Riddles

Personal collection

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Contents

1 What this is

This is a private collection of riddles and simple mathematical problems I've stumbled across in the past few years. I do not claim any copyright on them, in fact, some of them seem to be folklore. All of them can be solved, trust me. I only provide example solutions to some of them.

2 Problem statements

2.1 The infinite plane 01

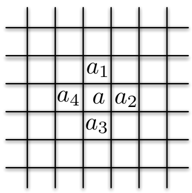
Consider the 2-dimensional plane. We randomly paint each point of that plane either black or white. Show that, regardless of the exact painting, there exists at least one pair of similarly colored points at distance 1 from each other.

2.2 The infinite plane 02

Consider the 2-dimensional plane. We randomly paint each point of that plane either black or white. Show that, regardless of the exact painting, there exist at least three similarly colored points forming the corners of an equilateral triangle.

2.3 The infinite grid

Consider an infinite 2-dimensional matrix (grid) with integer entries from 1 to 100. Each integer equals the arithmetic mean of it's four adjacent numbers (up, down, left & right). Show that all entries are equal or provide a counterexample.



$$a = \frac{a_1 + a_2 + a_3 + a_4}{4}$$

Figure 1: Illustration of the infinite integer grid.

2.4 5-doted triangle

Consider an equilateral triangle of side-length 1. Show that it is impossible to place 5 dots in the interior of that triangle so that they all have mutual distances greater than 0.5 .

2.5 Hat-puzzle

Consider 3 men and 5 hats (2 red ones & 3 white ones). Each man gets one of those 5 hats without seeing the color of his or the other's hats. They then stand in one line, the 2nd one facing the 1st one's back and the 3rd one facing the 2nd one's back. Everybody can only see the hats of those on their front. Obviously, the first one can't see anyone. Their task is to figure out what kind of hat they are wearing and announce it at the end of the current minute. After 3 minutes, the man on the front is the first to announce his color. What kind of hat was it and how did he figure it out?

2.6 Lion's lake

A man is standing on a rock at the centre of a circular lake. Outside that lake, a hungry lion awaits him. Though the lion is afraid of the water, it always tries to minimize its distance to the man. The lion can run 4 times as fast as the man can swim, but the man can run much quicker than the lion does.

How can the man leave the lake safely?

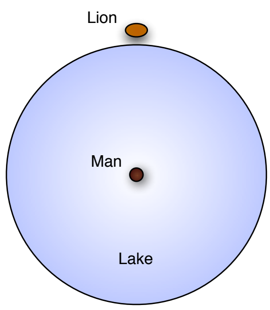


Figure 2: Man trapped within lake.

2.7 Ant-circus

Five ants stand on the corners of a rectangle of side-length 1, numbered clockwise 1 up to 4. Each ant constantly crawls directly towards the next ant (that is, ant 1 hunts ant 2, .., ant 4 hunts ant 1). If all ants crawl at the same constant speed, what distance will they cover until they meet at the centre?

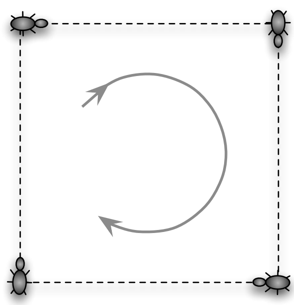
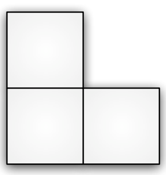


Figure 3: Ants chasing each other.

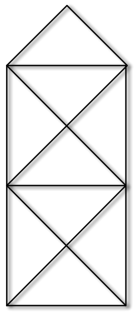
2.8 Chess-cover

Given a $2^n \times 2^n$ chessboard with one corner removed, show that it can be covered exactly without overlapping using only triominoes of the following shape (modulo rotation).



2.9 Double-house

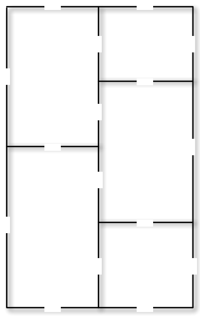
Consider the following task: Draw the shape below with a pencil without lifting the pencil from the paper while only drawing each line segment exactly once.



Show that this task is not possible.

2.10 Wicked house

The shape below represents the down-view on a house. Each line segment is a wall and each rectangle is a room. Each white spot on the wall represents a door between two rooms or between the interior & exterior of the house.



Show that there does not exist any route between the rooms, passing through each door exactly once, or provide a counterexample. The route may start and end in any room or even outside of the house.

2.11 Ghost-prison

Consider a prison consisting of 20 cells, numbered 1 to 20 each containing one prisoner. The prison's head decides to free some of the prisoners, so he sets up a little game. He starts by going through all cells, unlocking each door. He then goes again through cells 2,4,6... this time locking their doors. He continues by going through cells 3,6,9,12.. this time locking all unlocked doors he encounters and unlocking all locked ones. This process is repeated 20 times. I.e., on the 4th round, he visits cells 4,8,12,16,20 and on the 20th round he only visits cell 20. At the end of this game, prisoners in unlocked cells are given their freedom. How many prisoners get to be freed? What if there were 10^{20} cells with prisoners instead of 20?

2.12 Dwarf-island

Consider an island of dwarfs each of whom has either blue eyes (N dwarfs) or red eyes. None can see his own eye color nor can he tell others about theirs. The dwarfs may only look at each other during the common breakfast every morning. During breakfast, they receive the following, publicly announced task for everyone to fulfill:

> *As soon as you find out that your eye color is blue, kill your self on the night to come.*

This task is followed by the public announcement:

> *There is at least one blue-eyed dwarf among you.*

When do the blue-eyed dwarfs kill themselves, depending on their number N?

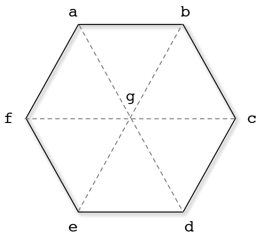
3 Problem solutions

3.1 The infinite plane 01

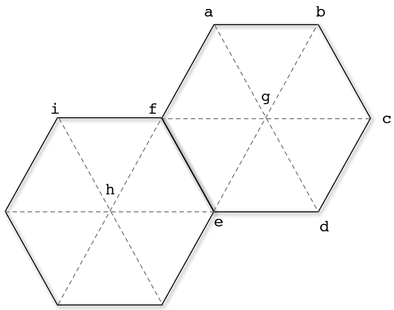
Keyword *pigeonhole principle*: Consider any equilateral triangle of side-length 1. Then at least two of its corners share the same color.

3.2 The infinite plane 02

Assume the contrary ($\star$). Now choose any two points (a) and (b) with similar color, say, w.l.o.g, black. Consider the equilateral hexagon illustrated in the figure below.



By assumption ($\star$), point (g) is white (note the triangle (abg)). By assumption ($\star$), at least one of (e) or (d) is black (note the triangle (ged)), say, w.l.o.g., (e). By assumption ($\star$), point (c) must be white (note the triangle (aec)). Thus (d) must be black (note the triangle (gcd)). Thus (f) must be white (note the triangle (fbd)). Now consider the two equilateral hexagons illustrated in the figure below.



By assumption ($\star$), point (h) is white (note the triangle (adh)). Thus point (i) must be black (consider the triangle (ihf)). But this implies that the equilateral triangle (aie) is monochromatic.

□

3.3 5-doted triangle

Hint: Pigeon hole principle.

3.4 Lion's lake

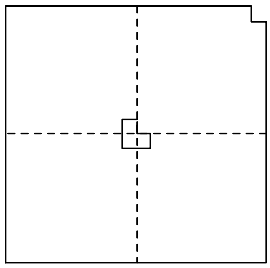
Hint: Find a trajectory in the lake for the man to swim, so that the man arrives at the lake's border at a strictly positive distance to the lion.

3.5 Ant-circus

Hint: What is the angle between their velocities (vectors)? What does this imply about the speed at which they approach each other?

3.6 Chess-cover

Divide the (originally intact) chessboard into 4 equal quadratic chessboards of dimension $2^{n-1} \times 2^{n-1}$, as indicated in the figure below. Covering the central three boxes with a triomino as indicated in the figure, reduces the problem of covering the rest of the broken chessboard with triominoes to the case $2^{n-1} \times 2^{n-1}$. The case $n = 1$ is trivial, so that the assertion follows by induction.



3.7 Double-house

Consider the crossing points of line-segments as *nodes* of a graph, connected by the *edges* defined by the line segments. Without loss of generality we assume drawing always to begin and end on nodes. Any node with an odd number of edges, must be either the start-node or the end-node. Since there are more than two such nodes, the task is impossible.

3.8 Wicked house

Hint: Same principle as in problem 2.9.

3.9 Ghost-prison

Cell doors are manipulated as many times as their numerical index has integer factors. All indices but the ones that are squares have an even number of integer factors. Thus, the eventually open cells will be the ones with a square index, that is cells 1, 4, 9 and 16. For the case 10^{20}, the answer is accordingly 10^{10}.

3.10 Dwarf-island

Clearly, all N dwarfs will, the case at hand, kill themselves on the same night. We show by induction the following two statements:

1. Blue-eyed dwarfs commit suicide on the N-th night.

2. Blue-eyed dwarfs commit suicide no sooner than the N-th night.

The case $N = 1$ is clear: If there was exactly one blue-eyed dwarf, he would conclude so already during the first breakfast, based on the second public announcement and the fact of him not seeing any other blue-eyed dwarf. The case $N = 2$ is also clear: As each blue-eyed dwarf sees his blue-eyed fellow, both still have the hope, at least during the first day, of not being blue-eyed themselves.

Now assume statement (1) to be true for all N up to a certain N_o and statement (2) to be true for all N up to $N_o + 1$. Now consider the case where $N = N_o + 1$. Then by (2), on the $N_o + 1$-th breakfast all blue-eyed dwarfs still live. Since all dwarfs are aware of (1), they conclude that there must be at least $N_o + 1$ blue-eyed dwarfs among them. Since each blue-eyed dwarf only sees N_o of his blue-eyed fellows, they all conclude that they should commit suicide on the following night. Thus statement (1) must also be true for $N = N_o + 1$. Now consider the case where $N = N_o + 2$. Then for all nights before $N_o + 2$, every blue-eyed dwarf will see $N_o + 1$ blue-eyed fellow dwarfs. For each one of them, the scenario of blue-eyed dwarfs being just $N_o + 1$ in number (thus, they themselves red-eyed) is up to breakfast $N_o + 1$ a valid one, since in that scenario their poor comrades would by (1) not commit suicide before night $(N_o + 1)$. They would thus all meet at breakfast $(N_o + 2)$, therefore verifying statement (2) for $N = N_o + 2$. This finishes our induction.