Problem solutions, 18 April 2012

D. E. Soper
University of Oregon
27 April 2012

Problem 5.1 is pretty simple, so I do not write out the solutions.

**Problem 5.2** The probability to find the unperturbed eigenstate $|k^{(0)}\rangle$ in the exact eigenstate $|k(\lambda)\rangle$ is

$$P = \frac{|\langle k^{(0)}|k(\lambda)\rangle|^2}{\langle k(\lambda)|k(\lambda)\rangle} . \quad (1)$$

Note that we need the denominator because, with the conventions we are using, $|k(\lambda)\rangle$ is not normalized. In fact, our normalization convention for $|k(\lambda)\rangle$ is $\langle k^{(0)}|k(\lambda)\rangle = 1$, so

$$P = \frac{1}{\langle k(\lambda)|k(\lambda)\rangle} . \quad (2)$$

Now with

$$|k(\lambda)\rangle = |k^{(0)}\rangle + \lambda|k^{(1)}\rangle + \lambda^2|k^{(2)}\rangle + \cdots \quad (3)$$

and $\langle k^{(0)}|k^{(0)}\rangle = 1$, $\langle k^{(n)}|k^{(0)}\rangle = 0$ for $n > 0$, we have

$$\langle k(\lambda)|k(\lambda)\rangle = 1 + \lambda^2\langle k^{(1)}|k^{(1)}\rangle + \cdots . \quad (4)$$

Thus

$$P = 1 - \lambda^2\langle k^{(1)}|k^{(1)}\rangle + \cdots . \quad (5)$$

Recall that

$$|k^{(1)}\rangle = \frac{Q_k}{E_k^{(0)} - H_0} V|n^{(0)}\rangle . \quad (6)$$

Thus

$$\langle k^{(1)}|k^{(1)}\rangle = \langle k^{(0)}|V\frac{Q_k}{[E_k^{(0)} - H_0]^2} V|k^{(0)}\rangle . \quad (7)$$
Inserting a complete set of unperturbed eigenstates, this is
\[
\langle k^{(1)} | k^{(1)} \rangle = \sum_{l \neq k} \langle k^{(0)} | V | l^{(0)} \rangle \frac{1}{[E_k^{(0)} - E_l^{(0)}]^2} \langle l^{(0)} | V | k^{(0)} \rangle .
\] (8)

Thus
\[
P = 1 - \lambda^2 \sum_{l \neq k} \frac{|\langle l^{(0)} | V | k^{(0)} \rangle|^2}{[E_k^{(0)} - E_l^{(0)}]^2} + \ldots.
\] (9)

**Problem 5.3** Our particle is in a two dimensional box. The energy eigenfunction for the ground state is
\[
\psi(x, y) = \frac{2}{L} \sin(\pi x/L) \sin(\pi y/L)
\] (10)

For the first excited energy level, there are two states
\[
\psi(x, y) = \frac{2}{L} \sin(2\pi x/L) \sin(\pi y/L),
\]
\[
\psi(x, y) = \frac{2}{L} \sin(\pi x/L) \sin(2\pi y/L).
\] (11)

Thus

The energy shift for the ground state is just
\[
\Delta = \int_0^L dx \int_0^L dy \, |\psi(x, y)|^2 \lambda xy
\]
\[
= \frac{4\lambda}{L^2} \left( \int_0^L dx \, x \sin^2(\pi x/L) \right)^2
\]
\[
= \frac{4\lambda}{L^2} \left( \frac{1}{2} \int_0^L dx \, (x + L - x) \sin^2(\pi x/L) \right)^2
\]
\[
= \frac{4\lambda}{L^2} \left( \frac{L^2}{4} \right)^2
\]
\[
= \frac{\lambda L^2}{4} .
\] (12)

The zeroth order energy eigenfunction is just the unperturbed ground state wave function.

For the first excited energy level, there is a twofold degeneracy at zeroth order so we need degenerate perturbation theory. However, we note that the
perturbation is invariant under the operator $P$ that interchanges $x$ and $y$. Thus we can diagonalize $P$ along with $H$. The eigenstates of $P$ within the space spanned by the first excited states are

$$
\psi_+(x,y) = \frac{\sqrt{2}}{L} \{ \sin(2\pi x/L) \sin(\pi y/L) + \sin(\pi x/L) \sin(2\pi y/L) \},
$$
\[ \text{(13)} \]

$$
\psi_-(x,y) = \frac{\sqrt{2}}{L} \{ \sin(2\pi x/L) \sin(\pi y/L) - \sin(\pi x/L) \sin(2\pi y/L) \}.
$$

Thus these are the zeroth order energy eigenstates. The corresponding energy shifts are

$$
\Delta_\pm = \int_0^L dx \int_0^L dy |\psi_\pm(x,y)|^2 \lambda xy
$$

$$
= \frac{2\lambda}{L^2} \left( \int_0^L dx \ x \sin^2(2\pi x/L) \right) \left( \int_0^L dy \ y \sin^2(\pi y/L) \right) + \frac{2\lambda}{L^2} \left( \int_0^L dx \ x \sin^2(\pi x/L) \right) \left( \int_0^L dy \ y \sin^2(2\pi y/L) \right)
$$

$$
\pm \frac{4\lambda}{L^2} \left( \int_0^L dx \ x \sin(\pi x/L) \sin(2\pi x/L) \right) \left( \int_0^L dy \ y \sin(\pi y/L) \sin(2\pi y/L) \right)
$$

$$
\times \left( \int_0^L dy \ y \sin(\pi y/L) \sin(2\pi y/L) \right)
$$

$$
= \frac{2\lambda}{L^2} \left( \frac{L^2}{4} \right) \left( \frac{L^2}{4} \right) + \frac{2\lambda}{L^2} \left( \frac{L^2}{4} \right) \left( \frac{L^2}{4} \right) \pm \frac{4\lambda}{L^2} \left( \frac{8L^2}{9\pi^2} \right) \left( \frac{8L^2}{9\pi^2} \right)
$$

$$
= \lambda L^2 \left\{ \frac{1}{4} \pm \frac{256}{81\pi^4} \right\}.
$$

\[ \text{(14)} \]

**Problem 5.4** We have a harmonic oscillator in the $x$-direction and a harmonic oscillator in the $y$-direction. The energies are

$$
E(n_x,n_y) = (n_x + n_y + 1)\omega,
$$
\[ \text{(15)} \]

where $n_x$ and $n_y$ are non-negative integers. Thus the lowest energy levels are

$$
E(0,0) = \omega,
$$
$$
E(1,0) = 2\omega,
$$
$$
E(0,1) = 2\omega.
$$
\[ \text{(16)} \]
The second energy level has a double degeneracy.

The first order energy shift for the ground state \( |0,0\rangle \) is just
\[
\Delta = \delta m \omega^2 \langle 0,0 | xy | 0,0 \rangle = 0 .
\] (17)

That is, the ground state energy remains equal to \( \omega \) at first order.

For the first excited energy level, there is a twofold degeneracy at zeroth order so we need degenerate perturbation theory. However, we note that the perturbation is invariant under the operator \( P \) that interchanges \( x \) and \( y \). Thus we can diagonalize \( P \) along with \( H \). The eigenstates of \( P \) within the space spanned by the first excited states are
\[
|\pm\rangle = \frac{1}{\sqrt{2}} \left\{ |1,0\rangle \pm |0,1\rangle \right\} .
\] (18)

Thus these are the zeroth order energy eigenstates. The corresponding first order energy shifts are
\[
\Delta_\pm = \frac{\delta m \omega^2}{2} \left\{ \langle 1,0 | xy | 1,0 \rangle + \langle 0,1 | xy | 0,1 \rangle \pm \langle 1,0 | xy | 0,1 \rangle \pm \langle 0,1 | xy | 1,0 \rangle \right\}
\]
\[
= \pm \delta m \omega^2 \left\{ \langle 1 | x | 0 \rangle \langle 0 | y | 1 \rangle + \langle 1 | y | 0 \rangle \langle 0 | x | 1 \rangle \right\}
\] (19)

where in the second line the first matrix element refers to the \( x \)-oscillator and the second matrix refers to the \( y \)-oscillator. These matrix elements are
\[
\langle 1 | x | 0 \rangle = \langle 0 | x | 1 \rangle = \langle 1 | y | 0 \rangle = \langle 0 | y | 1 \rangle = \frac{1}{\sqrt{2m\omega}}
\] (20)

Thus
\[
\Delta_\pm = \pm \frac{\delta m \omega}{m} .
\] (21)

The energy eigenvalues correct to first order in the perturbation are then
\[
E_{\pm} = 2\omega \pm \frac{\delta m \omega}{m} .
\] (22)

Now, let’s solve this exactly. We have
\[
H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2} (x^2 + y^2 + \frac{\delta m}{m} 2xy)
\] (23)
This is

\[ H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{(m + \delta m)\omega^2}{4}(x + y)^2 + \frac{(m - \delta m)\omega^2}{4}(x - y)^2 . \]  

(24)

Let us define new coordinates

\[
\bar{x} = \frac{x + y}{\sqrt{2}}, \\
\bar{y} = \frac{x - y}{\sqrt{2}},
\]

(25)

Then

\[ H = \frac{p_{\bar{x}}^2}{2m} + \frac{p_{\bar{y}}^2}{2m} + \frac{(m + \delta m)\omega^2}{2}\bar{x}^2 + \frac{(m - \delta m)\omega^2}{2}\bar{y}^2 . \]  

(26)

Now we have two independent oscillators. The total energy is

\[ E(n_{\bar{x}}, n_{\bar{y}}) = \left(n_{\bar{x}} + \frac{1}{2}\right)\omega\sqrt{1 + \frac{\delta m}{m}} + \left(n_{\bar{y}} + \frac{1}{2}\right)\omega\sqrt{1 - \frac{\delta m}{m}} . \]  

(27)

Up to first order in \( \delta m \), this is

\[ E(n_{\bar{x}}, n_{\bar{y}}) \approx (n_{\bar{x}} + n_{\bar{y}} + 1)\omega + (n_{\bar{x}} - n_{\bar{y}})\frac{\delta m}{m} . \]  

(28)

This agrees with what we found by applying first order perturbation theory directly.