Sakurai, problem 2.21 parts (a) and (b)

Part (a) is a standard exercise in solving the time independent Schrödinger equation by the separation of variables method. We use cylindrical coordinates $\rho, \phi, z$ and write the Hamiltonian as

$$ H = -\frac{1}{2m_e} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \left( \frac{\partial}{\partial \phi} \right)^2 + \left( \frac{\partial}{\partial z} \right)^2 \right] . $$

(1)

We look for an eigenfunction in the form

$$ \psi(\rho, \phi, z) = CR(\rho)Q(\phi)Z(z) . $$

(2)

Here $C$ is a normalization constant that will not concern us. (My notation is from Jackson, *Classical Electrodynamics.*) For $Z(z)$, we need an eigenfunction of $-(d/dz)^2$. To match the boundary conditions at $z = 0$ and $z = L$ we take

$$ Z(z) = \sin(k_l z) , $$

(3)

where

$$ k_l = \pi l / L $$

(4)

for $l = 1, 2, \ldots$. For $Q(\phi)$, we need an eigenfunction of $-(d/d\phi)^2$:

$$ Q(\phi) = e^{im\phi} , $$

(5)

for $m = 0, \pm 1, \pm 2, \ldots$.

With these choices, we have

$$ 0 = (H - E)R(\rho)Q(\phi)Z(z) $$

$$ = \frac{1}{2m_e} \left[ -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{m^2}{\rho^2} + k_l^2 - 2m_eE \right] R(\rho)Q(\phi)Z(z) . $$

(6)
Dividing by $Q(\phi)Z(z)/(2m_e)$ gives

$$0 = \left[ -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{m_e^2}{\rho^2} + k_i^2 - 2m_eE \right] R(\rho) . \quad (7)$$

The solutions of this are Bessel functions

$$R(\rho) = \alpha J_m(k\rho) + \beta N_m(k\rho) , \quad (8)$$

where

$$k = \sqrt{2m_eE - k_i^2} . \quad (9)$$

Now to match the boundary condition at $\rho = \rho_a$ we need

$$R(\rho) = N_m(k\rho_a) J_m(k\rho) - J_m(k\rho_a) N_m(k\rho) . \quad (10)$$

Then to match the boundary condition at $\rho = \rho_b$ we need

$$0 = N_m(k\rho_a) J_m(k\rho_b) - J_m(k\rho_a) N_m(k\rho_b) . \quad (11)$$

This equation has solutions $k_{mn}$, labelled by the $m$ that specifies the $\phi$ dependence and by a new index $n = 1, 2, \ldots$ that specifies which solution we pick. Then

$$E_{nm} = \frac{k_{mn}^2}{2m_e} + \frac{k_i^2}{2m_e} \quad (12)$$

We have found the corresponding eigenfunctions (up to the normalization) as the product $R(\rho)Q(\phi)Z(z)$.

Now we can turn to the case with a magnetic field. Here

$$H = \frac{1}{2m_e} \left( -i \vec{\nabla} - q \vec{A} \right)^2$$

$$= -\frac{1}{2m_e} \vec{\nabla}^2 + \frac{iq}{2m_e} (\vec{\nabla} \cdot \vec{A}) + \frac{iq}{m_e} \vec{A} \cdot \vec{\nabla} + \frac{q^2}{2m_e} \vec{A}^2 . \quad (13)$$

We can choose a vector potential such that $\vec{\nabla} \cdot \vec{A} = 0$. Then

$$H = -\frac{1}{2m_e} \vec{\nabla}^2 + \frac{iq}{m_e} \vec{A} \cdot \vec{\nabla} + \frac{q^2}{2m_e} \vec{A}^2 . \quad (14)$$

Now pick the vector potential, adapted to our coordinate system. We take

$$\vec{A}(\rho, \phi, z) = \frac{B\rho_a^2}{2\rho} \vec{n}_\phi \quad (15)$$
where $\vec{n}_\phi$ is a unit vector in the direction of increasing $\phi$. That is

$$A_x = -\frac{B\rho_a^2}{2} \frac{1}{x^2 + y^2} y ,$$

$$A_y = \frac{B\rho_a^2}{2} \frac{1}{x^2 + y^2} x ,$$

$$A_y = 0 .$$

We easily verify that $\vec{\nabla} \cdot \vec{A} = 0$. Then we verify that $\vec{\nabla} \times \vec{A} = 0$, so there is no magnetic field in the region between the shells. However

$$\int d\vec{x} \cdot \vec{A} = \pi \rho_a^2 B .$$

That is, we get the flux of magnetic field inside the inner shell, which is what we wanted. Since

$$\vec{n}_\phi \cdot \vec{\nabla} = \frac{1}{\rho} \frac{\partial}{\partial \phi}$$

we have

$$H = -\frac{1}{2m_e} \vec{\nabla}^2 + \frac{i q B\rho_a^2}{2m_e} \frac{1}{\rho^2} \frac{\partial}{\partial \phi} + q^2 B^2 \rho_a^4 \frac{1}{8m_e} \frac{1}{\rho^2}$$

$$= \frac{1}{2m_e} \left[ -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + i q B\rho_a^2 \frac{1}{\rho^2} \frac{\partial}{\partial \phi} + \frac{q^2 B^2 \rho_a^4}{4} \frac{1}{\rho^2} \right] .$$

Taking $\psi(\rho, \phi, z) = CR(\rho)Q(\phi)Z(z)$ with the same form for $Q(\phi)$ and $Z(z)$ as before, the eigenvalue equation becomes

$$0 = \left[ -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) - \frac{q B\rho_a^2 m}{\rho^2} + \frac{q^2 B^2 \rho_a^4}{4\rho^2} + \frac{m^2}{\rho^2} + k_l^2 - 2m_e E \right] R(\rho) .$$

This is

$$0 = \left[ -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{\nu^2}{\rho^2} + k_l^2 - 2m_e E \right] R(\rho) .$$

where

$$\nu = m - \frac{1}{2} q B\rho_a^2 .$$
The equation is the same as we had with $B = 0$, but now the integer $m$ is replaced by $\nu$, which is generally not an integer. The Bessel functions $J_\nu$ and $N_\nu$ are still defined. The energy eigenvalues are shifted.

**In Sakurai, problem 2.39 find the $n = 0$ wave function.**

Let’s just recall the solution to problem 2.39. We have a hamiltonian. For a particle with charge $q$ in a vector potential $\vec{A}$ and a scalar potential $\phi$, both time independent, we take

$$H = \frac{1}{2m} \vec{\Pi}^2 ,$$

(23)

where

$$\vec{\Pi} = \vec{p} - q\vec{A}(\vec{x})$$

(24)

and the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ is in the $z$ direction and has a constant value $B$. Using

$$[\Pi_i, \Pi_j] = iq \sum_k \epsilon_{ijk} B_k ,$$

(25)

we have

$$[\Pi_y, \Pi_z] = 0 ,$$

$$[\Pi_z, \Pi_x] = 0 ,$$

$$[\Pi_x, \Pi_y] = iqB .$$

(26)

We see from this that $[\Pi_z, H] = 0$, so that we can diagonalize $\Pi_z$ and $H$ at the same time. Thus we can look for states $|n, p_z, \xi\rangle$ where $H|n, p_z, \xi\rangle = E_n|n, p_z, \xi\rangle$ and $\Pi_z|n, p_z, \xi\rangle = p_z|n, p_z, \xi\rangle$.

Now let’s make this look like a harmonic oscillator by defining

$$a = \frac{1}{\sqrt{2qB}}(\Pi_x + i\Pi_y) ,$$

$$a^\dagger = \frac{1}{\sqrt{2qB}}(\Pi_x - i\Pi_y) ,$$

$$N = a^\dagger a .$$

(27)

We find that

$$H = \frac{qB}{m} \left( N + \frac{1}{2} \right) + \frac{1}{2m} \Pi_z^2 .$$

(28)
Let us choose states $|n, p_z, \xi\rangle$ that are eigenstates of $\Pi_z$ and $N$ with

\begin{align}
N|n, p_z, \xi\rangle &= n|n, p_z, \xi\rangle, \\
\Pi_z|n, p_z, \xi\rangle &= p_z|n, p_z, \xi\rangle.
\end{align}

(29)

Then

\begin{equation}
H|n, p_z, \xi\rangle = \left[ \frac{qB}{m} \left( n + \frac{1}{2} \right) + \frac{p_z^2}{2m} \right] |n, p_z, \xi\rangle
\end{equation}

(30)

This gives the eigenvalues of $H$ as soon as we know what the eigenvalues of $N$ and $\Pi_z$ are. There is no restriction on the eigenvalues $p_z$ of $\Pi_z$, so $p_z^2/(2m)$ can be any positive number.

To find the eigenvalues of $N$, we note that

\begin{equation}
[a, a^\dagger] = -\frac{i}{qB} [\Pi_x, \Pi_y] = 1.
\end{equation}

(31)

Thus

\begin{align}
Na &= a(N - 1), \\
Na^\dagger &= a^\dagger(N + 1).
\end{align}

(32)

That is, applying $a$ to an eigenstate of $N$ lowers the eigenvalue by 1, while applying $a^\dagger$ raises the eigenvalue by 1. Just as with the harmonic oscillator, we derive that the eigenvalues $n$ of $N$ are non-negative integers.

Now to continue on from this solution, we seek the wave function for $n = 0$. What $H|n, p_z, \xi\rangle = E|n, p_z, \xi\rangle$ says about the wave function is gauge dependent. Let us choose

\begin{align}
A_x &= 0, \\
A_y &= B x, \\
A_z &= 0.
\end{align}

(33)

Then $\vec{\nabla} \times \vec{A} = \vec{B}$ has only a $z$ component, with value $B$. With this choice, $H$ contains an $x$ but not a $y$. Therefore $H$ commutes with $p_y$. Thus we can diagonalize $p_y$,

\begin{equation}
p_y|n, p_z, \xi\rangle = \xi|n, p_z, \xi\rangle.
\end{equation}

(34)

We have already chosen to require

\begin{equation}
\Pi_z|n, p_z, \xi\rangle = p_z|n, p_z, \xi\rangle.
\end{equation}

(35)
To achieve both relations, we simply take
\[ \langle x, y, z | n, p_z, \xi \rangle = e^{i\xi y} e^{i p_z z} f(x) \] (36)

We now need to find \( f(x) \). The \( n = 0 \) states obey
\[ a|0, p_z, \xi \rangle = 0 . \] (37)

We have
\[
a = \frac{1}{\sqrt{2qB}} (\Pi_x + i \Pi_y ) \\
= \frac{-i}{\sqrt{2qB}} (\partial_x + i \partial_y + q B x) . \] (38)

We thus need
\[
0 = (\partial_x + i \partial_y + q B x) e^{i\xi y} e^{i p_z z} f(x) . \] (39)

That is
\[
0 = (\partial_x - \xi + q B x) f(x) . \] (40)

The solution to this is
\[ f(x) = C \exp \left( - \frac{qB}{2} x^2 + \xi x \right) , \] (41)

where \( C \) is a constant. That is
\[ f(x) = C' \exp \left( - \frac{qB}{2} \left( x - \frac{\xi}{qB} \right)^2 \right) , \] (42)

where \( C' \) is a different constant. Thus we get a gaussian wave function centered at \( x = \xi/(qB) \). If \( B \) is big, the width of the gaussian is small, corresponding to small orbits in the classical picture.