can assume that μ is maximal in the reverse lexicographic order among all the monomials of degree p with these properties. Clearly some subsequence of S is a broken circuit. Since $\mathscr{A}$ is supersolvable there exist i_r and i_s with $1 \leqslant r < s \leqslant p$ and such that $S_0 = (i_r, i_s, u)$ is a circuit for some u with $u > i_s$. Since S_0 is dependent we have

$$\mu_0 = x_{i_r} x_{i_s} x_u = (d\mu_0) x_u \in J_2 \subset \bar{J}.$$

If H_u is an element of S then μ_0 divides μ modulo $\bar{J}$, whence $\mu \in \bar{J}$. That is a contradiction. Suppose that u is not among the i_k for $k = 1, \ldots, p$. Then consider two other monomials μ' and μ'' substituting x_u for x_{i_r} and x_{i_s} respectively. Notice that μ', $\mu'' > \mu$ in the reverse lexicographic order and the respective sequences of hyperplanes are still dependent. Thus by choice of μ we have μ', $\mu'' \in \bar{J}$. But, modulo $\bar{J}$, the monomial μ is a linear combination (with coefficients ± 1) of μ' and μ'', whence again $\mu \in \bar{J}$. This contradiction completes the proof.

Now we want to deform U to a graph algebra. To do this, define the graph $\Gamma = \Gamma(\mathscr{A})$ on the vertices $\{1, 2, \ldots, n\}$ whose edges are exactly those 2-sets $\{i, j\}$ for which (H_i, H_j) is a broken circuit. Label every edge by 1. Then put $\tilde{A} = \tilde{A}(\mathscr{A}) = A(\Gamma)$. For every $\lambda \in F$ put $A_\lambda = A_\lambda(\mathscr{A}) = \Lambda / J_\lambda$ where Λ is the exterior algebra on n generators (as above) and J_λ is its ideal generated by the relations

$$x_i x_j - \lambda^{k-j} x_i x_k + \lambda^{k-i} x_j x_k$$

for every 3-circuit (H_i, H_j, H_k) with $i < j < k$. Let us sum up obvious properties of these algebras.

LEMMA 4.4. *With the above notation*,

(i) $A_1 = A$, $A_0 = \tilde{A}$,

(ii) *for every $\lambda \neq 0$ the algebra homomorphism defined by $x_i \mapsto \lambda^i x_i$ is an isomorphism of A onto A_λ.*

Our goal is to apply the Drinfeld theorem [**6**] to the family A_λ (cf. also [**20**]). First we need the following definition. If N is a natural number we call a quadratic algebra N-Koszul if its Koszul complex is exact in the first N terms from the right.

The formal distinction of our case from the main theorem of [**6**] is that the parameter λ is not real but belongs to the field F. However, using the Zariski topology it is easy to obtain the following form of the theorem.

THEOREM 4.5. *Suppose that for every $\lambda \in F$ we have a quadratic algebra A_λ whose quadratic relations depend on λ polynomially. Suppose that* $\dim (A_\lambda)_i$ *does not depend on λ for $i = 1$, 2, and* 3 *and A_0 is Koszul. Then for every natural number N there exists a Zariski open subset $W_N \subset F$ containing* 0 *and such that for every $\lambda \in W_N$ the algebra A_λ is N-Koszul.*

Now we are ready to prove the main result of this section.

THEOREM 4.6. *If an arrangement $\mathscr{A}$ is supersolvable then the algebras $A(\mathscr{A})$ and $U(\mathscr{A})$ are Koszul.*

Proof. It suffices to prove the statement for $A(\mathscr{A})$. Let us check the conditions of Theorem 4.5. By Theorem 3.2, $A_0 = \tilde{A}$ is Koszul. Consider the idea I_0 of Λ defining this algebra. The monomials of Λ contained in this ideal are those containing submonomials corresponding to broken circuits of length 2. Since $\mathscr{A}$ is supersolvable this is equivalent to containing any broken circuits. Thus due to the broken circuit basis theorem $H(\tilde{A}, t) = H(A, t) = H(A_\lambda, t)$ for every λ.

Now extend F to an infinite field if necessary. It follows from Theorem 4.5 that for every positive integer N there exists $\lambda_N \neq 0$ such that A_{λ_N} is N-Koszul. Now Part (ii) of Lemma 4.4 implies that A is Koszul.

COROLLARY 4.7. *If an arrangement $\mathscr{A}$ is supersolvable then its Aomoto-Kohno complex is exact.*

This result would have followed from a theorem in [**12**] but that theorem is false (see [**13**] for corrections). The following corollary was first proved in [**8**] using rational homotopy theory.

COROLLARY 4.8. *If an arrangement $\mathscr{A}$ is supersolvable then*

$$H(U(\mathscr{A}),t)\,H(A(\mathscr{A}),-t)=1.$$

5. *Examples*

In this section we consider three examples of non-supersolvable arrangements (or rather matroids). The first two arrangements have non-quadratic algebras A. The last example is quadratic. No example of a non-supersolvable arrangement with a Koszul algebra U is known to us. The last two examples are as close to that as we can find. To simplify notation we identify an arrangement with the respective set of linear functionals.

EXAMPLE 5.1. Let $\operatorname{char} F \neq 2$ and $\mathscr{A}=\{x,y,z,x+y,x+z,y+z\}$.

This is a formal arrangement whose algebra A is non-quadratic (and is the smallest such); here

$$H(A,t)=1+6t+12t^2+7t^3,\quad H(\bar{A},t)=1+6t+12t^2+8t^3+t^4.$$

The algebra U is not Koszul since some of the coefficients of the series $1/(1-6t+12t^2-8t^3+t^4)$ are negative, contradicting Corollary 2.4. (Note: the first negative coefficient occurs at t^{13}.)

EXAMPLE 5.2. Again $\operatorname{char} F \neq 2$ and $\mathscr{A}=\{z,x+y,x-y,x+z,x-z,y+z,y-z\}$. This is a representation of the celebrated non-Fano matroid.

The algebra A is again non-quadratic; here

$$H(A,t)=1+7t+15t^2+9t^3,\quad H(\bar{A},t)=1+7t+15t^2+10t^3+t^4.$$

The series $1/(1-7t+15t^2-10t^3+t^4)$ has all positive coefficients since the denominator has four positive real roots. There exist at least two different kinds of deformations $A(\lambda)$ of $\bar{A}$ (that is, $A(1)=\bar{A}$) such that $H(A(\lambda),t)=H(\bar{A},t)$ for all $\lambda\in F$. For the first kind, all algebras $A(\lambda)$ with $\lambda\neq 0$ are isomorphic to $\bar{A}$ but $A(0)$ is not Koszul. For the second kind, $A(0)$ is Koszul but the $A(\lambda)$ are not isomorphic to $\bar{A}$ anymore.

To describe the deformations of the second kind more explicitly notice that $\bar{A}$ is the quotient of the exterior algebra with generators $x_1,\dots,x_7$ (in the given order of the functionals) over the ideal generated by the six elements

$$R_1=x_1x_4-x_1x_5+x_4x_5,\quad R_2=x_1x_6-x_1x_7+x_6x_7$$
$$R_3=x_2x_4-x_2x_7+x_4x_7,\quad R_4=x_2x_5-x_2x_6+x_5x_6,$$
$$R_5=x_3x_4-x_3x_6+x_4x_6,\quad R_6=x_3x_5-x_3x_7+x_5x_7.$$

Considering the dual algebra $U=\bar{A}^!$ one notices that $z=x_1+\cdots+x_7$ is a central element. (A similar fact is true for any arrangement.) Thus changing the generators to $x_1,\dots,x_6,z$, one can make the identification $U=W\otimes F[z]$, where W is the subalgebra of U generated by $x_1, \ldots, x_6$. (In fact, W is the holonomy algebra of the affine arrangement induced in the hyperplane $x_7 = 1$.) Since U is Koszul if and only if W is Koszul we shall focus on W. The algebra $B = W^!$ is the quotient of the exterior algebra on the generators $x_1, \ldots, x_6$ over the ideal generated by the relations R_1, R_4, R_5 and also by $R_2' = x_1 x_6$, $R_3' = x_2 x_4$, and $R_6' = x_3 x_5$. Notice that

$$H(B,t) = H(\bar{A},t)/(1+t) = 1+6t+9t^2+t^3.$$

For every $\lambda \in F$ define $B(\lambda)$ by the last 3 relations and by

$$R_1(\lambda) = x_1 x_4 - \lambda x_1 x_5 + \lambda x_4 x_5, \quad R_4(\lambda) = x_2 x_5 - \lambda x_2 x_6 + \lambda x_5 x_6,$$
$$R_5(\lambda) = \lambda x_3 x_4 - x_3 x_6 + x_4 x_6.$$

Clearly $B(1) = B$. A straightforward computation shows that $H(B(\lambda),t)$ does not depend on λ.

Now we want to prove that the algebra $B(0)$ is Koszul. This is equivalent to its dual algebra $W(0) = B(0)^!$ being Koszul. Notice that $W(0)$ has the nine defining relations:

$$[x_1,x_2], \quad [x_1,x_3], \quad [x_1,x_5], \quad [x_2,x_3], \quad [x_2,x_6], \quad [x_3,x_4], \quad [x_4,x_5],$$
$$[x_5,x_6], \quad [x_3+x_4,x_6].$$

The Koszul complex of $W(0)$ has the form

$$0 \longrightarrow K_3 \longrightarrow K_2 \longrightarrow K_1 \longrightarrow K_0 \longrightarrow F \longrightarrow 0$$

and it is exact (as for every algebra) in terms K_0 and K_1. So to prove that it is exact it suffices to prove that the kernel of the map $\delta_2 : K_2 \to K_1$ is generated in degree 1. The map δ_2 can be represented by the following matrix

$$M = \begin{bmatrix} x_2 & -x_1 & 0 & 0 & 0 & 0 \\ x_3 & 0 & -x_1 & 0 & 0 & 0 \\ x_5 & 0 & 0 & 0 & -x_1 & 0 \\ 0 & x_3 & -x_2 & 0 & 0 & 0 \\ 0 & x_6 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & x_4 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & x_5 & -x_4 & 0 \\ 0 & 0 & 0 & 0 & x_6 & -x_5 \\ 0 & 0 & x_6 & x_6 & 0 & -x_3-x_4 \end{bmatrix}$$

that acts on the row-vectors from $K_2 = W(0)^9$ via the right multiplication. Denote the rows of M by $r_1, \ldots, r_9$. Then a row-vector $a = (a_1, \ldots, a_9) \in K_2$ belongs to the kernel of δ_2 if and only if the vector

$$\sum_{i=1}^{9} a_i r_i = 0 \qquad (*)$$

(in $K_1 = W(0)^6$).

Now we need a lemma.

LEMMA 5.3. *Let* $x, y, z \in W(0)$.

(i) *If* $xx_1 + yx_4 + zx_6 = 0$, *then* $x = y = z = 0$.

(ii) *If* $xx_1 + yx_3 + zx_6 = 0$, *then there exists* $u \in W(0)$ *such that* $x = ux_3, y = -ux_1$, *and* $z = 0$.

Proof. We can assume that x, y, z are homogeneous of a common degree d and apply induction on d. If $d = 0$ the result is obvious. We shall suppose that $d > 0$

and prove (i). The condition implies that $b = (x, 0, 0, y, 0, z)$ belongs to the kernel of $\delta_1: K_1 \to K_0$. Thus there exist elements $b_1, \dots, b_9$ of $W(0)_{d-1}$ such that $b = \sum_{i=1}^{9} b_i r_i$ in $W(0)^6$. In particular,

$$-b_1 x_1 + b_4 x_3 + b_5 x_6 = 0, \quad b_8 x_6 - b_7 x_4 - b_3 x_1 = 0, \quad -b_2 x_1 - b_4 x_2 + b_6 x_4 + b_9 x_6 = 0.$$

Using (i) and (ii) for $d-1$ we obtain from the first two inequalities that $b_3 = b_7 = b_8 = 0$, $b_1 = ux_3$, $b_4 = ux_1$, and $b_5 = 0$ for some $u \in W(0)$. Substituting this into the third equality we have

$$-(b_2 + ux_2) x_1 + b_6 x_4 + b_9 x_6 = 0.$$

Using again (i) for $d-1$ and the absence of zero divisors in $W(0)$ (see for example [**2**]), we have $b_1 = ux_3$, $b_2 = -ux_2$, $b_4 = ux_1$, and the other b_i vanish. Computing the vector b we obtain $x = y = z = 0$. The proof of (ii) is similar.

Now we can finish the computation of the kernel of δ_2. Using Lemma 5.3, from (*) we have $a_3 = a_5 = a_7 = a_8 = 0$ and $a_1 = ux_3$, $a_4 = ux_1$ for some $u \in W(0)$. Then using the absence of zero divisors again, we obtain $a_2 = -ux_2$ and $a_6 = a_9 = 0$. Thus $a = u(x_3, -x_2, 0, x_1, 0, \dots, 0)$ which completes the proof.

EXAMPLE 5.4. For this example it is more instructive to describe the matroid itself. This matroid is known in geometry as the plane of order 3. It can be given on nine elements $\{1, 2, \dots, 9\}$ as the collection of 3-circuits

$$\mathscr{X} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\}, \{1,5,9\}, \{2,6,7\}, \{3,4,8\}, \{1,6,8\}, \{2,4,9\}, \{3,5,7\}\}$$

and all the 4-sets are dependent. It can be represented over any field having a primitive cubic root of 1 (see [**5**]).

Let us first prove that for this matroid A is quadratic. Notice that every two elements i and j uniquely define a third one $k = \phi(i,j)$ such that $\{i,j,k\}$ is a circuit. Now fix an arbitrary 4-circuit $S = \{i_1, i_2, i_3, i_4\}$ (with the natural order) and consider six elements $k_{r,s} = \phi(i_r, i_s)$ for $1 \leqslant r < s \leqslant 4$. Since S is a circuit, none of $k_{r,s}$ belongs to S. Thus there exists a partition of S in two pairs (without any loss of generality $\{i_1, i_2\}$ and $\{i_3, i_4\}$) such that $k_{1,2} = k = k_{3,4}$. Now one easily checks that

$$R_S = R_{X_1}(x_{i_3} - x_{i_4}) + (x_{i_1} - x_{i_2}) R_{X_2}$$

in T, where $X_1 = (i_1, i_2, k)$, $X_2 = (k, i_3, i_4)$ and R_Z is the element of J corresponding to an ordered circuit Y. This implies that $R_S \in \bar{J}$ and A is quadratic.

We have $H(A,t) = (1+t)(1+4t)^2$, in particular all coefficients of the series

$$1/H(A, -t)$$

are positive. Using the rooted complex RC constructed in [**5**, Example 4.1(4)], it is easy to exhibit a family of quadratic algebras $A(\lambda)$ (with $\lambda \in F$) such that $A(1) = A$, $A(0) = A(RC_1)$, and $H(A(\lambda), t)$ does not depend on λ. To be more explicit we need to recall that the root complex is defined on the elements of the matroid and its 1-skeleton includes exactly two 2-subsets of each $X \in \mathscr{X}$. The omitted 2-subsets, written in the order of elements of $\mathscr{X}$ above, are $\{1,3\}$, $\{4,6\}$, $\{7,9\}$, $\{1,7\}$, $\{2,8\}$, $\{3,9\}$, $\{1,9\}$, $\{2,6\}$, $\{4,8\}$, $\{6,8\}$, $\{2,4\}$, $\{3,7\}$. Now $A(\lambda)$ is the quotient of the exterior algebra with nine generators $x_1, \ldots, x_9$ over the ideal generated by the relations $R_X(\lambda)$ (with $X \in \mathscr{X}$), where for $X = (i,j,k)$ with the omitted subset, say $\{i,j\}$, we have

$$R_X(\lambda) = x_i x_j - \lambda x_i x_k + \lambda x_j x_k.$$

It is not hard to prove that A is not isomorphic to $A(\lambda)$ with $\lambda \neq 1$. We suspect that A is Koszul but cannot prove this.

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