# ON KOSZUL ALGEBRAS AND A NEW CONSTRUCTION OF ARTIN-SCHELTER REGULAR ALGEBRAS

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Abstract. We prove the simple fact that the factor ring of a Koszul algebra by a regular, normal, quadratic element is a Koszul algebra. This fact leads to a new construction of quadratic Artin-Schelter regular algebras. This construction generalizes the construction of Artin-Schelter regular Clifford algebras.

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1991 *Mathematics Subject Classification.* 16W50, 14A22.
*Key words and phrases.* Regular algebra, quadratic algebra, Koszul algebra .

# INTRODUCTION

Our main goal here is a small step towards some problems in the theory of Artin-Schelter regular algebras of global dimension 4. Our main result is a new recipe for constructing previously unknown examples of these algebras. Along the way we prove a straightforward result about Koszul algebras. Throughout, $k$ is a field.

Artin-Schelter regular algebras were introduced in [1]. An Artin-Schelter regular algebra is a $k$-algebra of finite Gelfand-Kirillov and global dimensions which satisfies the Gorenstein condition, cf. section 2. These algebras play an important role in non-commutative algebraic geometry. The A-S regular algebras of global dimension 3 have been classified in [2], [3] and [10].

A Koszul algebra is a connected graded $k$-algebra whose Yoneda algebra is generated in homological degree one. Any connected quadratic graded algebra $A$ has an associated graded algebra, $A^!$, called its Koszul dual, defined by relations orthogonal to the relations of $A$. The algebra $A$ is then Koszul if and only if its Yoneda algebra is isomorphic to $A^!$.

Recall that an element $q$ in the ring $A$ is *normal* if $Aq = qA$ and regular if it is neither a left nor a right zero-divisor. A sequence of elements $q_1, \ldots, q_d$ is a *normal sequence* if for each $i$, $1 \leq i \leq d$, $q_i$ is normal in the factor ring $A/(q_1, \ldots, q_{i-1})$. A *normal regular sequence* is defined similarly.

Our main result, 2.4, takes advantage of the fact that certain types of Artin-Schelter regular algebras, known as quantum projective spaces, must be Koszul. Let $A$ be a quantum projective space of global dimension $d$. We prove that if there is a normal regular sequence $q_1, \ldots, q_d$ of quadratic elements in $A$ then the algebra $B := (A/(q_1, \ldots, q_d))^!$ is also a quantum projective space of global dimension $d$. Since there are many known examples of quantum projective spaces, such as skew polynomial rings, which admit lots of quadratic normal elements, it is easy to apply 2.4 to provide new examples of quantum projective spaces. We include an example in the final section.

We note that $B$ is dual to $A$ in the sense that there is a normal regular sequence of quadratic elements $q_d^!, \ldots, q_1^!$ in $B$ such that $A = (B/(q_d^!, \ldots, q_1^!))^!$.

The recipe given in 2.4 is seen to generalize the construction of Artin-Schelter regular Clifford algebras, [5], although the proof given here is different. To see this, let $R$ be the commutative polynomial ring on $d$ generators $x_i$ and let $C$ be an Artin-Schelter regular Clifford algebra on the linear generators $x_i$, as described in [5]. Then the elements $q_i = x_i^2$ in $C$ are central and form a regular sequence. Clearly $(C/(q_1, \ldots, q_d))^! = R$. Dually, we get $(R/(q_d^!, \ldots, q_1^!))^! = C$.

As a precursor to 2.4 we need to prove a general fact about Koszul algebras, 1.2. Let $A$ be a Koszul algebra and let $q \in A$ be a normal, regular, quadratic element. Then $A/(q)$ is Koszul. This generalizes, from commutative to non-commutative, part of a result of [4].

## 1. Koszul Properties

Let $k$ be a field. If $W = \oplus_{n \in \mathbb{Z}} W_n$ is a (locally finite) graded $k$-vector space then the Hilbert series of $W$ is the formal Laurent power series $H_W(t) = \sum_n \dim_k(W_n)t^n$. The $n^{th}$ shift of $W$, $W[n]$, is the graded $k$-vector space defined by $W[n]_m = W_{n+m}$.

Throughout this section, $V$ denotes a finite-dimensional $k$-vector space and $A$ denotes a graded $k$-algebra of the form $T(V)/I$ where $T(V)$ is the graded free $k$-algebra on $V$, i.e. $T(V)_n = V^{\otimes^n}$ and $I = \bigoplus_{n \geq 2} I_n$ is a homogeneous ideal of $T(V)$. We will always assume that $A$ is a *quadratic* $k$-algebra, that is $I$ is generated by its degree two elements: $I = \langle I_2 \rangle$.

Let $\mu_1$ be a fixed non-degenerate $k$-bilinear form on $V$. Then $\mu_1$ multiplicatively induces a non-degenerate bilinear form $\mu_2$ on $T(V)_2 = V \otimes V$. For any subspace $J \subseteq V \otimes V$ we write $J^\perp$ for the orthogonal complement of $J$ with respect to $\mu_2$. We define the *Koszul dual* of $A$ to be the quadratic $k$-algebra $A^! := T(V)/\langle I_2^\perp \rangle$. Notice that $(A^!)^! = A$. For future use we also extend the bilinear form $\mu_1$ to a non-degenerate bilinear form $\mu_n$ on $T(V)_n$.

Let $T = {}_AT$ and $T_A$ be the left and right trivial graded $A$-modules both defined as $A/A_+$. If the graded $A$-module $M$ has a minimal projective resolution by finitely generated modules, then the groups $\mathrm{Ext}_A^n(M,T)$ and $\mathrm{Ext}_A^n(M,A)$ are naturally graded. We have the formulas $\mathrm{Ext}_A^n(M[r],T) = \mathrm{Ext}_A^n(M,T[-r]) = \mathrm{Ext}_A^n(M,T)[-r]$. Following the usual convention, we will write $\mathrm{Ext}_A^{n,j}(M,T)$ for the degree $-j$ part of $\mathrm{Ext}_A^n(M,T)$. We recall that $\mathrm{Ext}_A^{n,j}({}_AT, {}_AT) = 0$ for $j < n$.

If the Ext-groups $\mathrm{Ext}_A^n(T,T)$ are all finite dimensional, then the Poincare series of $A$ is $P_A(t) = \sum_n \dim(\mathrm{Ext}_A^n(T,T))t^n$.

**Definition 1.1.** The algebra $A$ is said to be *Koszul* if any of the following equivalent conditions holds. Proofs, as well as a more comprehensive list of equivalent conditions, can be found in [4].

1. The Yoneda algebra, $\mathrm{Ext}^*_A(T,T)$, is generated as a $k$-algebra in homological degree one.
2. For all $n$, $\mathrm{Ext}^{n,j}_A(T,T) = 0$ unless $n = j$.
3. The Yoneda algebra is isomorphic, as a graded algebra, to $A^!$.
4. The trivial module ${}_AT$ has a minimal free-graded resolution $\ldots P_k \to P_{k-1} \ldots \to P_0 \to T \to 0$ in which $P_n$ is generated in degree $n$.
5. $P_A(t) = H_A(-t)^{-1}$.
6. $A^!$ is Koszul.

If $A$ is Koszul then $H_A(t) \cdot H_{A^!}(-t) = 1$. This property, which is sometimes referred to as *numerically Koszul*, is not sufficient to prove that $A$ is Koszul, [7] and [6].

If $A$ is Koszul then $A$ has finite global dimension if and only if $A^!$ is a finite dimensional algebra. In that case the global dimension of $A$ is the degree of the polynomial $H_{A^!}(t)$.

The following theorem should be well-known to experts. The use of spectral sequences in the proof is, at most, a convenience.

**Theorem 1.2.** *Let $A$ be a quadratic $k$-algebra and $q \in A_2$ a regular and normal element. Let $B = A/Aq$. If $A$ is Koszul then $B$ is Koszul.*

**Proof.** Assume $A$ is Koszul. Considering $B$ as a left $A$ module, we have a minimal projective resolution by free-graded modules $0 \to A[-2] \to A \to B \to 0$. Thus $\mathrm{Ext}^n_A({}_AB, {}_AT)$ is ${}_BT$ for $n = 0$, ${}_BT[2]$ for $n = 1$, and 0 for $n > 1$. We plug this information into the standard spectral sequence arising from the ring homomorphism $A \to B$:

$$E_2^{p,q} := \mathrm{Ext}^p_B({}_BT, \mathrm{Ext}^q_A({}_AB, {}_AT)) \Rightarrow \mathrm{Ext}^{p+q}_A({}_AT, {}_AT)$$

Since this spectral sequence has only two non-zero rows we get the long exact sequence:

$$\cdots \to \mathrm{Ext}^{n+1}_A({}_AT, {}_AT) \to \mathrm{Ext}^n_B({}_BT, {}_BT[2]) \to \mathrm{Ext}^{n+2}_B({}_BT, {}_BT) \to \mathrm{Ext}^{n+2}_A({}_AT, {}_AT) \to \cdots$$

The maps in this long exact sequence are all graded. By hypothesis, $\mathrm{Ext}^m_A(T,T) = \mathrm{Ext}^{m,m}_A(T,T)$ for all $m$. It is standard that $\mathrm{Ext}^m_B(T,T) = \mathrm{Ext}^{m,m}_B(T,T)$ for $m = 0$ and $m = 1$. Now we conclude by induction on $n$. Since $A$ is Koszul, the group $\mathrm{Ext}^{n+2}_A(T,T)$ is zero except in degree

$-n-2$. By induction, $\mathrm{Ext}^n_B(T, T[2]) = \mathrm{Ext}^n_B(T,T)[2]$ is zero except in degree $-n-2$. By the long exact sequence $\mathrm{Ext}^{n+2}_B(T,T)$ must also be zero except in degree $-n-2$. Hence $B$ is Koszul. ■

We continue this section with some comments on the converse of theorem 1.2. We do not know if the converse is true. The reader who is primarily interested in Artin-Schelter regular algebras may wish to skip to the next section.

For a moment, assume $A$ is commutative and $q \in A_2$ is regular. Then $A$ is Koszul if and only if $B := A/Aq$ is Koszul. This result of Backelin and Fröberg, [4], is a simple consequence of a theorem of Tate, [11], which states that $P_A(t) = (1-t^2)P_B(t)$.

It is worth observing that even when $A$ is not commutative, Tate's formula, $P_A(t) = (1-t^2)P_B(t)$, must hold whenever $A$ and $B$ are both Koszul. This is a consequence of 1.2. Moreover, if Tate's formula holds for $A$ and $B$ then theorem 1.2, as well as the converse, follow immediately. This is seen directly from definition 1.1.

Another approach to the converse of 1.2 can be attempted by switching to the algebras $A^!$ and $B^!$. Assume now only that $q \in A_2$ and that $B = A/\langle q\rangle$. Choose an element $Q \in T(V)_2$ such that $q = Q + I \in A_2$. Let $J_2 = (kQ \oplus I_2)^\perp \subset T(V)_2$ and $J = \langle J_2 \rangle \subset T(V)$. Then $B^! = T(V)/J$. There must then be an element $Q^! \in T(V)_2$ for which $I_2^\perp = J_2 + kQ^!$. Let $q^! = Q^! + J \in B_2$. The following is standard:

**Lemma 1.3.** *The element $q$ is normal in $A$ if and only if the element $q^!$ is 1-regular in $B$. In particular, if $q$ is normal and regular then $q^!$ is normal and 1-regular.*

**Proof.** It is clear that $(q^!)^! = q$, so the second statement follows from the first. We will omit tensor notation for multiplication in $T(V)$. Suppose $q$ is normal in $A$. This means $QV + I_3 = VQ + I_3$ in $T(V)_3$. Since $I_3 = VI_2 + I_2V$ and $kQ + I_2 = J_2^\perp$ we can rewrite this as $J_2^\perp V + VI_2 = VJ_2^\perp + I_2V$. Taking the orthogonal complement with respect to $\mu_3$ yields: $J_2V \cap VI_2^\perp = VJ_2 \cap I_2^\perp V$. Since $I_2^\perp = J_2 + kQ^!$, this can be rewritten as $J_2V \cap (VJ_2 + VQ^!) = VJ_2 \cap (J_2V + Q^!V)$. This equality can only be true if both sides of the equality are equal to $J_2V \cap VJ_2$, and this in turn gives us $J_3 \cap VQ^! = (J_2V + VJ_2) \cap VQ^! = J_3 \cap Q^!V = 0$, i.e., $q^!$ is 1-regular in $B$. This argument is obviously reversible. ■

**Corollary 1.4.** *Let $A$, $q$ and $B$ be as in 1.2.*

1. *If $A$ is Koszul, then the element $q^!$ in $B^!_2$ is normal and regular.*
2. *If $B$ is Koszul and $q^!$ is regular in $B^!$ then $A$ is Koszul.*

**Proof.** Suppose $A$ is Koszul. Then, by 1.1 and 1.2, $A^!$, $B$ and $B^!$ are all Koszul. We know that $q^!$ is normal and that $A^! = B^!/B^!q^!$. Moreover we get $H_{A^!}(t) = (H_A(-t))^{-1} = (H_B(-t)/(1-t^2))^{-1} = (1-t^2)H_{B^!}(t)$. Thus $q^!$ is regular.

Suppose $B$ is Koszul and $q^!$ is regular. Then we can apply 1.2 to $B^!$ and $q^!$ to conclude that $A^! = B^!/B^!q^!$ is Koszul. Hence $A$ is Koszul. ■

Finally, suppose that $x$ is a normal and regular element of $A$ in degree one and let $C = A/Ax$. Backelin and Fröberg prove, in [4], that if $A$ is commutative then $A$ is Koszul if and only if $C$ is Koszul. As in the quadratic case this is an easy application of another Poincare series formula of Tate, [11], $P_A(t) = (1+t)P_C(t)$. We can prove easily that the commutative hypothesis of Backelin and Fröberg's result is not necessary.

**Theorem 1.5.** *Let $A$ be as above and let $x \in A_1$ be normal and regular. Let $C = A/Ax$. Then $A$ is Koszul if and only if $C$ is Koszul. If either $A$ or $C$ is Koszul then Tate's formula, $P_A(t) = (1+t)P_C(t)$, holds.*

**Proof.** Exactly as in 1.2 we have a two row spectral sequence associated to the ring homomorphism $A \to C$. The corresponding graded long exact sequence is

$$\cdots \to \mathrm{Ext}^{n+1}_A({}_AT, {}_AT) \to \mathrm{Ext}^n_C({}_CT, {}_CT[1]) \xrightarrow{\Delta_n} \mathrm{Ext}^{n+2}_C({}_CT, {}_CT) \to \mathrm{Ext}^{n+2}_A({}_AT, {}_AT) \to \cdots$$

Suppose $A$ is Koszul, so that for all $n$, $\mathrm{Ext}^n_A(T,T)$ is zero except in graded degree $-n$. We know $\mathrm{Ext}^n_C(T,T)$ is zero except in degree $-n$ for $n=0$ and $n=1$. Proceeding by induction, we assume $n \geq 0$ and $\mathrm{Ext}^n_C(T,T)$ is zero except in degree $-n$. Since $\mathrm{Ext}^n_C(T,T[1])$ lives only in degree $-n-1$ and $\mathrm{Ext}^{n+2}_C(T,T)$ lives only in degrees $k$ for $k \leq -n-2$, we see that $\Delta_n$ is zero. But then $\mathrm{Ext}^{n+2}_C(T,T)$ embeds in $\mathrm{Ext}^{n+2}_A(T,T)$, which is zero except in degree $-n-2$. Hence $C$ is Koszul.

Conversely, if $C$ is Koszul then it is immediate that $\Delta_n$ is zero for all $n$. It follows as above that $\mathrm{Ext}^{n+1}_A(T,T)$ lives only in degree $-n-1$.

In either case $\Delta_n$ is zero, from which Tate's formula follows immediately. ■

## 2. GENERATING ARTIN-SCHELTER REGULAR ALGEBRAS.

For the general definition of an Artin-Schelter regular algebra the reader is referred to [2]. We give here only the more restrictive definition of a quantum projective space.

**Definition 2.1.** A quantum $\mathbb{P}^{d-1}$ is an $\mathbb{N}$-graded $k$-algebra $A$ which satisfies:

1. $H_A(t) = \dfrac{1}{(1-t)^d}$.
2. The global dimension of $A$ is $d$.
3. The trivial module ${}_AT$ admits a minimal projective resolution by finitely generated modules, $\mathrm{Ext}^n_A({}_AT, A) = 0$ for $n \neq d$ and $\mathrm{Ext}^d_A({}_AT, A) = T_A$. (The Gorenstein condition.)

It is well known that a quantum $\mathbb{P}^2$ or $\mathbb{P}^3$ must be a Koszul algebra. Note that we do not assume a quantum projective space is Noetherian, is a domain, or even that it is finitely generated or generated in degree one.

**Theorem 2.2.** *A quantum* $\mathbb{P}^{d-1}$ *is a Koszul algebra. In particular, it is generated in degree one and is quadratic.*

**Proof.** Let $A$ be a quantum $\mathbb{P}^{d-1}$. Let $P_* \to T \to 0$ be a minimal projective resolution of the trivial module. Then $P_*$ will be a free-graded resolution and $P_0 = A$. By the global dimension and Gorenstein properties we have $P_d = A[-s]$ for some $s \geq d$.

For each $k$, $0 \leq k \leq d$, define $G_k = \{n \in \mathbb{N} | \mathrm{Hom}(P_k, {}_AT)_{-n} \neq 0\}$. The set $G_k$ gives the degrees of the graded generators of $P_k$. Since the resolution is minimal, if $n \in G_k$ then $n \geq k$. Let $Q_{d-k} = \mathrm{Hom}_A(P_k, A[-s])$. The Gorenstein condition tells us that the complex $Q_*$ is a minimal projective resolution of the module $T_A$. Let $G'_k = \{n \in \mathbb{N} | \mathrm{Hom}(Q_k, T_A)_{-n} \neq 0\}$. As above, if $n \in G'_k$ then $n \geq k$. By unravelling definitions we see that $n \in G_k$ if and only if $s - n \in G'_{d-k}$. In particular, if $n \in G_k$, then $s - n \geq d - k$, i.e. $s \geq d - k + n$. This shows that $s$ is strictly larger than any $n$ in any $G_k$ for $k < d$.

Let $H_{P_k}(t) = p_k(t) H_A(t)$ for some polynomial $p_k(t)$. The previous paragraph shows that $p_k(t)$ is a polynomial of degree less than $s$ for $k < d$ and $p_d(t) = t^s$. Since $P_*$ is a resolution of ${}_AT$, we get $1 = H_A(t)((-1)^d t^s + \sum_{k<d} (-1)^k p_k(t))$. Since $H_A(t) = 1/(1-t)^d$ we obtain $s = d$. The inequalities $n \geq k$ and $s \geq d - k + n$ for $n \in G_k$ then imply $G_k = \{k\}$. This is precisely the statement that $A$ is generated in degree one and is a Koszul algebra. ■

**Remark 2.3.** We see that every quantum $\mathbb{P}^{d-1}$ is generated in degree one with $d$ generators and $\binom{d}{2}$ relations. On the other hand, a Koszul algebra on $d$ generators with $\binom{d}{2}$ relations need not be a quantum $\mathbb{P}^{d-1}$, even if it has the correct Hilbert series. The algebra $k\langle x, y\rangle/(xy)$, for example, is Koszul with Hilbert series $1/(1-t)^2$. But this algebra does not satisfy the Gorenstein condition.

Now we can give a recipe for forming new quantum projective spaces from old ones.

**Theorem 2.4.** *Let $A$ be a quantum $\mathbb{P}^{d-1}$. Suppose that $q_1, \dots, q_d$ is a normal regular sequence of quadratic elements of $A$. Let $B := (A/(q_1, \dots, q_d))^!$. Then $B$ is a quantum $\mathbb{P}^{d-1}$.*

**Proof.** Let $\hat{A} = A/(q_1, \dots, q_d)$ so that $B = \hat{A}^!$. By 2.2, $A$ is Koszul. By 1.2, $\hat{A}$ is Koszul and hence $B$ is Koszul. Since the elements $q_i$ form a normal regular sequence, $H_{\hat{A}}(t) = (1-t^2)^d H_A(t) = (1+t)^d$. Thus $H_B(t) = 1/(1-t)^d$ and the global dimension of $B$ is $d$. We need only prove the Gorenstein condition for $B$.

By the comments of the previous section there exist quadratic elements $q_d^!, \dots, q_1^!$ in $B$ which form a normal regular sequence and such that $(B/(q_d^!, \dots, q_1^!))^! = A$. Let $\hat{B} = B/(q_d^!, \dots, q_1^!)$. By the Rees Lemma we have $\mathrm{Ext}^n_B({}_BT, B) = \mathrm{Ext}^{n-d}_{\hat{B}}({}_{\hat{B}}T, \hat{B})$. Thus we have $\mathrm{Ext}^n_B({}_BT, B) = 0$ for $n \neq d$. Let $P_* \to {}_BT \to 0$ be a minimal projective (free-graded) resolution. Then the dual, $\mathrm{Hom}_B(P_*, B)$, is an acyclic complex of free-graded modules. Since $P_*$ resolves a one-dimensional module, so must $\mathrm{Hom}_B(P_*, B)$. Thus $\mathrm{Ext}^d_B({}_BT, B)$ is one-dimensional, as required. ■

## 3. EXAMPLES.

We conclude with one example in global dimension four, i.e. a quantum $\mathbb{P}^3$. We assume $k$ is the complex field. Before we give the examples we recall the definition and basic structure of the scheme of point modules associated to our graded algebras. One of our motivations is the study of the structure of this scheme, especially when the scheme is finite, i.e. zero-dimensional.

To simplify everything, assume throughout that $A$ is a quantum $\mathbb{P}^3$. We can choose a representation of $A$ as $k\langle x_1, x_2, x_3, x_4\rangle/(f_1, \dots, f_6)$ where the $f_i$ are linearly independent 2-tensors. We consider $x_1, x_2, x_3, x_4$ as homogeneous coordinate functions on $\mathbb{P}^3$.

Let $\Gamma = \Gamma_A$ be the closed subscheme of $\mathbb{P}^3 \times \mathbb{P}^3$ defined by the tensors $f_1, \dots, f_6$. $\Gamma$ is the *point scheme* of the algebra $A$ and parameterizes the point modules of $A$, a point module being a cyclic graded $A$-module of Hilbert series $1/(1-t)$. We know from [8], 1.4, that if $A$ is Noetherian and Cohen-Macaulay, then $\Gamma$ is the graph of an automorphism, $\sigma$, of a closed subscheme, $E$, of $\mathbb{P}^3$. The proof of [8], 1.4., shows that $\Gamma$ is also the graph of an automorphism anytime that the dimension of $\Gamma$ is zero (without the Noetherian and Cohen-Macaulay hypotheses). It is also well-known that when $\Gamma$ is zero-dimensional (the minimum possible) then $\Gamma$ has total multiplicity 20. It is proved in [9] that in this zero-dimensional case, $\Gamma$ determines $A$ in the sense that the space of 2-tensors which vanish on the scheme $\Gamma$ is spanned by $f_1, \dots, f_6$. This may fail if $\Gamma$ is not zero-dimensional, cf. [8] and [12]

Now we can give our example.

**Example 3.1.** Let $B$ be the $k$-algebra with generators $x_1, x_2, x_3, x_4$ and relations:

$$\begin{aligned} f_1 &= x_3 \otimes x_1 - x_1 \otimes x_3 + x_2 \otimes x_2, & f_2 &= ix_4 \otimes x_1 + x_1 \otimes x_4, \\ f_3 &= x_4 \otimes x_2 - x_2 \otimes x_4 + x_3 \otimes x_3, & f_4 &= ix_3 \otimes x_2 + x_2 \otimes x_3, \\ f_5 &= x_1 \otimes x_1 - x_3 \otimes x_3, & f_6 &= x_2 \otimes x_2 - x_4 \otimes x_4, \end{aligned} \tag{1}$$

Then $B$ is a quantum $\mathbb{P}^3$ whose point scheme $\Gamma$ consists of 20 distinct points. The associated automorphism $\sigma$ has order 4 with 2 orbits of order 2 and 4 orbits of order 4.

**Proof.** Let $a$ and $f$ be arbitrary non-zero complex parameters and let $A$ be the skew polynomial $k$-algebra on the generators $x_1, x_2, x_3, x_4$ with the relations:

$$\begin{aligned} g_1 &= x_2 \otimes x_1 - ax_1 \otimes x_2, & g_2 &= x_3 \otimes x_1 + x_1 \otimes x_3, \\ g_3 &= x_4 \otimes x_1 - ix_1 \otimes x_4, & g_4 &= x_3 \otimes x_2 - ix_2 \otimes x_3, \\ g_5 &= x_4 \otimes x_2 + x_2 \otimes x_4, & g_6 &= x_4 \otimes x_3 - fx_3 \otimes x_4, \end{aligned} \tag{2}$$

Then $A$ is a graded iterated Ore extension and hence a quantum $\mathbb{P}^3$. Let $Q_1 = x_1 \otimes x_2$, $Q_2 = x_3 \otimes x_4$, $Q_3 = x_1 \otimes x_1 + x_3 \otimes x_3 + x_2 \otimes x_4$ and $Q_4 = x_2 \otimes x_2 + x_4 \otimes x_4 + x_1 \otimes x_3$ and let $q_i$ be the image of $Q_i$ in $A$, $1 \leq i \leq 4$. It is clear that $q_1$ and $q_2$ are normal in $A$ and it is straightforward to check that $q_3$ and $q_4$ are normal in $A/(q_1, q_2)$. Thus we have a normal sequence in $A$. Let $\hat{A} = A/(q_1, \dots, q_4)$. A quick computer calculation shows that $H_{\hat{A}}(t) = (1+t)^4$, which shows that $q_1, \dots, q_4$ is a normal regular sequence in $A$.

Let $\mu_2$ be the standard inner product on the space of 2-tensors in $x_1, \ldots, x_4$ whose orthonormal basis is the monomial basis. With respect to this inner product, $B = \hat{A}^!$. By 2.4, $B$ is a quantum $\mathbb{P}^3$.

The computation of the point scheme of $B$ is a simple computer calculation which we omit. We describe only the result. Let $[a_1, a_2, a_3, a_4]$ be homogeneous coordinates on $\mathbb{P}^3$. Then $E$ consists of the points $e_1 = [1,0,0,0]$, $e_2 = [0,1,0,0]$, $e_3 = [0,0,1,0]$, $e_4 = [0,0,0,1]$ and 16 points of the form $[1, a_2, a_3, a_4]$ with:

$$\begin{aligned} &1 - 4a_4^4 + a_4^8 = 0 \\ &a_2^2 - a_4^2 - 4ia_2a_4^3 + ia_2a_4^7 = 0 \\ &a_3 = 8ia_4^2 - 15a_2a_4^3 + 2ia_4^6 - 4a_2a_4^7 \end{aligned} \tag{3}$$

The automorphism $\sigma$ is given by $\sigma(e_1) = e_2$, $\sigma(e_2) = e_1$, $\sigma(e_3) = e_4$, $\sigma(e_4) = e_3$ and $\sigma([1, a_2, a_3, a_4]) = [1, ia_2/a_3^2, 1/a_3, -ia_4]$. This is clearly seen to have order 4 on points of the latter form.

■

**Remark 3.2.** In the example above, the elements $q_3$ and $q_4$ are clearly not normal in $A$ for generic choices of $a$ and $f$. In fact it is not possible to choose values of $a$ and $f$ which make both $q_3$ and $q_4$ normal in $A$.

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