REPRESENTATION THEORY OF TWO FAMILIES OF QUANTUM PROJECTIVE 3-SPACES

Pete Goetz
Department of Mathematics
Bucknell University
Lewisburg, Pennsylvania 17837
pgoetz@bucknell.edu

Brad Shelton
Department of Mathematics
University of Oregon
Eugene, Oregon 97403-1222
shelton@math.uoregon.edu

Abstract. D. Stephenson and M. Vancliff recently introduced two families of quantum projective 3-spaces (quadratic and Artin-Schelter regular algebras of global dimension 4) which have the property that the associated automorphism of the scheme of point modules is finite order, and yet the algebra is not finite over its center. This is in stark contrast to theorems of Artin, Tate, and Van den Berg in global dimension 3. We analyze the representation theory of these algebras. We classify all of the finite dimensional simple modules and describe some zero-dimensional elements of Proj, i.e. so called fat point modules. In particular, we observe that the shift functor on zero-dimensional elements of Proj, which is closely related to the above automorphism, actually has infinite order.

1991 *Mathematics Subject Classification.* 16W50, 16S38, 16S37.
Key words and phrases. Artin-Schelter regular algebras, quantum $\mathbb{P}^3$, fat point modules.

1. Introduction

Let K be an algebraically closed field of characteristic zero and fix scalars $a, b, c, d, \rho \in K$. Stephenson and Vancliff, in [6], introduced two families of iterated Ore extensions, which we will denote by $S = S(a, b, c, d, \rho)$ and $S' = S'(a, b, c, d)$. These algebras were introduced as examples of quadratic Artin-Schelter regular algebras (cf. [1]) of global dimension 4 with the following two properties. Each algebra is infinite as a module over its center. The automorphism of the associated scheme of point modules has finite order. In contrast, it is a well-known and important result of Artin, Tate and Van den Bergh, [2], that these two properties are equivalent for an Artin-Schelter regular algebra of global dimension 3.

The purpose of this paper is to study some of the representation theory of the Stephenson-Vancliff examples, with an eye towards understanding the extent to which the geometry associated to each algebra is mirrored by ring-theoretic properties. First we give the definitions of the algebras.

Let R be the quadratic K-algebra on generators x, y, and z given by the relations:

$$xy + yx = 2z^2 \qquad xz - zx = 0 \qquad yz - zy = 0.$$

Then $S = S(a, b, c, d, \rho) = R[w; \sigma, \delta]$ where σ is the K-linear automorphism given by $\sigma(x) = -x$, $\sigma(y) = -y$ and $\sigma(z) = z$ and δ is the σ-derivation determined by the formulas:

$$\delta(x) = y^2 + ayz + bz^2 + \rho xz \qquad \delta(z) = xy - z^2$$
$$\delta(y) = x^2 + cxz + dz^2 + (\rho - 2)yz.$$

(Please note that we are using *left* Ore extensions and *left* σ-derivations, as carefully described in [5].) Similary, let R' be the K-algebra generated by elements x, y and z with relations:

$$xy + yx = 0 \qquad xz - zx = 0 \qquad yz - zy = 0.$$

We let $S' = S'(a, b, c, d) = R'[w; \sigma', \delta']$, where σ' is the K-algebra automorphism of R' given by $\sigma'(x) = -x$, $\sigma'(y) = -y$ and $\sigma'(z) = z$ and δ' is the K-linear σ'-derivation of R' defined by:

$$\delta'(x) = y^2 + ayz + bz^2 \qquad \delta'(y) = x^2 + cxz + dz^2 \qquad \delta'(z) = xy.$$

It is apparent that S and S' become graded (quadratic, connected) algebras if the generators x, y, z and w are all given degree 1. Throughout the paper, all modules will be right modules. This is justified by the fact that S and S' are isomorphic to their opposite algebras.

The vast majority of this paper is devoted to classifying the finite dimensional simple modules for S and S' under certain generic assumptions on the parameters a, b, c, d. Given scalars λ, μ and ζ with

$\lambda\mu = \zeta^2$ (respectively $\lambda\mu = 0$), let $F(\lambda, \mu, \zeta)$ (respectively $F'(\lambda, \mu, \zeta)$) be the one-dimensional R-module (respectively R'-module) on which the generators x, y and z act as λ, μ and ζ respectively.

Theorem 1.1. *Assume the parameters a, b, c, d are S-generic (definition 3.4). Then there are no simple S-modules of even dimension. For each odd number n, the isomorphism classes of nontrivial finite dimensional simple S-modules of dimension n are in one-to-one correspondence with nontrivial solutions (λ, μ, ζ) to the pair of equations $\lambda\mu - \zeta^2 = 0$ and $(\mu^3 - \lambda^3) + n(a\mu^2 + 2\zeta^2 - c\lambda^2)\zeta + (b\mu - \lambda d)\zeta^2 = 0$. The correspondence is realized by taking (λ, μ, ζ) to the unique simple quotient of $F(\lambda, \mu, \zeta) \otimes_R S$.*

Theorem 1.2. *Assume $abcd \neq 0$ and $ad^2 - cb^2 \neq 0$. Then there are no simple S'-modules of even dimension. For each odd number n, the isomorphism classes of nontrivial finite dimensional simple S'-modules of dimension n are in one-to-one correspondence with nontrivial solutions (λ, μ, ζ) to one of the two pairs of equations $\mu = 0$ and $\lambda^2 + nc\lambda\zeta + d\zeta^2 = 0$ or $\lambda = 0$ and $\mu^2 + na\mu\zeta + b\zeta^2 = 0$. The correspondence is realized by taking (λ, μ, ζ) to the unique simple quotient of $F'(\lambda, \mu, \zeta) \otimes_{R'} S'$.*

Of course both families of algebras admit one additional simple finite dimensional module, the trivial module.

Let T be any Noetherian, connected, graded K-algebra. We recall from [3] the category $\mathrm{Proj}(T)$ of finitely generated graded T-modules modulo finite length graded T-modules, and its shift operation $s : Ob(\mathrm{Proj}(T)) \to Ob(\mathrm{Proj}(T))$, $s(M) = M(1)$. This shift operator, in the context of our algebras, is related to the automorphism of the point scheme mentioned earlier. It is the underlying function on closed points for that scheme automorphism. It is not difficult to see that s is the identity when restricted to the point modules of S and S'.

In [3], Artin introduced the notion of fat point modules, which play the role of zero-dimensional objects in $\mathrm{Proj}(T)$ and generalize point modules. In section 63 we prove the following:

Theorem 1.3. *For generic values of a, b, c, d, the algebras S and S' each admit a one-parameter family of fat point modules of multiplicity 2. This family is closed under the shift operation s and on this family s has infinite order.*

This theorem restores at least some of the connection between the geometry of zero-dimensional objects in $\mathrm{Proj}(S)$ (respectively $\mathrm{Proj}(S')$) and the internal ring structure of the algebra.

We have chosen to include the proofs of Theorem 1.1 and the S-part of Theorem 1.3. We do not include the proofs of Theorem 1.2 or the S'-part of Theorem 1.3. We do this in part to save space, but primarily because there are no ideas in the latter proofs that are not already apparent in the former proofs. In fact, the formulas and verifications of the S-proofs are substantially more intricate, but have the positive feature that they do not bifurcate into cases, as the S'-proofs do.

The proof of Theorem 1.1 is spread out over sections 3 through 5. In section 3 we show that if M is an S-module induced from a simple finite dimensional R-module, then M is almost certainly already simple and has a unique simple quotient. This reduces the only possible finite dimensional simple modules to be factors of M when M is associated to the parameters (λ, μ, ζ) as given in the theorem. The only problem that remains is to show that these induced modules do in fact have appropriate finite dimensional factors. At that point it is easier to prove the existence of the finite dimensional modules directly, and this is carried out in section 5. In between, in section 4, we need to discuss some simple combinatorics of odd by odd integer matrices. Using these combinatorial formulas we are able to give explicit power-series type formulas for the finite dimensional simple S modules. Those formulas are given in the proof of Theorem 5.1 and complete the proof of 1.1. The proof of 1.3 is a straightforward calculation which we sketch in section 6.

2. The centers of S and S'.

Stephenson and Vancliff proved that the centers of S and S' are contained in R and R' respectively, but they did not explicitly calculate the centers. Many of our later calculations are motivated by exact knowledge of the center.

Lemma 2.1. *The center of S (respectively S') is the K-subalgebra of S (respectively S') generated by the quadratic elements $\omega_1 = x^2 + az^2$ and $\omega_2 = y^2 - cz^2$ and is a polynomial ring on these generators.*

Proof. Let $Z(S')$ (respectively $Z(R')$) be the center of S' (respectively R'). We know from [6] that $Z(S')$ is contained in the subalgebra R', although they do not specifically tell us what the center is. However, from $Z(S') \subset R'$, it becomes clear that $Z(S')$ consists of those elements of $Z(R')$ which are both σ'-invariant and in the kernel of δ'. We observe that $Z(R')$ is the polynomial subalgebra $K[x^2, y^2, z]$, all elements of which are σ'-invariant. From the calculation: $\delta'(x^2) = \sigma'(x)\delta'(x) + \delta'(x)x = (-x)(y^2 + ayz + bz^2) + (y^2 + ayz + bz^2)x = -2axyz$ and the similar calculations: $\delta'(y^2) = 2cxyz$ and $\delta'(z^2) = 2xyz$, we see that $\delta'(x^2 + az^2) = 0$ and $\delta'(y^2 - cz^2) = 0$. Thus ω_1 and ω_2 are in $Z(S')$. But $Z(R')$ can also be written as the polynomial ring $K[\omega_1, \omega_2, z]$. Since $\delta'(z) = xy \neq 0$, it follows immediately that the kernel of the derivation δ' on $Z(R')$ is exactly $K[\omega_1, \omega_2]$. This proves the result for S'. The proof for S is almost identical. □

3. R-induced S-modules

The algebra R has finite dimensional simple modules of two types, one dimensional and two dimensional. If $\lambda\mu = \zeta^2$, for $\lambda, \mu, \zeta \in K$, then there is a one dimensional module $F(\lambda, \mu, \zeta)$ on which x, y and z act as λ, μ and ζ respectively. If $\pm\lambda\mu \neq \zeta^2$ then there is a two dimensional simple module $F(\lambda, \mu, \zeta)$ with the actions:

$$x = \begin{pmatrix} \lambda & 0 \\ 2\zeta^2 & -\lambda \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 1 \\ \mu^2 & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}.$$

This exhausts the finite dimensional simple R-modules. There are some obvious isomorphisms amongst the 2-dimensional $F(\lambda, \mu, \zeta)$, but that is not relevant to our discussion (cf. 3.2).

For each (λ, μ, ζ) as above, let $V(\lambda, \mu, \zeta)$ be the induced right S-module $V(\lambda, \mu, \zeta) = F(\lambda, \mu, \zeta) \otimes_R S$. We identify the K-subalgebra of S generated by w with the polynomial algebra $K[w]$ and we note that as a vector space, $S \cong R \otimes_K K[w]$. In particular, the induced module $V(\lambda, \mu, \zeta)$ is a free right $K[w]$ module of rank one or two.

Notation 3.1. In the ring S, as well as the modules $V(\lambda, \mu, \zeta)$, elements are polynomials in w with (left) coefficients from R (respectively $F(\lambda, \mu, \zeta)$). The *degree* of these polynomials in w is well-defined. We use the symbol $O(n)$ (respectively $O_V(n)$) to denote an arbitrary element of S (respectively $V = V(\lambda, \mu, \zeta)$) of degree less than or equal to n. The symbols $O(-n)$ and $O_V(-n)$ represent the element 0 for any $n > 0$.

Lemma 3.2. *If $\pm\lambda\mu \neq \zeta^2$, then the S-module $V = V(\lambda, \mu, \zeta)$ is simple.*

Proof. In S we have the formula (a) $w^n z = zw^n + n(xy - z^2)w^{n-1} + O(n-2)$, for all positive integers n. This is proved easily by induction. Let $p(w) = \sum_{i=0}^{n} v_i \otimes w^i$ be any nonzero element of V of degree $n > 0$. By (a), $p(w)z - \zeta p(w) = n(\lambda\mu - \zeta^2)(v_n \otimes w^{n-1}) + O_V(n-2)$. This element has degree exactly $n - 1$. This is enough to prove V is simple. □

For the remainder of the section we may assume $\lambda\mu - \zeta^2 = 0$. The S-module $V = V(\lambda, \mu, \zeta)$ is then induced from a one-dimensional module and so we may identify it with $K[w]$. We use the notation $p(w) \to p_V(w)$ for this identification.

The following complicated lemma requires some notation. First we define $\bar{N} \in R$ by $\bar{N} = \mu x - \lambda y$ and note that this element will act by 0 on the cyclic vector of V. This element plays a critical role in all of our calculations. We also define the following homogeneous elements of R.

$$F_1 = y^3 - x^3 + (by - dx)z^2, \quad F_2 = \mu y^2 - \lambda x^2 + (b\mu - d\lambda)z^2$$

$$G_1 = (ay^2 + 2z^2 - cx^2)z, \qquad G_2 = ((a\mu - (\rho - 2)\lambda)y + (\rho\mu - c\lambda)x)z$$

Lemma 3.3. *In R, $\sigma(xy - z^2) = xy - z^2$ and $\delta(xy - z^2) = F_1 + G_1$. Moreoever, for all $n \geq 0$,*

(1) $w^{2n+1}z = zw^{2n+1} + (2n+1)(xy - z^2)w^{2n} + n(F_1 + (2n+1)G_1)w^{2n-1} + O(2n-2)$,

(2) $w^{2n+2}z = zw^{2n+2} + (2n+2)(xy - z^2)w^{2n+1} + (n+1)(F_1 + (2n+1)G_1)w^{2n} + O(2n-1)$,

(3) $w^{2n+1}\bar{N} = -\bar{N}w^{2n+1} + (F_2 + (2n+1)G_2)w^{2n} + O(2n-1)$,

(4) $w^{2n+2}\bar{N} = \bar{N}w^{2n+2} - (2n+2)G_2w^{2n+1} + O(2n)$.

Proof. The first equation is obvious. For the second we calculate: $\delta(xy) = -x(x^2 + cxz + dz^2 + (\rho - 2)yz) + (y^2 + ayz + bz^2 + \rho xz)y = F_1 + (ay^2 + 2xy - cx^2)z$. Similarly, $\delta(z^2) = 2xyz - 2z^3$. Combining these yields the required formula for $\delta(xy - z^2)$.

Equation (1), for $n = 0$, is just the relation $wz = zw + xy - z^2$. From this we get $w^2z = w(zw + xy - z^2) = (zw + xy - z^2)w + \sigma(xy - z^2)w + \delta(xy - z^2)$. This is exactly (2) for $n = 0$. We now prove (1) and (2) by induction. Note that $\sigma(F_1) = -F_1$ and $\sigma(G_1) = G_1$. For $n > 0$, inductively, $w^{2n+1}z = w(zw^{2n} + 2n(xy - z^2)w^{2n-1} + n(F_1 + (2n-1)G_1)w^{2n-2} + O(2n-3)) = (zw + (xy - z^2))w^{2n} + 2n(xy - z^2)w^{2n} + 2n(F_1 + G_1)w^{2n-1} + n(-F_1 + (2n-1)G_1)w^{2n-1} + O(2n-2) = zw^{2n+1} + (2n+1)(xy - z^2)w^{2n} + n(F_1 + (2n+1)G_1)w^{2n-1} + O(2n-2)$. This is (1). Then (2) follows from (1) by the same type of manipulation.

For $n = 0$, equation (3) is just the statement that $\delta(\bar{N}) = \mu\delta(x) - \lambda\delta(y) = F_2 + G_2$, a straightforward calculation. Equations (3) and (4) for all n follow by induction and the facts that $\sigma(F_2) = F_2$ and $\sigma(G_2) = -G_2$. □

To facilitate our discussion, let

$$f = \mu^3 - \lambda^3 + (b\mu - d\lambda)\zeta^2 \qquad \text{and} \qquad g = (a\mu^2 + 2\lambda\mu - c\lambda^2)\zeta$$

be the scalars by which F_1 and G_1 act on the one-dimensional module $F(\lambda, \mu, \zeta)$, i.e. the evaluations of the elements at $x = \lambda$, $y = \mu$ and $z = \zeta$. It is very important then to note that f and g are also the scalars by which F_2 and G_2 act respectively on $F(\lambda, \mu, \zeta)$ (since $\lambda\mu = \zeta^2$). This small miracle is the reason why everything works.

Definition 3.4. Consider λ, μ, ζ as projective coordinates on $\mathbb{P}^2$. We say that the parameters a, b, c, d are S-generic if the three curves in $\mathbb{P}^2$ (a) $\lambda\mu - \zeta^2 = 0$, (b) $g = 0$ and (c) $f = 0$ have no common intersection.

One can easily see that the open condition S-generic is nonempty.

Theorem 3.5. *Let $\lambda\mu = \zeta^2$ and assume λ, μ and ζ are not all 0. Let $W = p_V(w)K[w]$ be a proper S-invariant subspace of $V = V(\lambda, \mu, \zeta)$, of codimension $n = deg(p(w))$. Assume the parameters a, b, c, d are S-generic. Then n is odd and (λ, μ, ζ) satisfies the cubic polynomial:*

$$(\mu^3 - \lambda^3) + n(a\mu^2 + 2\zeta^2 - c\lambda^2)\zeta + (b\mu - \lambda d)\zeta^2 = 0.$$

Proof. We may write $p(w) = w^n + \alpha w^{n-1} + O(n-2)$. Suppose first that n is even. Then by (3) and (4) of 3.3 and recalling that $\bar{N}$ acts as 0 on $F(\lambda, \mu, \zeta)$, we get $p_V(w)\bar{N} = (w^n + \alpha w^{n-1})_V \bar{N} + O_V(n-2) = -ngw^{n-1} + O_V(n-2)$. This has lower degree than $p(w)$, a contradiction, unless $g = 0$. Similarly, by (1) and (2) of 3.3, $p_V(w)z - \zeta p_V(w) = (n/2)(f + (n-1)g)w^{n-2} + O_V(n-3)$. We conclude that f must also be 0, which contradicts the assumption that the parameters are generic. This proves n must be odd.

Now that we know n is odd we can calculate, again by (3) and (4) of 3.3, $p_V(w)\bar{N} = (f + ng)w^{n-1} + O_V(n-2)$. Degree considerations then force $f + ng = 0$, which is exactly the required equation. □

Remark 3.6. It is not difficult to show that if the parameters a, b, c, d are not S-generic, and λ, μ, ζ nontrivially solve the three equations of Definition 3.4, then the induced module $V(\lambda, \mu, \zeta)$ has a one-parameter family of 2-dimensional simple factor modules.

In view of the lemmas of this section we may make the following definition.

Definition 3.7. Assume that a, b, c, d are S-generic. Then for any $\lambda, \mu, \zeta \in K$ we let $L(\lambda, \mu, \zeta)$ be the unique simple factor module of $V(\lambda, \mu, \zeta)$.

4. Odd by Odd Integer Matrices

In this section we record combinatorial formulas relating some specific odd by odd integer matrices.

Throughout this section, fix a nonnegative integer k and let I be the identity element of the ring $M_{2k+1}(\mathbb{Z})$. We define $N = (x_{i,j})$ and $R = (r_{i,j})$ in $M_{2k+1}(\mathbb{Z})$ by $x_{i,j} = \delta_{i,j+1}$ and $r_{i,j} = (-1)^{i+1}\delta_{i,j}$. We note that $NR = -RN$.

For $n \geq 0$, let f_n, d_n, g_n be the linear operators on $M_{2k+1}(\mathbb{Z})$ given by $f_n(T) = N^nT + TN^n$, $d_n(T) = N^nT - TN^n$, and $g_n(T) = N^nTN^n$. We note that $f_ng_m = g_mf_n$ and $d_ng_m = g_md_n$. It is easy to check that $f_n = f_{n-1}f_1 - g_1f_{n-2}$ and $d_n = d_{n-1}f_1 - g_1d_{n-2}$.

Lemma 4.1. *Let $J_1 = (s_{i,j}) \in M_{2k+1}(\mathbb{Z})$ be the matrix given by $s_{i,j} = 0$ if $i - j \neq 1$, $s_{k-j+1,k-j} = (-1)^{k-j+1}(2j+1)$ for $0 \leq j \leq k-1$ and*

$s_{k+j+2,k+j+1} = (-1)^{k+j+1}(2j+1)$ for $0 \leq j \leq k-1$. Let $J_{2n+1} = (-1)^n N^n J_1 N^n$ for $n \geq 0$. Then for all n, $NJ_{2n+1}+J_{2n+1}N = 2N^{2n+2}R$. Moreover:

(1) $N^{2m}J_{2n+1} + J_{2n+1}N^{2m} = 2J_{2m+2n+1}$ for all $m \geq 0$.

(2) $N^{2m}J_{2n+1} - J_{2n+1}N^{2m} = 4mN^{2m+2n+1}R$ for all $m \geq 0$.

Proof. The formula $f_1(J_1) = 2N^2R$ is a straightforward calculation. For $n > 0$, $f_1(J_{2n+1}) = f_1((-1)^n g_n(J_1)) = (-1)^n g_n(f_1(J_1)) = 2N^{2n+2}R$.

Both formulas (1) and (2) hold trivially for $n = m = 0$. By double induction we get

$$\begin{aligned} f_{2m}(J_{2n+1}) &= (-1)^n g_n(f_{2m}(J_1)) \\ &= (-1)^n g_n(f_{2m-1}(f_1(J_1)) - Nf_{2m-2}(J_1)N) \\ &= (-1)^n g_n(f_{2m-1}(2N^2R) - N(f_{2m-2}(J_1)N) \\ &= (-1)^n g_n(0 - N(2J_{2m-1})N) \\ &= 2J_{2m+2n+1} \end{aligned}$$

This is (1). Equation (2) is similar. □

We will need two additional specific matrices and their properties. Let $H = (1/2)(I-R) \in M_{2k+1}(\mathbb{Z})$. This is the diagonal matrix with diagonal entries: $(0,1,0,1,0,\ldots,1,0)$. We define $B = (b_{i,j}) \in M_{2k+1}(\mathbb{Z})$ by the two conditions: (1) $b_{i,j} = 0$ if $j - i \neq 1$, (i.e. B is supported on the first superdiagonal) and (2) $NB + BN = kI + H$. The pattern for the first k elements along the superdiagonal of B is $(k, 1, k-1, 2, k-2, 3, \ldots)$ and B, like most of the matrices we use, is symmetric around its sinister diagonal.

Lemma 4.2. *For all $n \geq 0$:*

(1) $BN^{2n} - N^{2n}B = nN^{2n-1}R$,

(2) $HN^{2n+1} - N^{2n+1}H = N^{2n+1}R$.

Proof. Both formulas are clear for $n = 0$. For $n > 0$ we have, inductively,

$$\begin{aligned} d_{2n}(B) &= d_{2n-1}(f_1(B)) - Nd_{2n-2}(B)N \\ &= d_{2n-1}(kI + H) + N(n-1)N^{2n-3}RN \\ &= -N^{2n-1}R - (n-1)N^{2n-1}R \\ &= -nN^{2n-1}R. \end{aligned}$$

This is (1). For (2), first notice that $f_1(H) = N$ and thus, inductively, $d_{2n+1}(H) = d_{2n}(f_1(H)) - Nd_{2n-1}(H)N = d_{2n}(N) + NN^{2n-1}RN = 0 - N^{2n+1}R$, as required. □

5. Finite dimensional simple S-modules

Theorem 5.1. *Assume that the parameters a, b, c, d are S-generic and let $\lambda, \mu, \zeta \in K$ satisfy $\lambda\mu = \zeta^2$, with not all three parameters zero.*

Suppose

$$(*) \qquad (\mu^3-\lambda^3)+(2k+1)(a\mu^2+2\zeta^2-c\lambda^2)\zeta+(b\mu-d\lambda)\zeta^2=0,$$

for some nonnegative integer k. Then there are matrices X, Y, Z and W in $M_{2k+1}(K)$ such that:
(a) $XY+YX=2Z^2$,
(b) $XZ-ZX=0$,
(c) $YZ-ZY=0$,
(d) $WX+XW=Y^2+aYZ+bZ^2+\rho XZ$,
(e) $WY+YW=X^2+cXZ+dZ^2+(\rho-2)YZ$,
(f) $WZ-ZW=XY-Z^2$,
and moreover these matrices define a simple $2k+1$-dimensional S-module isomorphic to $L(\lambda,\mu,\zeta)$.

Proof. We use the matrix notation N, J_{2n+1}, H and B from Section 4. Set $h=a\mu^2+2\zeta^2-c\lambda^2$ and $f=\mu^3-\lambda^3+(b\mu-d\lambda)\zeta^2$. Since our parameters are S-generic, f and $h\zeta$ cannot both be zero. But our hypothesis $(*)$ is that $f+(2k+1)h\zeta=0$, hence f and $h\zeta$ are both non-zero. In particular, ζ and h (and thus also λ and μ) are nonzero.

Fix $\alpha=(a\mu+\lambda)/h$ and $\beta=(c\lambda-\mu)/h$.

Let ζ_n be the sequence of scalars defined by the formulas:

$$\zeta_0=\zeta, \qquad \zeta_1=-1/(2\zeta h), \qquad \sum_{i+j=n}\zeta_i\zeta_j=0,\ n\geq 2.$$

Let $\phi_0=1$. Let $\phi_1=-(ac+1)/(2h^2)$. By direct calculation, one finds that ϕ_1 is also equal to the following three quantities:

$$(a-\alpha^2h)/(2\lambda^2h)=(-c-\beta^2h)/(2\mu^2h)=(-1-\alpha\beta h)/(2\zeta^2h).$$

For example: $1+\alpha\beta h=(h+(a\mu+\lambda)(c\lambda-\mu))/h=(a\mu^2+2\zeta^2-c\lambda^2+(ac\zeta^2-a\mu^2+c\lambda^2-\zeta^2))/h=(ac+1)\zeta^2/h$. From this we get $\phi_1=(-1-\alpha\beta h)/(2\zeta^2h)$. The other two equalities follow from the calculations: $\alpha^2h-a=(ac+1)\lambda^2/h$ and $\beta^2h+c=(ac+1)\mu^2/h$. We now define the rest of the sequence ϕ_n by $\sum_{i+j=n}\phi_i\phi_j=0$ for $n\geq 2$.

Before proceeding we note that ϕ_1 was defined to simultaneously solve the following three equations.

$$2\lambda^2\phi_1+\alpha^2-a/h=0,$$

$$2\mu^2\phi_1+\beta^2+c/h=0,$$

$$4\zeta^2\phi_1+2\alpha\beta=-2/h.$$

By $(*)$, we have the equality

$$\frac{\mu^2+b\zeta^2+(2k+1)(a\mu+\lambda)\zeta}{2\lambda}=\frac{\lambda^2+d\zeta^2+(2k+1)(c\lambda-\mu)\zeta}{2\mu}$$

and we take δ_0 to be this common value. Let δ_1 and γ be the simultaneous solutions to the two equations:

$$(2\lambda)\delta_1 + (2\alpha)\gamma = -2\lambda\delta_0\phi_1 - (c+b)/h$$

$$(2\mu)\delta_1 + (2\beta)\gamma = -2\mu\delta_0\phi_1 + (a-d)/h$$

We note that δ_1 and γ are well-defined and unique because of the equation $\mu\alpha - \lambda\beta = 1$. We extend δ_0, δ_1 to the sequence δ_n by the recursive definition $\sum_{i+j=n} \delta_i\phi_j = 0$ for $n \geq 2$.

Next, we define the sequence ϵ_n by the formula

$$\epsilon_n = \left(\frac{h}{2}\right) \sum_{i+j=n} \phi_i\zeta_j, \quad n \geq 0.$$

A standard argument using power series yields the following two formulas:

$$(A) \qquad \sum_{i+j=n} i\phi_i\epsilon_{j+1} = \left(\frac{h}{2}\right)(\phi_1\zeta_n - (n+1)\phi_{n+1}\zeta)$$

and

$$(B) \qquad \sum_{i+j=n} i\zeta_i\epsilon_{j+1} = \left(\frac{h}{2}\right)(\zeta\zeta_1\phi_n - (n+1)\zeta_{n+1}\zeta)$$

for all $n \geq 0$.

We are now ready to define the matrices X, Y, Z, two auxiliary matrices D and E, and, finally, W, by the following formulas:

$$\begin{aligned} X &= \lambda \sum_{n\geq 0} \phi_n N^{2n}R + \alpha N \\ Y &= \mu \sum_{n\geq 0} \phi_n N^{2n}R + \beta N \\ Z &= \sum_{n\geq 0} \zeta_n N^{2n} \\ D &= \sum_{n\geq 0} \delta_n N^{2n}R + \gamma N \\ E &= \sum_{n\geq 0} \epsilon_{n+1} J_{2n+1} \\ W &= D + E + \left(\tfrac{\rho-1}{2}\right) Z - 2h\zeta B \end{aligned}$$

Since N^{2i} commutes with $N^{2j}R$ and N, we see that X and Y each commute with Z, proving equations (b) and (c). We also immediately get:

$$Z^2 = \sum_{p\geq 0} \sum_{n+m=p} \zeta_n\zeta_m N^{2p} = \zeta^2 I - (1/h)N^2.$$

Since $N^{2n}R$ and N anticommute, we get the formulas:

$$\begin{aligned} X^2 &= \lambda^2 \sum_{p\geq 0} \sum_{n+m=p} \phi_n\phi_m N^{2p} + \alpha^2 N^2 \\ &= \lambda^2 I + (2\lambda^2\phi_1 + \alpha^2)N^2 = \lambda^2 I + (a/h)N^2 \end{aligned}$$

and similarly

$$Y^2 = \mu^2 I + (\mu^2\phi_1 + \beta^2)N^2 = \mu^2 I - (c/h)N^2.$$

Moreover,

$$\begin{aligned} XY + YX &= 2\lambda\mu \sum_{p\geq 0}\sum_{n+m=p} \phi_n\phi_m N^{2p} + 2\alpha\beta N^2 \\ &= 2\zeta^2(I + 2\phi_1 N^2) + 2\alpha\beta N^2 \\ &= 2\zeta^2 I - (2/h)N^2 \\ &= 2Z^2. \end{aligned}$$

This is equation (a).

We need a good expression for $Y^2 + aYZ + bZ^2 + \rho XZ$.

$$\begin{aligned} Y^2 + aYZ + bZ^2 + \rho XZ &= (Y^2 + bZ^2) + (aY + \rho X)Z \\ &= (\mu^2 + b\zeta^2)I - \left(\tfrac{c+b}{h}\right) N^2 + (a\beta + \rho\alpha)ZN \\ &\quad +(a\mu + \rho\lambda) \sum_{p\geq 0}\sum_{n+m=p} \phi_n\zeta_m N^{2p} R \\ &= (\mu^2 + b\zeta^2)I - \left(\tfrac{c+b}{h}\right) N^2 + (a\beta + \rho\alpha)ZN \\ &\quad + \tfrac{2(a\mu+\rho\lambda)}{h} \sum_{p\geq 0} \epsilon_p N^{2p} R \end{aligned}$$

Similarly we have the formula:

$$\begin{aligned} X^2 + cXZ + dZ^2 + (\rho-2)YZ &= (\lambda^2 + d\zeta^2)I + \left(\tfrac{a-d}{h}\right) N^2 \\ &\quad +(c\alpha + (\rho-2)\beta)ZN \\ &\quad +\tfrac{2(c\lambda+(\rho-2)\mu)}{h} \sum_{p\geq 0} \epsilon_p N^{2p} R. \end{aligned}$$

We turn to equation (d) and begin by calculating $WX + XW$ in four steps. The first calculation is similar to that of $XY + YX$ and uses the defining equations of δ_1 and γ.

$$\begin{aligned} DX + XD &= 2\lambda \sum_{p\geq 0}\sum_{n+m=p} \phi_n\delta_m N^{2p} + 2\alpha\gamma N^2 \\ &= 2\lambda(\delta_0 I + (\delta_0\phi_1 + \delta_1)N^2) + 2\alpha\gamma N^2 \\ &= 2\lambda\delta_0 I + (2\lambda(\delta_0\phi_1 + \delta_1) + 2\alpha\gamma)N^2 \\ &= 2\lambda\delta_0 I - \left(\tfrac{c+b}{h}\right) N^2 \end{aligned}$$

Calculating $EX+XE$ requires Lemma 4.1 and formula (A). Recall that J_{2n+1} anticommutes with R.

$$\begin{aligned} EX+XE &= \lambda \sum_{p\geq 0}\sum_{n+m=p} \phi_n\epsilon_{m+1}(J_{2m+1}N^{2n}R + N^{2n}RJ_{2m+1}) \\ &\qquad +\alpha\sum_{p\geq 0}\epsilon_{p+1}(J_{2p+1}N + NJ_{2p+1}) \\ &= \lambda\sum_{p\geq 0}\sum_{n+m=p}\phi_n\epsilon_{m+1}(-4n)N^{2p+1} + \alpha\sum_{p\geq 0}\epsilon_{p+1}2N^{2p+2}R \\ &= -4\lambda\left(\tfrac{h}{2}\right)\sum_{p\geq 0}(\phi_1\zeta_p - (p+1)\phi_{p+1}\zeta)N^{2p+1} + 2\alpha\sum_{p\geq 1}\epsilon_pN^{2p}R \\ &= -2\lambda h\phi_1 ZN + 2\lambda h\zeta\sum_{p\geq 0}(p+1)\phi_{p+1}N^{2p+1} + 2\alpha\sum_{p\geq 1}\epsilon_pN^{2p}R \end{aligned}$$

Since X and Z commute, the next term is simple:

$$\begin{aligned} \left(\tfrac{\rho-1}{2}\right)(ZX+XZ) &= (\rho-1)ZX \\ &= (\rho-1)\lambda\sum_{p\geq 0}\sum_{n+m=p}\phi_n\zeta_mN^{2p}R + (\rho-1)\alpha ZN \\ &= \tfrac{2(\rho-1)\lambda}{h}\sum_{p\geq 0}\epsilon_pN^{2p}R + (\rho-1)\alpha ZN \end{aligned}$$

Finally, we use 4.2 in the following calculation. Recall that $BN+NB = kI+H = (k+\frac{1}{2})I - \frac{1}{2}R$.

$$\begin{aligned} -2\zeta h(BX+XB) &= -2\zeta h\lambda\sum_{p\geq 0}\phi_p(BN^{2p}R + N^{2p}RB) - 2\zeta h\alpha(BN+NB) \\ &= -2\zeta h\lambda\sum_{p\geq 0}\phi_p pN^{2p-1} - 2\zeta(a\mu+\lambda)(kI+H) \\ &= -2\zeta h\lambda\sum_{p\geq 0}\phi_{p+1}(p+1)N^{2p+1} - \zeta(a\mu+\lambda)(2k+1)I \\ &\qquad +\zeta(a\mu+\lambda)R \end{aligned}$$

The quantity $WX+XW$ is the sum of the last four expressions. The coefficient of ZN is thus

$$\begin{aligned} -2\lambda h\phi_1 + (\rho-1)\alpha &= \tfrac{\lambda(ac+1)}{h} + \tfrac{(\rho-1)(a\mu+\lambda)}{h} \\ &= \tfrac{a(c\lambda-\mu)+\rho(a\mu+\lambda)}{h} \\ &= a\beta + \rho\alpha. \end{aligned}$$

The coefficient of $N^{2p}R$, for $p\geq 1$, is

$$\begin{aligned} 2\alpha\epsilon_p + \tfrac{2(\rho-1)\lambda}{h}\epsilon_p &= \tfrac{2((a\mu+\lambda)+(\rho-1)\lambda)}{h}\epsilon_p \\ &= \tfrac{2(a\mu+\rho\lambda)}{h}\epsilon_p. \end{aligned}$$

Since $\zeta = 2\epsilon_0/h$, the coefficient of R, from the third and fourth terms, is $(\rho-1)\lambda\zeta + (a\mu+\lambda)\zeta = (a\mu+\rho\lambda)\zeta = (2(a\mu+\rho\lambda)/h)\epsilon_0$. The coefficient of N^{2p+1}, obtained from the second and fourth terms, is 0. The coefficient of the identity matrix, I, is $2\lambda\delta_0 - (2k+1)(a\mu+\lambda)\zeta = \mu^2 + b\zeta^2$. Finally, the coefficient of N^2 is apparent in the first term. Combining all of this gives the formula

$$\begin{aligned} WX + XW &= (\mu^2 + b\zeta^2)I - \left(\tfrac{c+b}{h}\right) N^2 + (a\beta + \rho\alpha)ZN \\ &+ \tfrac{2(a\mu+\rho\lambda)}{h} \sum_{p\geq 0} \epsilon_p N^{2p} R. \end{aligned}$$

This is exactly the expression we computed for $Y^2 + aYZ + bZ^2 + \rho XZ$, and we have proved equation (d).

The proof of equation (e) is essentially the same as (d) and we omit it.

The verification of (f) is only slightly different and somewhat easier. Notice that only the terms E and $-2h\zeta B$ in W do not commute with Z. By using 4.1, 4.2 and (B), one easily verifies that

$$WZ - ZW = \sum_{p\geq 0} \phi_p N^{2p+1} R.$$

Using the facts $\mu\alpha - \lambda\beta = 1$ and $2\zeta^2\phi_1 + \alpha\beta = -1/h$, and our previous calculation of Z^2, it is very easy to also calculate

$$XY - Z^2 = \sum_{p\geq 0} \phi_p N^{2p+1} R$$

This proves equation (f).

The matrices X, Y, Z and W now define a $2k+1$-dimensional (right) S-module, say L'. But the matrices X, Y and Z are all lower triangular matrices with the scalars λ, μ and ζ, respectively, in the upper left corner. This shows that $F(\lambda, \mu, \zeta)$ is an R-submodule of L'. It follows that L' contains a nonzero (finite dimensional) factor module of $V(\lambda, \mu, \zeta)$. By Theorem 3.5, the only possible dimension of such a factor module is $2k + 1$ and so we conclude that $L' \cong L(\lambda, \mu, \zeta)$. □

Remark 5.2. In view of Lemma 3.2, Theorem 3.5 and Theorem 5.1 there is only one step left in the proof of Theorem 1.1. It simply remains to prove that two $2k + 1$-dimensional modules of the form $L(\lambda, \mu, \zeta)$ and $L(\lambda', \mu', \zeta')$ are not isomorphic unless $(\lambda, \mu, \zeta) = (\lambda', \mu', \zeta')$. To see this, we observe from the formulas for X, Y and Z of Theorem 5.1, that the R-composition factors of $L(\lambda, \mu, \zeta)$ consist of $k + 1$ copies of $F(\lambda, \mu, \zeta)$ and k copies of $F(-\lambda, -\mu, \zeta)$. By the S-generic hypothesis, $(-\lambda, -\mu, \zeta)$ is not a solution of the appropriate equations. This proves the assertion, and completes the proof of Theorem 1.1.

6. Multiplicity 2 Fat Point Modules

We recall the shift operator s on $\mathrm{Proj}(S)$ described in the introduction. For our purposes, it is easier to work with the equivalent operator $\hat{s}$ defined by $\hat{s}(M) = M(1)_{\geq 0}$. In this section we construct infinite families of multiplicity two fat point modules (cf. [3]) for the algebra S.

It is a straightforward calculation to show that $\hat{s}$ is the identity on the set of point modules of S and S'. In contrast to this, we show that $\hat{s}$ has infinite order on the set of multiplicity two fat point modules.

Assume throughout this section that $a \neq 0$ and $c + a^{-1} \neq 0$. Fix scalars λ and μ such that $\lambda^2 = -a$ and $\mu^2 = c + a^{-1}$. Let $V = \bigoplus_{n \in \mathbb{N}} V_n$ be an $\mathbb{N}$-graded vector space with $\dim_K(V_n) = 2$ for all n. Fix a basis $\{e_n, f_n\}$ for each $n \in N$. We can easily make V into a right graded R-module by letting x, y and z map V_n to V_{n+1}, in the given bases, by the matrices $X = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, $Y = \begin{pmatrix} \lambda^{-1} & \mu \\ \mu & -\lambda^{-1} \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. It is easy to see that this is a well-defined right R-module.

Definition 6.1. Let $\gamma = (d - a + (c+b)/a)/(2\mu)$. For $h \in K$ and $n \in \mathbb{N}$, let

$$W_n(h) = \left(\frac{c+b}{-2a}\right) X + \left(\frac{\rho - 1}{2}\right) Z + \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (h - n\lambda\mu) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Theorem 6.2. *For each $h \in K$, the R-module V extends to a graded right S-module, denoted $V(h)$, where the action of w from V_n to V_{n+1} is given by the matrix $W_n(h)$. These modules represent distinct multiplicity two fat points in* $\operatorname{Proj}(S)$.

Proof. The fact that $V(h)$ is a well defined S-module is straightforward and we omit the proof. Since V is already homogeneous and GK-critical as an R-module, so is $V(h)$, and hence $V(h)$ represents a fat point in $\operatorname{Proj}(S)$. Suppose that $V(h)$ and $V(h')$ define the same fat point, i.e. that they have isomorphic tails. We must show that $h = h'$. Since $\hat{s}(V(h)) = V(h - \lambda\mu)$, an easy induction argument allows us to assume that $V(h)$ and $V(h')$ are themselves isomorphic. Since z acts via the identity matrix on both modules (in the standard basis), we see that $W_0(h)$ and $W_0(h')$ must have the same characteristic polynomial. From this we see $h^2 = (h')^2$. But the same holds for $W_1(h)$ and $W_1(h')$, from which we easily conclude $h = h'$, as required. □

We note that the center of S acts trivially on $V(h)$ and that $\hat{s}(V(h)) = V(h - \lambda\mu)$, so that s has infinite order on this set of fat point modules.

References

1. M. Artin, J. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 33–85. MR **MR1086882 (92e:**14002)
2. ______, *Modules over regular algebras of dimension* 3, Invent. Math. **106** (1991), no. 2, 335–388. MR **MR1128218 (93e:**16055)

3. Michael Artin, *Geometry of quantum planes*, Azumaya algebras, actions, and modules (Bloomington, IN, 1990), Contemp. Math., vol. 124, Amer. Math. Soc., Providence, RI, 1992, pp. 1–15. MR **MR1144023 (93b:**14004)
4. Michael Artin and William F. Schelter, *Graded algebras of global dimension* 3, Adv. in Math. **66** (1987), no. 2, 171–216. MR **88k:**16003
5. K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, second ed., London Mathematical Society Student Texts, vol. 61, Cambridge University Press, Cambridge, 2004. MR MR2080008
6. Darin Stephenson and Michaela Vancliff, *Some finite quantum* P^3*'s that are infinite modules over their centers*, To appear, Journal of Algebra.