Notes from Craigfest

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1 Fibrations and cofibrations

References: [1, 2, 7]

1.1 Fibrations

Definition 1.1. Let $E, B$ be topological spaces. A map $p : E \to B$ has the homotopy lifting property (HLP) with respect to $Y$ if there exists $\tilde{h}$ that makes the following diagram commute:

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & E \\
i_0 \downarrow & & \downarrow p \\
Y \times I & \xrightarrow{h} & B
\end{array}
\]

where $i_0 : Y \hookrightarrow Y \times I$ is given by $y \mapsto (y, 0)$.

Definition 1.2. A (Hurewicz) fibration is a map $p : E \to B$ that has HLP for all spaces $Y$.

Remarks:

1. We have defined a Hurewicz fibration. There is also the concept of a Serre fibration which has ‘all spaces $Y$’ replaced with ‘all $I^n$’.
2. If $b_0 \in B$, then we call $p^{-1}(b_0)$ the fibre.
3. The fibres need not be be homeomorphic, but they are in fact homotopic. (See eg [2])
4. We can denote a fibration by $F \hookrightarrow E \xrightarrow{p} B$. $F$ is the fibre, $E$ is the total space and $B$ is the base space.

Examples:

1. Covering spaces $p : \tilde{X} \to X$ are obviously fibrations. In fact, they satisfy the unique homotopy lifting property.
2. $E = B \times F$, $p : E \to B$ is the trivial fibration.
3. $S^1 \hookrightarrow S^3 \to S^2$ is the Hopf fibration.

1.2 Pull-back of a fibration

Given a fibration $p : E \to B$ and a map $g : A \to B$, define

\[A \times_g E := \{(a, e) \in A \times E | g(a) = p(e)\} \subset A \times E.\]

Then the following diagram commutes:

\[
\begin{array}{ccc}
A \times_g E' & \xrightarrow{p} & E \\
\downarrow \pi & & \downarrow p \\
A & \xrightarrow{g} & B
\end{array}
\]

NOTE: $A \times_g E$ is the pullback of the data given by $g$ and $p$. In particular it satisfies the following universal property: If $\pi' : W \to A$ and $p' : W \to E$ are two other maps that commute with $g$ and $p$, then they factor
Lemma 1.3. $\pi : A \times_g E \to A$ is a fibration.

Proof. We want to show that, if $Y$ is a topological space, and $h : Y \times I \to A$ is a homotopy, then we can find a lift $\tilde{h} : Y \times I \to A \times_g E$. We can use the universal property of a pullback to give the existence of a lift. Pictorially:

1.3 Fibre bundles

Definition 1.4. A fibre bundle $E$ over $B$ with fibre $F$ consists of a surjection $p : E \to B$ with the property:

If $x \in B$, then there exists an open set $U$ containing $x$ and a homeomorphism $\varphi : U \times F \to \varphi^{-1}(U)$ such that the following diagram commutes:

Theorem 1.5. For paracompact base space $B$, a fibre bundle is a fibration.

Proof. See [7]. Here is a sketch.

Let $p : E \to B$ be a continuous map such that $B$ is paracompact. Let $\{U_\alpha\}_\alpha$ be an open cover of $B$ such that $p_\alpha : p^{-1}(U_\alpha) \to U_\alpha$ is a fibration for all $\alpha$. This can be done by local triviality of fibre bundles. Then we can use paracompactness to glue everything together to get that $p : E \to B$ is a fibration.

Remarks:

- The converse is false. Fibre bundles have homeomorphic fibres but fibrations only need to have homotopy equivalent ones.
- Proposition 4.4.8 in [1] proves the result for Serre fibrations without the paracompactness condition.

Definition 1.6. If $X$ and $Y$ are topological spaces, then $X^Y$ is the set of all continuous functions $Y \to X$.

We can endow $X^Y$ with the compact-open topology to turn $X^Y$ into a topological space. The compact-open topology on $X^Y$ is the topology having subbasis being all subset $(K,U)$ where

$$(K,U) := \{f \in X^Y | K \text{ is compact, } U \text{ is open, } f(K) \subset U\}.$$ 

Exercise 1.7. Show that $X^Y$ together with the topology having subbasis being all the subsets of the form $(K,U)$ is a topological space.
**Definition 1.8.** Let \((Y, y_0)\) be a based space. The path space \(P_{y_0}Y\) is the space of all paths in \(Y\), starting at \(y_0\). That is,

\[
P_{y_0}Y := \text{Map}((Y, y_0), (I, 0)).
\]

We sometimes drop the subscript \(y_0\) and just write \(PY\).

Let \(p : PY \to Y\) be given by \(p(\alpha) = \alpha(1)\) be evaluation at the end of that path.

**Theorem 1.9.**

1. The map \(p : PY \to Y\) is a fibration. The fibre is \(\Omega Y\).

2. \(PY\) is contractible.

**Proof.** See [2].

Remark: The fact that \(PY\) is contractible allows us to use the Serre spectral sequence to compute the cohomology of \(\Omega Y\) if no know the cohomology of \(Y\).

### 1.4 Cofibrations

**Definition 1.10.** Let \(A\) and \(X\) be topological spaces. A map \(i : A \to X\) satisfies the homotopy extension property (HEP) with respect to \(Y\) if there exists a \(\tilde{h}\) that makes the following diagram commute:

\[
\begin{array}{ccc}
A & \xrightarrow{h} & Y' \\
i & \downarrow & \downarrow p_0 \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \(p_0(\xi) = \xi(0)\).

**Definition 1.11.** A map \(i : A \to X\) is a cofibration if it satisfies the homotopy extension property for all spaces \(Y\).

Remark: We often think of \(A\) as a subset of \(X\) with \(i\) the inclusion map. Then \(i : A \to X\) is a cofibration if it satisfies: given a \(Y\) and a homotopy \(h : A \times I \to Y\), then \(h\) can be extended to a homotopy on all of \(X\).

### 1.5 Pushout of a cofibration

Given a map \(g : A \to B\) and a map \(i : A \to X\) we can form the pushout \(B \cup_g X := (B \cup X)/(i(a) \sim g(a))\). Pushouts are dual to pull-backs and satisfy a similar universal property. Here is the associated diagram showing the universal property (you can guess how to write it down explicitly):

\[
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
g \downarrow & \uparrow & \uparrow \downarrow \\
B & \xrightarrow{g'} & B \cup_g X & \xrightarrow{W}
\end{array}
\]

**Lemma 1.12.** The pushout of a cofibration is a cofibration.

**Proof.** Dualise the proof for fibration.
1.6 Thinking of \((X, A)\) as a cofibration

Often we consider pairs \((X, A)\) where \(A\) is a subspace of \(X\). Then the inclusion \(i : A \to X\) is a cofibration if it satisfies: if \(f : X \to Y\) is a map of topological spaces and \(h : A \times I \to X\) is a homotopy, then there exists a map making the following diagram commute:

\[
\begin{array}{ccc}
X \times \{0\} \cup A \times I & \xrightarrow{(f, h)} & Y \\
\downarrow & & \\
X \times I & \rightarrow & \end{array}
\]

**Definition 1.13.** A pair \((X, A)\) is called a neighbourhood deformation retract (NDR) pair if there exists a \(u : X \to I\) and a \(h : X \times I \to X\) such that

1. \(A = u^{-1}(0)\);
2. \(h(\cdot, 0) = \text{id}_X\);
3. if \(t \in I\) and \(a \in A\), then \(h(a, t) = a\);
4. if \(x \in X\) and \(u(x) < 1\), then \(h(x, 1) \in A\).

Moreover, \((X, A)\) is a DR pair if it satisfies: if \(x \in X\) then \(u(x) < 1\). We say that \(A\) is a deformation retract of \(X\).

Remark: We can think of these conditions meaning that there is a neighbourhood of \(A\) that deformation retracts to \(A\). In particular, all CW-pairs \((X, A)\) where \(X\) is a CW-complex and \(A\) is a subcomplex are NDR pairs.

**Theorem 1.14.** Let \(A\) be a closed subspace of \(X\). The following are equivalent.

1. \((X, A)\) is a NDR pair.
2. \((X \times I, X \times \{0\} \cup A \times I)\) is a DR pair.
3. \(X \times \{0\} \cup A \times I\) is a retract of \(X \times I\).
4. The inclusion \(i : A \to X\) is a cofibration.

**Proof.** Exercise.

2 Cofibre sequences and higher homotopy groups.

**Definition 2.1 (Mapping cylinder).** Let \(f : A \to X\). The mapping cylinder \(M_f\) is

\[
M_f := A \times I \sqcup X/((a, 1) \sim f(a)).
\]

Remarks:

1. \(M_f\) is the pushout of \(f : A \to X\) and \(A \looparrowright A \times I\).
2. \(M_f\) deformation retracts onto \(X\) by sliding each point \((a, t)\) along \(\{a\} \times I \subset M_f\) to \(f(a) \in X\).

**Definition 2.2 (Mapping cone).** Let \(f : A \to X\). The mapping cone \(C_f\) is

\[
C_f := A \times I \sqcup X/((a, 0) \sim f(a), (a, 0) \sim (a', 0)) = M_f/(A \times 0).
\]

Remark: The mapping cone is the pushout \(C_f = X \sqcup_f CA\), where \(CA\) is the cone of \(\text{id}_A : A \to A\).
Definition 2.3 (Smash product). Let $X$ and $Y$ be topological spaces. Define

$$X \land Y := \frac{X \times Y}{X \lor Y}.$$  

If $X$ and $Y$ are based spaces, then we have

$$X \land Y = \frac{X \times Y}{X \times \{y_0\} \cup Y \times \{x_0\}}.$$  

Remarks:

1. If you are familiar with the category of $R$-modules, then you may recall that there is an adjunction from between the functors $- \otimes B$ and $\text{Hom}(B, -)$. This means that if $A$ and $C$ are $R$-modules, then there is an isomorphism $\text{Hom}(A \otimes B, C) \simeq \text{Hom}(A, \text{Hom}(B, C))$.

2. By analogy, (so we aren’t being too precise here), there is an adjunction between $- \land A$ and $\text{Maps}_*(A, -)$, giving isomorphisms $\text{Maps}_*(X \land A, Y) \simeq \text{Maps}_*(X, \text{Maps}_*(A, Y))$. The analogy is to think of smash product like tensor product. If we want to make this more precise, we need to show that the category of modules with tensor product is symmetric monoidal and so is the category of pointed topological spaces with smash product (!! - can clarify in future).

Definition 2.4 (Reduced suspension). Let $X$ be a topological space. The reduced suspension of $X$ is

$$\Sigma X := \frac{X \times I}{(X \times \{0, 1\}) \cup \{x_0\} \times I}.$$  

Theorem 2.5. $X \land S^1 = \Sigma X$, where $S^1 = I/\partial I$.

Proof. Exercise.

Theorem 2.6. The suspension functor is adjoint to the loop space functor. That is $[\Sigma X, Y] \simeq [X, \Omega Y]$, where $[A, B]$ are the maps from $A$ to $B$ up to homotopy.

Proof. We have $\text{Maps}_*(X \land A, Y) \simeq \text{Maps}_*(X, \text{Maps}_*(A, Y))$ from the above comment. Taking $A = S^1$ and using Theorem 2.5 we have

$$\text{Maps}_*(\Sigma X, Y) \simeq \text{Maps}_*(X, \text{Maps}_*(S^1, Y))$$

$$\text{Maps}_*(\Sigma X, Y) \simeq \text{Maps}_*(X, \Omega Y)$$

And taking $\pi_0$ of both sides gives the result.

Theorem 2.7. Let $f : X \to Y$ and $C_f$ be the cone of $f$. Let $i : Y \hookrightarrow C_f$ be the inclusion of $Y$ into $C_f$ given by $y \mapsto y$. Let $C_i$ be the cone of the inclusion map.

$C_i$ is homotopy equivalent to $\Sigma X$, and we get a long exact sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \to \Sigma X \to \Sigma Y \to \ldots$$

called the cofibre sequence.

Proof. There is an inclusion $C_f \to C_i$. Thus we have a sequence $X \to Y \to C_f \to C_i$. Once we have shown that $C_i$ is homotopy equivalent to $\Sigma X$, we are done since we have

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \to C_i \simeq \Sigma X \to \Sigma Y \to \ldots$$

[I STILL WANT TO FLESH THIS OUT A BIT MORE FOR MYSELF]

Theorem 2.8. If $X$ and $Y$ be pointed, path connected, CW complexes and $Z$ is any pointed space, then the cofibre sequence induces a long exact sequence:

$$[X, Z] \xleftarrow{f^*} [Y, Z] \xleftarrow{i^*} [C_f, Z] \leftarrow [\Sigma X, Z] \leftarrow [\Sigma Y, Z] \leftarrow \ldots$$

This is a LES of groups to the right of $[C_f, Z]$ and is abelian for $[\Sigma^2, -]$ or pointed sets for the first three sets.
2.1 Fibre sequences

For a based map $f : X \to Y$, define the homotopy fibre to be

$$F_f := X \times_f PY = \{(x, \lambda) | (f(x) = \lambda(1)) \} \subset X \times PY.$$  

Equivalently, it is the pullback of the data $f : X \to Y$ and $p_1 : PY \to Y$, where $p_1(\lambda) = \lambda(1)$.

$$
\begin{array}{ccc}
F_f \times_g E & \longrightarrow & PY \\
\pi \uparrow & & \downarrow p_1 \\
A & \longrightarrow & B
\end{array}
$$

Note that $\pi : F_f \to X$ is a fibration since $PY \to Y$ is a fibration. Let $m : \Omega Y \to F_f$ be the inclusion $m(x) = (\ast, x)$. We have the fibre sequence

$$
\Omega F_f \xrightarrow{\Omega m} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{m} F_f \xrightarrow{f} X \xrightarrow{f} Y.
$$

**Theorem 2.9.** For any based space $Z$, the induced sequence

$$
\ldots \rightarrow [Z, \Omega F_f] \rightarrow [Z, \Omega X] \rightarrow [Z, \Omega Y] \rightarrow [Z, \Omega F_f] \rightarrow [Z, X] \rightarrow [Z, Y].
$$

is a long exact sequence. This is called the Puppe sequence. The last three terms are pointed sets, while after that the terms are groups.

**Proof.** No proof in notes... you do it!

2.2 Higher homotopy groups

We can define the higher homotopy groups similar to how we define $\pi_1$.

**Definition 2.10.** $\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]$, where $s_0 = (1, 0, \ldots, 0)$.

Usually we write $\pi_n(X)$ for any path-connected space $X$ since in this case, $\pi_n(X)$ is independent of choice of basepoint. Observe that

$$
\pi_n(\Omega X) = [S^n, \Omega X] = \Sigma S^n, X] = [S^{n+1}, X] = \pi_{n+1}(X).
$$

**Theorem 2.11.** For a fibration $F \to E \to B$, we have a long exact sequence in homotopy given by

$$
\ldots \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \rightarrow \pi_{n-1}(F) \rightarrow \ldots \rightarrow \pi_0(F) \rightarrow \pi_0(E) \rightarrow \pi_0(B)
$$

**Proof.** Putting $Z = S^0$ in the Puppe sequence gives

$$
\ldots \rightarrow [S^0, \Omega^m F] \rightarrow [S^0, \Omega^m E] \rightarrow [S^0, \Omega^m B] \rightarrow \ldots.
$$

Then using the above observation gives the result:

$$
\ldots \rightarrow [S^n, F] \rightarrow [S^n, E] \rightarrow [S^n, B] \rightarrow \ldots
$$

Remarks:
1. Covering spaces $p : (X) \to X$ are fibrations with discrete fibres. Therefore $p_\gamma : \pi_n((X)) \to \pi_n(X)$ is an isomorphism for $n \geq 1$ and injective for $n = 1$.

2. The Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ can be used to show that $\pi_n(S^3) = \pi_n(S^2)$ for $n \geq 3$.

**Theorem 2.12.** If $n \geq 2$, then $\pi_n(X)$ is abelian.

**Proof.** INSERT PICTURE PROOF (THE ONE YOU SEE ALL THE TIME WITH $g$ and $f$ IN A BOX MOVING IN A CIRCLE)

### 3 Cohomology theories

#### 3.1 Definitions

**Definition 3.1.** A reduced cohomology theory $\hat{E}^*$ is a collection of contravariant functors

$$\hat{E}^q : h\text{Top} \to \text{Ab},$$

for $q \in \mathbb{Z}_{\geq 0}$ satisfying

1. If $i : A \to X$ is a cofibration, then there exists an exact sequence

$$\hat{E}^2(X/A) \to \hat{E}^q(X) \to \hat{E}^q(A)$$

induced by $X/A \leftarrow X \xleftarrow{i} A$.

2. There exists a natural isomorphism $\Sigma^{-1} : \hat{E}^{q+1}(\Sigma A) \xrightarrow{\cong} \hat{E}^q(A)$.

3. If $\{X_i\}_{i \in I}$ is a set of based spaces, then the inclusions $X_i \xrightarrow{j_i} \bigvee_{i \in I} X_i$ induces an isomorphism

$$\pi(j_i)^* : \pi_i \hat{E}^q(X_i) \cong \hat{E}^q(\bigvee_{i \in I} X_i).$$

4. If $f : X \to Y$ is a weak equivalence (ie. an isomorphism in $\pi_q$ for all $q$) then $f^* : \hat{E}^*X \to \hat{E}^*Y$ is an isomorphism.

Example: $H^*(-; A)$, singular cohomology with coefficients in $A$.

**Definition 3.2.** A contravariant functor $F : C \to \text{Set}$ is representable if there exists and object $X_F$ in $C$ and a natural isomorphism

$$\eta : F(Y) \to \text{Hom}_C(Y, X_F).$$

Example: Let $C = k\text{-Vect}$, $F : C \to \text{Set}$ given by $V \mapsto V^* = \text{Hom}(V, k)$. This is the dual of a vector space we see in first year linear algebra. To make it match with our definition of a representable functor, we can compose $F$ with the forgetful functor to $\text{SET}$.

**Definition 3.3.** A covariant functor $F : C \to \text{Set}$ is corepresentable if there exists an object $Z_F$ in $C$ and a natural isomorphism

$$\eta : F(Y) = \text{Hom}_C(Z_F, Y).$$

Note: In the cases we consider, the morphisms of $C$ will be maps up to homotopy. In this case, a functor $F$ is corepresentable if there is a natural transformation

$$\eta : F(Y) = \text{Hom}_C(Z_F, Y) = [Z_F, Y]$$

Example: $C = h\text{Top}$, $F = \pi_n(-) = [S^n, -]$, so $Z_{\pi_n} = S^n$.

**Theorem 3.4** (Brown). If $F$ satisfies the wedge and M.V axiom, then it is representable.
The MV axiom is: If \(X = U \cup V\) and \(\alpha_u \in F(U), \alpha_v \in F(V)\) and \(\alpha_{U \cap V} = \alpha_U|_{U \cap V} = \alpha_V|_{U \cap V}\), then there exists \(\alpha \in F(X)\) such that \(\alpha_U = \alpha|_U\) and \(\alpha_V = \alpha|_V\).

**NEED A REFERENCE.**

**Theorem 3.5** (Yoneda Lemma). If \(F\) and \(G\) are representable functors and \(\eta : F \to G\) is a natural transformation, then there exists \(f_n : X_F \to X_G\) such that the following diagram commutes.

\[
\begin{array}{ccc}
F(Y) & \xrightarrow{\eta} & G(Y) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\Hom_C(Y, X_F) & \longrightarrow & \Hom_C(Y, X_G).
\end{array}
\]

**NEED A REFERENCE.**

### 3.2 Spectrum

**Definition 3.6.** A prespectrum \(T\) is a collection \((T_i)_{i \in \mathbb{Z}}\) of based spaces with the homotopy type of a CW complex and maps

\[f_i : \Sigma T_i \to T_{i+1}.\]

- \(T\) is a spectrum if the adjoint maps \(\tilde{f}_i : T_i \to \Omega T_{i+1}\) are homeomorphisms.
- \(T\) is a \(\Omega\)-spectrum if the adjoint maps are homotopy equivalences.

**CAUTION:** In this section, we are using \(\Sigma\) to mean the suspension (not the reduced suspension).

**Theorem 3.7.** \(\tilde{T}^*\) is a cohomology theory. Moreover, every cohomology theory \(E^*\) is built this way from a \(\Omega\)-spectrum \(E\).

**Proof.** Two parts:

\((\Rightarrow)\) • Homotopy invariance: automatic from [ , ].
- Abelian group: \(\tilde{T}^q(X) = [X, T_q] \cong [X, \Omega^2 T_{q+2}]\) and the last of these is clearly an abelian group.
- Axiom 1: Given \(A \to X \to X_i = X/A\), we have \([A, T_q] \leftarrow [X, T_q] \leftarrow [X/A, T_q]\) which is what we want.
- Axiom 2: \(\tilde{T}^q(X) = [X, T_q] = [X, \Omega T_{q+1}] = [\Sigma X, T_{q+1}] = \tilde{T}^{q+1}(\Sigma X)\).
- Axiom 3: \([\vee_{i \in I} X_i, T_q] = \pi_i[X_i, T_q]\).
- Axiom 4: Whitehead theorem.

\((\Leftarrow)\) Suppose we are give \(E^*\). The wedge axiom and the LES together with Brown’s Theorem give \(\tilde{E}^Q(X) = [X, E_q]\). Here we use the fact the we can derive the MV axiom from LES.

Suspension: \(\tilde{E}^{q+1}(\Sigma X) \cong \tilde{E}^q(X)\) gives \([X, E_q] = [\Sigma X, E_{q+1}] = [X, \Omega E_{q+1}]\).

Lastly, Yoneda implies that \(E_q \to \Omega E_{q+1}\) gives \(f_* : \pi_n E_q \cong \pi_n \Omega E_{q+1}\), taking \(X = S^n\).
4 Simplicial objects

Definition 4.1. Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ consists of $(X_n)_{n \geq 0}$ where $X_n \in \mathcal{C}$ together with morphisms
$$\partial^i_n : X_n \to X_{n-1},$$
and
$$s^i_n : X_n \to X_{n+1},$$
for $0 \leq i \leq n$ such that
1. $\partial_i \partial_j = \partial_{j+1} \partial_i$, if $i < j$
2. $\text{del}_i s_j = \begin{cases} s_{j-1} \partial_i & i < j \\ \text{id} & i = j, j + 1 \\ s_{j+1} \partial_i & i > j \end{cases}$
3. $s_i s_j = s_{j+1} s_i$ if $i \leq j + 1$.

The $\partial_i$ are called boundary maps and the $s_i$ are called degeneracy maps.

Definition 4.2. Let $\mathcal{C}$ be a category. The simplicial category $s\mathcal{C}$ is the category with
- Objects = simplicial objects in $\mathcal{C}$;
- $\text{Hom}_{s\mathcal{C}}(X,Y) = \{(f_n)_{n \geq 0} | f_n \in \text{Hom}_\mathcal{C}(X_n, Y_n), \partial_i f_n = f_{n-1} \partial_i, s_i f_n = f_{n+1} s_i\}$.

Another way we can think about the simplicial category is as follows.

Let $[n]$ be the category whose objects are the integers $0, \ldots, n$, with morphisms given by $i \to j$ if $i \leq j$.

We can draw this category as
$$0 \to 1 \to \ldots \to n.$$

Let $\Delta$ be the category with
- Objects $= [n]$;
- $\text{Hom}([n],[m]) = \text{order preserving maps from } [n] \to [m]$.

Then an object of $s\mathcal{C}$ is a contravariant functor $\Delta \to \mathcal{C}$. In particular, $X_n = X([n])$.

Lemma 4.3. Every morphism in $\Delta$ can be factored as $f = \partial \ldots \partial s \ldots s$ where $\partial^i : [n-1] \to [n]$ is given by
$$j \mapsto \begin{cases} j & j < i \\ j + 1 & j \geq i \end{cases},$$
and $s^i : [n+1] \to [n]$ is given by
$$j \mapsto \begin{cases} j & j \leq i \\ j - 1 & j > i \end{cases}.$$

To get the boundary and degeneracy maps for $s\mathcal{C}$ in this picture, we have $\partial^i = X(\partial^i)$ and $s^i = X(s^i)$. The morphisms of $s\mathcal{C}$ are natural transformations and the relations they satisfy correspond to commuting squares.

Homework: Complete the story. Remember you must also show that given an object in $s\mathcal{C}$ as given in the first definition, we get contravariant functors $\Delta \to \mathcal{C}$.
4.1 Nerve functor

Let $\text{Cat}$ be the category of small categories.

**Definition 4.4.** The nerve (in $\mathcal{C}$) is a functor

$$N : \text{Cat} \to s\mathcal{C}$$

$$\mathcal{D} \mapsto \text{Func}(\mathcal{D}, \Delta) : \Delta \to \mathcal{C}$$

In particular, $\mathcal{N} \mathcal{D} : \Delta \to \mathcal{C}$ is given by $\mathcal{N} \mathcal{D}([n]) = \text{Func}(\Delta, \mathcal{D})([n]) = \{T(0) \to \ldots \to T(n)\}$.

Let $G$ be a group. Consider the category $\hat{G}$ where $\text{Ob}(\hat{G}) = \ast$ and $\text{Hom}(\ast, \ast) = G$ and composition is $h \circ g = gh$.

We have $\mathcal{N} \hat{G}([n]) = \{\ast \xrightarrow{g_1} \ast \xrightarrow{g_2} \ldots \xrightarrow{g_n} \ast | g_i \in G\} = G^n$.

Bar notation: we can instead write $\mathcal{N} \hat{G}([n]) = \{[g_1] \ldots [g_n] | g_i \in G\}$. Using this notation, the $\partial_i$ and $s_i$ are easy to describe.

We have $\partial_i : G^n \to G^{n-1}$ is given by

$$[g_1] \ldots [g_n] \mapsto [g_1] \ldots [g_i-1] [g_{i+1}] [g_i-1] \ldots [g_n]$$

For $\partial_0$ we just get rid of $g_1$, ie

$$\partial_0 [g_1] \ldots [g_n] = [g_2] \ldots [g_n]$$

Similarly, $s_i : G^n \to G^{n+1}$ is given by

$$[g_1] \ldots [g_n] \mapsto [g_1] \ldots [g_{i-1}] [g_i] [g_i+1] \ldots [g_n]$$

This is called the nerve of $G$.

[COMMENT: IF WE WANT THIS TO BE A FUNCTOR, WE SHOULD SAY WHAT IT DOES TO MORPHISMS]

4.2 Geometric realization

**Definition 4.5.** Let $X$ be a simplical space. The geometric realization of $X$, denoted $|X|$, is the space

$$|X| : = \bigsqcup (X_n \times \Delta^n)/\sim,$$

where $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | t_i \geq 0, \sum t_i = 1\}$ and $\sim$ is the equivalence relation generated by

- if $x \in X_n$, $u \in \Delta^{n-1}$, $v \in \Delta^{n+1}$, then $(x, \delta^i u) \sim (\partial_i x, u)$,
- if $x \in X_n$, $u \in \Delta^{n-1}$, $v \in \Delta^{n+1}$, then $(x, \sigma^i u) \sim (s_i x, u)$,

where

$$\delta^i (t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{n-1});$$

and

$$\sigma^i (t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}).$$

**Definition 4.6.** Let $G$ be a group. The classifying space of $G$ is $BG := |\hat{G}|$.

With some work [REFERENCE SOMETHING], we can get a fibre bundle $G \hookrightarrow EG \xrightarrow{p} BG$, with $EG$ weakly contractible. But this means $EG$ is simply connected, so $EG$ is the universal cover of $BG$, so $\pi_1(BG) = G$.

If we use the long exact sequence of homotopy groups, we can also show that $\pi_i(G) \cong \pi_{i-1}(G)$ If $G$ is a discrete group, then $\pi_i(G) = G$ if $i = 0$ and is 0 is $i \neq 0$.

If $G$ is abelian, we can iterate this process to get the spaces $K(G,n)$.
5 Operads and loop spaces

References: [3]

NOTATION: In this section, we use Σ's for the symmetric group and S's for suspension.

5.1 Statements of recognition principle

Motivation: Let $X$ be a topological space. How can we tell if there exists an $n \in \mathbb{Z}_{\geq 0}$ and a space $Y$ such that $X = \Omega^n Y$?

This question is essentially answered by the following theorem:

**Theorem 5.1** (Recognition principle). There exist $\Sigma$-free operads $C_n$ for $1 \leq n \leq \infty$ such that every $n$-fold loop space is a $C_n$-space and every connected $C_n$-space has the weak homotopy type of an $n$-fold loop space.

There is some history about the development of the recognition principle in the preface to [3] that may be of interest.

In the next section, we define what an operad $C$ and a $C$-space is.

5.2 Operads

Let $U$ be the category of topological spaces (compactly generated, Hausdorff) with continuous maps, and let $\mathcal{T}$ be the same category, but with based spaces and based maps.

**Definition 5.2.** An operad $C$ is a collection of spaces $C(j) \in U$ for $j \geq 0$, with $C(0) = \ast$ with maps

$$\gamma : C(K) \times C(j_1) \times \ldots \times C(j_k) \to C(j_1 + \ldots j_k).$$

such that:

1. An associativity formula holds:

2. There is an identity $1 \in C(1)$ such that $\gamma(1; d) = d$ and $\gamma(c; 1, \ldots, 1) = c$.

3. There is a right action of the symmetric group $\Sigma_j$ on $C(j)$

[BE MORE EXPLICIT LATER, MAYBE WITH A DIAGRAM]

**Definition 5.3.** A morphism of operads $\psi : C \to C'$ is a sequence of $\Sigma_j$ equivariant maps $\psi_j : C(j) \to C'(j)$ such that $\psi(1) = 1$ and the following diagram commutes:

$$\begin{array}{ccc}
C(K) \times C(j_1) \times \ldots \times C(j_k) & \xrightarrow{\gamma} & C(j) \\
\psi_1 \times \cdots \psi_k \downarrow & & \downarrow \psi_j \\
C'(K) \times C'(j_1) \times \ldots \times C'(j_k) & \xrightarrow{\gamma'} & C'(j),
\end{array}$$

where $j = j_1 + \ldots + j_s$

**Example 5.4** (The endomorphism operad). Let $X \in \mathcal{T}$ The endomorphism operad $\mathcal{E}_X$ of $X$ is the operad given by

- $(E)_x(j) = \{ \text{based maps from } X^j \to X \}$.

- $\gamma(f, g_1, \ldots, g_k) = f(g_1 \times \ldots g_k)$

- $1 \in \mathcal{E}_X(1) = id : X \to X$.

- The $\Sigma_j$ action permutes the coordinates of $X^j$. 

Example 5.5 (Little $n$-disks operad). The little $n$-disks operad is the operad with

$$C_n(j) = \{j\text{-tuples of ‘nice’ embeddings of an } n\text{-disk into a single } n\text{-disk such that their images are disjoint}\}.$$  

Exercise: How is $\gamma$ defined? What do we mean by nice?

In some sense, the endomorphism operad is the mother of all operads. The associativity, unit and symmetry axioms for an operad are defined precisely to mimic what happens in the case of the endomorphism operad. If we are given a $j$-tuple of elements in $X$, then elements of $\mathcal{E}_X(j)$ can act on it by plugging in the tuple to the $j$ functions. For a general operad, we can define actions by going through $\mathcal{E}_X$.

Definition 5.6. A $C$-space is a pair $(X, \theta)$ where $X \in \mathcal{T}$, $\theta : C \to \mathcal{E}_X$ is a morphism of operads.

As an aside, we can define morphisms of $C$-spaces and get a category of $C$-spaces. We don’t use this language much in this section so this can be ignored.

Definition 5.7. A morphism of $C$-spaces is a map $f : (X, \theta) \to (X', \theta')$ such that $f$ is a based map $f : X \to X'$ and $f \circ \theta_j(c) = \theta'_j(c) \circ f^j$ where $c \in C(j)$.

The category of $C$-spaces is denoted by $[C\mathcal{T}]$.

Now, we can think of $C$-spaces in another way. Given a morphism of $\theta : C \to \mathcal{E}_X$, we can consider the maps $\theta_j : C(j) \to \mathcal{E}_X(j) = \{X^j \to X\}$. Noting the adjunction

$$\text{Hom}(C(j), \mathcal{E}_X(j)) \xrightarrow[\cong]{\sim} \text{Hom}(C(j) \times X^j, X),$$

we can specify a $C$-space by maps $\theta_j : C(j) \times X^j \to X$ satisfying certain conditions. We state this more precisely as a lemma.

In particular, given $\theta_j : C(j) \to \mathcal{E}_X(j)$, we get a map $C(j) \times X^j \to X$ by

$$(x_1, \ldots, x_j) \mapsto \theta_j(c)(x_1, \ldots, x_j), \text{ for } c \in C(j).$$

We now state the above discussion a bit more precisely in the form of a lemma.

Lemma 5.8. An action $\theta : C \to \mathcal{E}_X$ determines and is determined by maps $\theta_j : C(j) \times X^j \to X$, for $j \geq 0$ such that

1. $\theta_0 : * \to X$.

2. The following commutes

$$(C(K) \times C(j_1) \times \ldots \times C(j_k) \times X^j) \xrightarrow{\gamma} C(j) \times X^j \xrightarrow{\theta_j} X$$

3. If $x \in X$ then $\theta_1(1; x) = x$.

4. If $c \in C(j)$, $\sigma \in \Sigma_j, y \in X^j$, then $\theta_j(c \sigma ; y) = \theta_j(c ; \sigma y)$.

We can now prove one half of the recognition principle. That is, we can show that ever $\Omega^n Y$ is a $C_n$-space, where $C_n$ is the little $n$-disks operad. In particular, we are claiming that the operads in the recognition principle are the little disks operad. To show that $\Omega^n Y$ is a $C_n$-space, we need to define the action.
Given a map \( f : S^n \to Y \), we can think of it as a map \( D^n \to Y \) that sends the boundary of \( D^n \) to the basepoint in \( Y \). Then \( \theta_j : \mathcal{C}_n(j) \times (\Omega^n Y)^j \to Y \) is given by

\[
\theta_j(e_1, \ldots, e_j; f_1, \ldots, f_j) = \begin{cases} f_i(y), & y \in e_i(D^n), \\ \ast, & y \in D \setminus \{e_1(D^n), \ldots, e_j(D^n)\}, \end{cases}
\]

where \((e_1, \ldots, e_j) \in \mathcal{C}_n(j), f_i : (D^n, \partial D^n) \to (Y, \ast)\) is in \( \Omega^n Y \). This can be more easily understood if we draw a map of the domain of the new function: [INSERT PICTURE]

In [3], May uses the little \( n \)-cubes operad instead of the little \( n \)-disks operad. Both of these work fine and in fact, the proof of the recognition principle allows for operads that are homotopy equivalent.

### 5.3 Monads and \( C \)-algebras

Operads are closely related to monads. In fact, it turns out that an operad \( \mathcal{C} \) determines a monad \( \mathcal{C} \) and a \( \mathcal{C} \) space becomes an algebra over the monad \( \mathcal{C} \) (a \( \mathcal{C} \)-algebra).

We are particularly interested in this because of the following.

**Theorem 5.9 (Approximation theorem).** For the operads \( \mathcal{C}_n \) of the recognition principle, there is a natural map of \( \mathcal{C}_n \)-spaces

\[
\alpha_n : \mathcal{C}_n \to \Omega^n S^n X
\]

for \( 1 \leq n \leq \infty \) and \( \alpha_n \) is a weak homotopy equivalence.

In summary, \( C_n X \simeq \Omega^n S^n X \), which is pretty cool, because we are interested in loop spaces. In particular, while \( \Omega^n S^n X \) is a priori hard to understand, the monads \( C_n \) will be quite concrete, and yield to computations more easily. We will see some applications later on.

We begin with some definitions.

**Definition 5.10.** A monad in a category \( \mathcal{T} \) is a triple \((C, \mu, \eta)\), where \( C : \mathcal{T} \to \mathcal{T} \) is a contravariant functor, and \( \mu : C^2 \Rightarrow C \), \( \eta : 1 \Rightarrow C \) are natural transformations such that the following diagrams commute.

\[
\begin{array}{ccc}
CX & \xrightarrow{C\mu} & C^2X \\
\downarrow & \mu(X) & \downarrow \mu(X) \\
CX & \xrightarrow{C\eta} & CX
\end{array}
\]

\[
\begin{array}{ccc}
C^3X & \xrightarrow{C\mu} & C^2X \\
\downarrow & \mu(X) & \downarrow \mu(X) \\
C^2X & \xrightarrow{C\eta} & CX
\end{array}
\]

The first diagram tells us that \( \eta \) acts as the unit (in particular we can think of it as saying \( C\eta = \eta C = C \)), while the second diagram says that \( \mu \) is associative (think \( \mu(C^2)C = C\mu(C^2) \)). Thus, we can think of monads as monoids in the functor category.

**Definition 5.11.** A monad \( \psi : (C, \mu, \eta) \to (C', \mu', \eta') \) of monads is a natural transformation of functors \( \psi : C \Rightarrow C' \) such that the following commute for all \( X \in \mathcal{T} \).

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & C'X \\
\downarrow & \psi & \downarrow \eta' \\
CX & \xrightarrow{\psi} & C'X
\end{array}
\]
Definition 5.12. An algebra \((X, \xi)\) over a monad \((C, \mu, \eta)\) is an object \(X \in T\) together with a map \(\xi : CX \rightarrow X\) in \(T\) such that the following commute.

\[
\begin{array}{ccc}
CCX & \xrightarrow{\psi^2} & C'C'X \\
\mu \downarrow & & \mu' \downarrow \\
CX & \xrightarrow{\psi} & C'X
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & CX \\
\downarrow & \searrow \downarrow & \downarrow \xi \\
& = & \downarrow \\
& & X\end{array}
\]

\[
\begin{array}{ccc}
CCX & \xrightarrow{\mu} & CX \\
\downarrow C\xi & & \downarrow \xi \\
CX & \xrightarrow{\xi} & X
\end{array}
\]

For brevity, we will often just call these objects \(C\)-algebras. So why the ‘algebra’? If we instead pretend that \(C\) were a group \(G\), and \(\xi : G \times X \rightarrow X\) was such that it satisfied the above diagrams, then this would mean precisely that \(X\) was a space with a \(G\) action. Thus we can think of a \(C\)-algebra as a space \(X\) with some notion of an action of \(C\) on it.

Definition 5.13. A morphism \(f : (X, \xi) \rightarrow (X', \xi')\) of \(C\)-algebras is a map \(f : X \rightarrow X'\) in \(T\) such that the following diagram commutes.

\[
\begin{array}{ccc}
CX & \xrightarrow{Cf} & CX' \\
\downarrow \xi & & \downarrow \xi' \\
X & \xrightarrow{f} & X'
\end{array}
\]

Given an operad \(C\) we can construct a monad \(C\). Then for the little disks operads \(C_n\), we get associated monads \(C_n\). Recalling that monads are functors (in our case from \(T \rightarrow T\)), we can apply \(C_n\) on a space \(X\). When \(X\) is connected, the approximation theorem is the statement that

\[C_nX \simeq \Omega^nS^nX,\]

where \(\simeq\) here means weak homotopy equivalence. In the next section, we discuss how to construct these monads.

5.4 Construction a monad from an operad

Let \(C\) be an operad. Define maps \(\sigma_i : C(j) \rightarrow C(j - 1), 0 \leq i < j\) by \(\sigma_i(c) := \gamma(c; s_i)\), where \(s_i = (1, \ldots, 1, *, 1, \ldots, 1)\).

Example: In \(E_X, \sigma_i f)(y) = f(s_i y)\) where \(s_i : X^{j-1} \rightarrow X^j\) given by

\[s_i(x_1, \ldots, x_{j-1}) = (x_1, \ldots, x_i, *, x_{i+1}, \ldots, x_{j-1}).\]

Thus, we \(s_i\) puts an element of \(X^{j-1}\) into \(X^j\) in the simplest way possible. In particular, \(\sigma_i\) turns the \(i^\text{th}\) part of \(c \in C(j)\) to the basepoint \(*\).
Now construct $C$ as follows:

For $X \in T$,

$$CX = \bigsqcup_{j \geq 0} C(j) \times X^j / \sim$$

where $\sim$ is the equivalence relation given by

1. $(\sigma_i c, y) \sim (c, s_i y)$ for $c \in C(j), 0 \leq i < j, y \in X^{j-1}$
2. $(\sigma y, c) \sim (c, \sigma y)$ for $c \in C(j), \sigma \in \Sigma_k, y \in X^j$

Note: Sometimes, if we want to be explicit about condition 2, then we will write $CX = \bigsqcup_{j \geq 0} C(j) \times \Sigma_k X^j / \sim$, using $\sim$ only for relation 1.

We can think of the first condition as a kind of degeneracy condition. For example, in the endomorphism operad, we would regard $(f_1, f_2, f_3) \times (x_1, *, x_3) \sim (f_1, f_3) \times (x_1, x_3)$. The second condition says that the quotient respects the $\Sigma_j$ action on $C(j)$. For example $(f_1, f_2, f_3) \times (x_1, x_2, x_3) \sim (f_2, f_1, f_3) \times (x_2, x_1, x_3)$.

**Exercise:** What does $CX$ look like for the little disks operad? For $C$ to be a monad, we need to check that $CX \in T$. We can topologise $CX$ as follows.

Let $F_k CX$ be the image of $\bigsqcup_{j=0}^k C(j) \times X^j$ in $CX$. Note that $\bigsqcup_{j=0}^k C(j) \times X^j$ is a topological space, and we can give $F_k CX$ the quotient topology. Observe that $F_{k-1} CX$ is a closed subspace of $F_k CX$ and so we can give $CX$ the topology of the union of $F_k CX$. Then $F_0 CX$ is a single point which will be our base point for $CX$.

Check $C$ is functorial: If $c \in C(j), y \in X^j$, let $[c, y]$ denote the image of $(c, y)$ in $CX$. For a map $f : X \to X'$ in $T$, define $Cf : CX \to CX'$ by

$$Cf[x; y] = [c; f(y)].$$

We should then check that $C(f \circ g) = Cf \circ Cg$. [EXERCISE]

We also need to define $\mu$ and $\eta$ for our monad $C$.

Define $\mu : C^2 X \to XC$ by

$$\mu[c, [d_1, y_1], \ldots [d_k, y_k]] = [\gamma(c; d_1, \ldots, d_k), y_1, \ldots, y_k]$$

for $c \in C(k), d_s \in C(j), y_s \in X^{j_s}$.

Define $\eta : X \to CX$ by

$$\eta(x) = [1, x]$$

for $x \in X$. We can check that $\mu$ and $\eta$ satisfy the diagrams in the definition of $(C, \mu, \eta)$. In particular, the definition of an operad shows that $\mu$ is well defined and gives the associativity square. Moreover, we can use the unit formulas for operads give the unit square for $\eta$.

**Example 5.14.** If $C(j) = *$, then $CX = (\sqcup_k * \times \Sigma_k X^k) / \sim$. Elements of $CX$ can be represented as tuples of elements of $x$. The multiplication is just concatenation, and the $\Sigma_k$ action makes things commutative. The equivalence relation says $(x_1, * x_3) = (x_1, x_3)$. In particular, $CX$ is the free commutative monoid on $X$ with unit $*$.  

**Example 5.15.** If $C(j) = \Sigma_k$, then $CX = (\sqcup_k \Sigma_k \times \Sigma_k X^k) / \sim$ is the free monoid on $X$.
5.5 Approximation Theorem

In this section, $C_n$ will refer to the operads of the recognition principle. In particular, we should think of $C_n$ as the little $n$-disks operad. We will use $C_n$ to denote the monad associated to $C_n$ by the construction in the previous section.

**Theorem 5.16** (Approximation Theorem). If $X$ is connected, then there is a natural map $\alpha_n : C_nX \to \Omega^n S^n X$, which is a weak homotopy equivalence. In fact, $\Omega^n S^n$ is a monad, and $\alpha_n : C_n \to \Omega^n S^n$ is a morphism of monads.

For a general pair $L, R$ of adjoint functors, we can get a monad. Observing that $S^n$ and $\Omega^n$ are adjoint functors this is how we get the monad $\Omega^n S^n$. In briefly describe this process below.

Let $L : C \to D$ and $R : D \to C$ be adjoint functors. That is, for $X \in C$ and $Y \in D$, $\text{Hom}_D(LX, Y) \cong \text{Hom}_C(X, RY)$. Then $M = RL$ is a monad. We can explain how this works as follows. Consider

$$\text{Hom}(LX, LX) \cong \text{Hom}(X, RLX)$$

which we obtain from replacing $Y$ with $LX$. There is and identity element $id \in \text{Hom}(LX, LX)$, which we can push through to $\text{Hom}(X, RLX)$ to get a natural transformation

$$\eta : I \to RL.$$

$\eta$ is the called the unit of the adjunction.

Similarly, consider

$$\text{Hom}(LRY, Y) \cong \text{Hom}(RY, RY)$$

which we obtain from replacing $X$ with $RY$. We can similarly get the co-unit of the adjunction by taking $id \in \text{Hom}(RY, RY)$ and pushing it to the left to get a natural transformation

$$\epsilon : LR \to I.$$

We can define the natural transformation $\mu : RLRL \to RL$ by

$$RLRL \to \text{Re}(RL)L.$$

Then we can check that $(RL, \mu, \eta)$ is a monad.

5.5.1 Idea of proof

We did not prove the recognition principle in the seminar. The full proof can be found in gils. The crux of the argument is to construct a quasi-fibration

$$C_nX \to E_n(TX, X) \to C_{n-1}(SX)$$

such that the following diagram commutes.

$$\begin{array}{ccc}
C_nX & \xrightarrow{\alpha_n} & E_n(TX, X) \\
\downarrow & & \downarrow \pi_{n-1} \\
\Omega^n S^n X & \xrightarrow{\Omega^n S^n} & \Omega^{n-1} S^{n-1}(SX)
\end{array}$$

The maps $\alpha_n$ are the morphisms of monads $C_n \to \Omega^n S^n$ we discussed (but did not construct) earlier. The bottom row is the fibration discussed in 1.8. Lastly, although we do not describe it here, the $E_n$ is an attempt to construct a ‘path’ version of the little disks algebra. Given this, and the fact the when $n = 1$, $\alpha_0 : SX \to SX$ is an isomorphism, we can apply the long exact sequence of homotopy groups together with the 5-lemma to get the approximation theorem.

**Exercise:** Fill in the details.
5.6 The bar construction

Recall the recognition principle (Theorem 5.1). We have already described how one side of the theorem works. In this section, we aim to make headway into showing the other direction. In particular, we want to show the following.

Every connected $C_n$-space has the weak homotopy type of an $n$-fold loop space.

We will do this via the following string of equivalences.

$$X \simeq B(\Omega^n S^n, C, X) \cong \Omega^n B(S^n, C, X)$$

In particular, we will construct the space $B(S^n, C, X)$ which we can think of as a de-looping of $X$. The way we construct the $B$ is similar to the discussion earlier on the construction of classifying spaces of groups, and hence the use of the name ‘bar construction’.

We will build the space $B$ similar to the nerve construction used to build classifying spaces in section 4.

Recall that an object $X \in s\mathcal{T}$ is a sequence of objects $X_q \in \mathcal{T}$, $q \geq 0$, together with maps

$$\partial_i : X_q \to X_{q-1}$$

and

$$s_i : X_q \to X_{q+1}$$

in $\mathcal{T}$, for $0 \leq i \leq q$. These maps were called boundary and degeneracy maps respectively and they satisfied the relations given below.

1. $\partial_i \partial_j = \partial_{j+1} \partial_i$, if $i < j$

2. $del_i s_j = \begin{cases} s_{j-1} \partial_i & i < j \\ id & i = j, j + 1 \\ s_j \partial_i & i > j \end{cases}$

3. $s_i s_j = s_{j+1} s_i$ if $i \leq j + 1$.

We also defined morphisms $f : X \to Y$ in $s\mathcal{T}$ to be a sequence $f_q : X_q \to Y_q$ of maps in $mc\mathcal{T}$ such that $\partial_i f_q = f_{q-1} \partial_i$ and $s_i f_q = f_{q+1} s_i$.

**Definition 5.17.** A homotopy $h : f \to g$ in $s\mathcal{T}$ between maps $f, g : X \to Y$ consists of maps

$$h_i : X_q \to Y_{q+1}$$

for $0 \leq i \leq q$ such that

1. $\partial_0 h_0 = f_q$ and $\partial_{q+1} h_q = g_q$

2. $\partial_i h_j = \begin{cases} h_{j-1} \partial_i & i < j \\ \partial_j h_{j-1} & i = j > 0 \\ h_{j+1} \partial_{i-1} & i > j + 1 \end{cases}$

3. $s_i h_j = \begin{cases} h_{j-1} \partial_i & i \leq j \\ h_j s_{j+1} & i > j \end{cases}$

**NOTE:** The important thing here is that we have notions of homotopy equivalence, deformation retracts etc. When $\mathcal{T} = Top$, these things translate to the usual notion of homotopy after we take geometric realizations.

Chapter 9 of [3] goes through key properties of the simplicial category which we use in the construction of $B$. In this section, we will outline the main arguments.
**Definition 5.18.** Let \((C, \mu, \eta)\) be a monad in \(T\). A \(C\)-functor \((F, \lambda)\) in a category \(V\) is a functor \(F : T \to V\) together with a natural transformation \(\lambda : FC \to F\) such that the following commute.

\[
\begin{align*}
\xymatrix{ F & FC \\
\downarrow F \eta & \downarrow \lambda \\
FC & F. }
\end{align*}
\]

\[
\begin{align*}
\xymatrix{ FCC & FC \\
\downarrow \lambda & \downarrow \lambda \\
FC & C }
\end{align*}
\]

Compare this with the definition of a \(C\)-algebra \((X, \xi)\). In particular, one can think of a \(C\)-functor as something with a right action of \(C\).

[PUT SOME EXAMPLES HERE]

We can now define the categorical 2-sided bar construction.

Let \(B(T, V)\) be the category whose objects are triples \(((F, \lambda), (C, \mu, \eta), (X, \xi))\) where \((F, \lambda)\) is a \(C\)-functor, \((C, \mu, \eta)\) is a monad in \(T\) and \((X, \xi)\) is a \(C\)-algebra. The morphisms are triples of morphisms \((\pi, \psi, f)\) of \(C\)-functors, *monads* and \(C\)-algebras respectively.

Then we can define a functor

\[ B_* : B(T, V) \to sV \]

by \(B_q(F, C, X) = Fe^qX\). There are \(q + 1\) action maps with which to form a map \(FX^qX \to FC^{q-1}X\).

- The right action of \(C\) on \(F\) induced by \(\lambda\) is \(\partial_0\).
- The \(q - 1\) multiplication of \(C\) induced by \(\mu\) are the \(\partial_i\) for \(0 < i < q\).
- The left action of \(C\) on \(X\) induced by \(\xi\) is \(\partial_q\).

The \(s_i\) correspond to the \(q + 1\) insertions of the unit of \(C\) to get a map \(FC^qX \to FC^{q+1}X\).

The morphisms \(B_*(\pi, \psi, f)\) are given by \(B_q(\pi, \psi, f) = \pi \psi^q f : FC^qX \to F'(C')^qX'\).

With these definitions, one can check that \(B_*\) is well defined by checking the conditions required to be in \(sV\).

### 5.7 Proof of recognition principle

We didn’t actually complete the proof in the seminar. I might type this up slowly in my spare time. Note: \(S\) and \(\Sigma\) have switched again!

Here is an outline. We want to show the following.

\[ X \simeq B(\Omega^n S^n, C, X) \cong \Omega^n B(S^n, C, X) \]

We claim that \(\Omega^n S^n\) is a \(C\)-functor, so \(B(\Omega^n S^n, C, X)\) makes sense. Lemma 9.7 of [3] gives the second equivalence. It states the following.

**Lemma 5.19.** Let \(G : V \to V'\) be a functor. Then \(GF : TT \to V'\) is a \(C\)-functor and for a \(C\)-algebra \(X\),

\[ B_*(GF, C, X) = G_* B_*(F, C, X). \]

In particular, \(B(\Omega^n S^n, C, X) \cong \Omega^n B(S^n, C, X)\). The first inequality is analogous to \(\Omega B G \cong G\) for a group \(G\).

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6 Operads in action

In this section, we illustrate some of the cool things you can do with the approximation theorem and the recognition principle. Instead of using the little \( n \)-disks operad, we will use the little \( n \)-cubes operad. We note that this does not change our two theorems - in their proofs, we can replace the operads that are equivalent.

**Definition 6.1** (Little \( n \)-cubes operad). The little \( n \)-disks operad is the operad with
\[
C_n(j) = \{ j \text{-tuples of ‘nice’ embeddings of an } n \text{-cube into a single } n \text{-disk such that their images are disjoint} \}.
\]

Exercise: How is \( \gamma \) defined? What do we mean by nice?

Claim: the map \( C_n(k) \to PConf_n(I^n) \) that maps a little cubes to its lower left hand corner is a homotopy equivalence.

**Definition 6.2.** \( PConf_n(X) = \{(x_1, \ldots, x_k) | x_i \neq x_j \text{ if } i \neq j \} \subset X^k \)

\( PConf_n(X) \) is called the pure configuration space of \( X \). Notice that the symmetric group \( S_k \) acts on \( PConf_n(X) \) by permuting coordinates. In particular
\[
\sigma.(x_1, \ldots, x_k) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)}).
\]

The configuration space of \( X \) is \( Conf_n.X := PConf_n(X)/S_k \). We can think of \( Conf_n.X \) as subsets of \( k \) distinct elements of \( X \). To get an idea of what these look like, we study \( Pconf_n(I^2) \). We start with the case \( n = 1 \).

**Proposition 6.3.** \( PConf_n(I) \simeq S_k \times \hat{\Delta}^k \). As a set, it is \( k! \) copies of \( \hat{\Delta}^k \).

Proof. A \( k \)-tuple of distinct elements in \( I \) is a set \( \{x_1, \ldots x_k\} \subset I \) and a permutation \( \sigma \) such that
\[
0 < x_{\sigma(1)} < \ldots < x_{\sigma(k)} < 1.
\]
In particular, \( \{(x_1, \ldots, x_k) \subset I^k | 0 < x_1 \ldots < x_k < 1\} \cong \hat{\Delta}^k. \)

Now consider the case \( n = 2 \). That is, we now look at \( Pconf_k(I^2) \). Observe that \( C_2(k) \simeq PConf_k(I^2) \).

We now want to give an example of the approximation theorem in action. Recall that the approximation theorem says the following. If \( X \) is a connected based space, then
\[
C_n.X \simeq \Omega^n \Sigma^n X,
\]
where \( \simeq \) here denotes weak homotopy equivalence.

Recall,
\[
C_n.X = \bigsqcup_{k \geq 0} C_n(k) \times S_k \times X^k/\sim.
\]
In particular, if $n = 1$, then

$$C_1X = \bigsqcup_{k \geq 0} C_1(k) \times S_k X^k / \sim$$

$$\simeq \bigsqcup_{k \geq 0} (S_k \times \Delta^k) \times S_k X^k / \sim$$

$$= \bigsqcup_{k \geq 0} \Delta^k \times X^k / \sim$$

$$\simeq \bigsqcup_{k \geq 0} X^k / \sim$$

$$=: MX$$

**Definition 6.4.** $MX$ is the free topological monoid on $X$ with unit $\ast$.

Observe that the name is justified, since an arbitrary element of $MX$ is just a tuple of elements of $X$, $(x_1, \ldots, x_k)$ for some $k$, say. The multiplication in $MX$ (when we look back at the original definition) simply becomes concatenation. i.e. $(x_1, \ldots, x_k) \cdot (y_1, \ldots, y_l) = (x_1, \ldots, x_k, y_1, \ldots, y_l)$. The equivalence relation gives

us the unit $\ast$, satisfying $(x_1, \ast, \ldots, \ast, x_k) = (x_1, \ldots, \ast, \ldots, \ast, x_k)$, where $\ast$ denotes removing $\ast$ from the tuple.

The upshot of the recognition principle is that we now know

$$\Omega \Sigma X \simeq MX.$$ 

So what was a priori hand to understand $(\Omega \Sigma X)$ is actually something quite concrete. Actually, it is also true that the equivalence also preserves the multiplication, which we have not shown, but is easy to see if we draw a picture.

**Corollary 6.5.** If the ring of coefficients is such that $\hat{H}_* X$ is flat over $R$, then $H_*(\Omega \Sigma X) \cong T(\hat{H}_* X)$ as rings, where $T(\hat{H}_* X)$ is the tensor algebra over $\hat{H}_* X$.

**Proof.** We will inductively show:

$$H_*(\bigsqcup_{k=0}^n X^k / \sim) = \bigoplus_{k=0}^n \hat{H}_*(X)^{\otimes k}.$$ 

**Base:** $H_*(MX) = H_*(X) = H_0(X) \oplus \hat{H}_* X = R \oplus \hat{H}_* X$

**Inductive step:** Consider the cofibration $M_n X \to M_{n+1} X \to X_i$. The cofibre $C(i) = M_{n+1} X / M_n X = X^{\times (n+1)} / \sim$, where $(x_1, \ldots, x_i, *, x_{i+2}, \ldots, x_n) \sim *$.

Therefore $C(i) = X^{\wedge (n+1)}$. Moreover, by the flatness assumption, $\hat{H}_*(X^{\wedge n}) = \hat{H}_*(X)^{\otimes n}$.

Thus we have the long exact sequence

$$\ldots \to H_*(M)nX \to H_* M_{n+1} X \to H_* C_i \to \ldots$$

Checking that this splits gives the result. [FILL IN THE DETAILS]

**Corollary 6.6.** For $n \geq 2$, $H_*(\Omega S^n) = R[x]$, where $\dim X = n - 1$.

**Proof.** $\Omega S^n = \Omega \Sigma S^{n-1}$. $H_*(\Omega S^n) = T(\hat{H}_*(S^{n-1})) = R[ \text{generator in dim } n - 1]$ [QUESTION: How is this failing?]
Here is an example.
Let $M$ be a closed smooth manifold. Let $X_i = \bigcup_{k \geq 0} \Emb(M_i^{id}, F^n)$ and $X' = \bigcup_{k \geq 0} \Emb(M_{i,k}^{id}, F^n)/\text{Diff}(M_{i,k})$. This is an algebra over $C_n X$. For example, in the case where $M = S^1$, we can define $\theta$ in the usual way. [Draw picture].

This leads us into considering the group completion theorem.

**Theorem 6.7.** If $X$ is a $C_1$-algebra, then there exists a space $BX$ (it is the same $B$ as earlier!) such that

$$H_*(X)[\pi_0^{-1}] \cong H_*(\Omega BX)$$

Note that previously we had $X = \Omega [BX]$ if $X$ were connected. The group completion theorem tells us part the story when $X$ is no longer connected.

## 7 Group Completion Theorem

References: [4]

### 7.1 Two-sided bar construction

We have already seen many incarnations of the bar construction. They are all essentially the same thing. Nevertheless, we will now describe the bar construction for a topological monoid.

Let $M$ be a topological monoid. Let $X$ be a right $M$-space. Let $Y$ be a left $M$-space. For $p \in \mathbb{Z}_{\geq 0}$, define $B_p(X, M, Y) = X \times M^p \times Y$, with maps

$$\partial^i(x, m_1, \ldots, m_p, y) = \begin{cases} (x m_1, m_2, \ldots, m_p, y) & i = 0 \\ (x, m_1, \ldots, m_i m_{i+1}, \ldots, m_p, y) & 0 < i < p \\ (x, m_1, \ldots, m_p y) & i = p \end{cases}$$

for $0 \leq i \leq p$. Intuitively, we think of $\partial^i$ as removing the $i$'th comma (start counting at 0).

Then $B_*(X, M, Y)$ is a (semi-)simplicial space.

Define $B_p M := B_p(\ast, M, \ast)$ and $E_p M := B_p(\ast, M, M)$. Then we can define the classifying space of $M$ to be the geometric realization $BM := |B_\ast M|$. Moreover $EM := |E_\ast M|$.

In particular,

$$BM = \bigsqcup_{n=0}^{\infty} M^n \times \Delta^n / \sim$$

where $\sim$ is the equivalence relation generated by

1. $(m_0, \ldots, m_{i-1}, 1, m_{i+1}, \ldots, m_n, t_0, \ldots, t_n) \sim (m_0, \ldots, m_{i-1}, m_{i+1}, t_0, \ldots, t_i + t_{i+1}, \ldots, t_n)$, for $i = 0, \ldots, n - 1$.

2. $(m_0, \ldots, m_n, 0, t_1, \ldots, t_n) \sim (m_1, \ldots, m_n, t_1, \ldots, t_i + t_{i+1}, \ldots, t_n)$

3. $(m_0, \ldots, m_n, 0, t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \sim (m_1, \ldots, m_{i-1} t_{i+1}, \ldots, m_n, t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$

4. $(m_0, \ldots, m_n, t_0, \ldots, t_{n-1}, 0) \sim (m_0, \ldots, m_{n-1}, t_1, \ldots, t_{n-1})$

These equivalence relations precisely reflect the simplicial structure of $B$. In particular, we can summarise the last three relations as $(\partial^i(x, y), y) \sim (x, d_i(y))$, where $d_i : \Delta^n \to \Delta^{n-1}$ is the inclusion of the $i$'th face. If we had defined degeneracy maps and had a simplicial space, then the first condition would reflect the degeneracy maps, $(\sigma^i(x, y), y) \sim (x, s_i(y))$.

Note: $EM/M \cong BM$.

If $M$ were a group, we have a fibration $X \to EM \times_M X \to BM$. However, we $M$ is a monoid, we do not always get $X$ as the fibre back.
7.2 Homology fibrations

Homology fibrations are a weakening of the notion of a fibration. That is, if we consider a map \( p: E \rightarrow B \).

It can be one of the following.

1. A fibre bundle.
2. Fibrations.
3. Quasi-fibrations.

If \( p: E \rightarrow B \) satisfied is one of the things on the list, then it is also everything below it. A homology fibrations is something at the very bottom of the list, which we describe below. We first give the definition of a quasi-fibration.

If \( b \in B \), the define the homotopy fibre of \( F \) at \( b \) to be \( F_{p,b} := E \times_B B \), where \( B = \text{Map}_*((I, 0), (B, b)) \).

The following diagram commutes.

\[
\begin{array}{ccc}
F_{p,b} & \xrightarrow{\varphi_b} & E \\
p^{-1}(b) & \xrightarrow{p} & B \\
\{b\} & \xrightarrow{\{b\}} & B
\end{array}
\]

In particular, if we have \( \gamma: I \rightarrow B \) such that \( \gamma(0) = b \) and \( \gamma(1) = b \), then \( \varphi_b(e) = (\gamma, e) \).

**Definition 7.1.** A map \( p: E \rightarrow B \) is a quasi-fibration if it satisfies the following condition: if \( b \in B \), then there exists \( \varphi_b: p^{-1}(b) \rightarrow F_{p,b} \) which is a weak homotopy equivalence.

**Definition 7.2.** A map \( p: E \rightarrow B \) is a homology-fibration if \( \varphi_b \) is a homology equivalence.

**Theorem 7.3.** Let \( M \) be a monoid, \( X \) a topological space with a \( M \) action, \( x \mapsto mx \) be an isomorphism in \( H_* \) for every \( m \in M \). Then

\[
EM \times_M X \rightarrow BM
\]

is a homology fibration.

**Theorem 7.4.** If \( X \) is a weakly contractible space with a free action of \( M \), then there exists an \( M \)-equivariant homotopy equivalence \( X \rightarrow EM \). So \( X/M \overset{\sim}{\rightarrow} EM/M = BM \)

**Example 7.5.** Let \( M = \mathbb{Z}_{\geq 0} \). Then \( \mathbb{R} \) is a contractible space with a free action of \( M \). Thus \( \mathbb{R} \) is a model for \( EM \). Then \( BM = EM/\mathbb{Z}_{\geq 0} = S^1 \).

Here, when we mod out by \( M \), we need to turn the monoid relations \( x \sim mx \) into an equivalence relation. (Essentially, the smallest equivalence relation that turns \( \sim \) into an equivalence relation). After doing this, we see the fibres are \( \mathbb{Z} \) instead of \( \mathbb{Z}_{\geq 0} \) as one might expect.

7.3 The telescope

**Definition 7.6.** Let \( M \) be a topological monoid. Assume \( \pi_0 M = \mathbb{Z}_{\geq 0} \). Let \( m \in M \) such that \( m \in 1 \), that is, \( m \) is in the component \( 1 \) of \( \pi_0 M \). Define

\[
M_{\infty} = \text{Tel}_m M = \bigcup_{n=1}^{\infty} M_n \times I / \sim,
\]

where the \( \sim \) is given by \( (x, 1)_n \sim (mx, 0)_{n+1} \) and \( M_n = M \).

Note that this is a homotopy colimit. [insert joke about microscope + picture]

**Proposition 7.7.** If \( p: EM \rightarrow BM \), then \( H_*(F_b) \cong H_*(\text{Tel} M) \), where \( F_b \) is the homotopy fibre.
7.4 The theorem

**Theorem 7.8** (Group Completion Theorem [4]). Let $M$ be a topological monoid and $BM$ be its classifying space. Let $\pi = \pi_0 M$. If $\pi$ is in the centre $Z(H,M)$ of $M$, then

$$H_*(M)[\pi^{-1}] \cong H_* (\Omega BM).$$

We do this in 3 steps which are as follows.

1. Show that $M_\infty \times_M EM \to BM$ is a homology fibration.
2. Show that $\text{hofib}(p) \cong_{w.e} \Omega BM$.
3. Show that $H_* M_\infty = H_* M[\pi_0^{-1}]$.

Given these three, the first implies that $H_* (\text{hofib}(p)) \cong H_* M_\infty$. The second then give $H_* M \cong_{w.e} H_* (\Omega BM)$. Lastly the third give $H_* M[\pi_0^{-1}] \cong H_* (\Omega BM)$ which is the theorem.

The rest of this section will be to describe the three steps.

7.5 Step 1: $M_\infty \times_M EM \to BM$ is a homology fibration

We use another definition of homology fibration that is stronger than our earlier definition (ie. it implies the other definition) and will show that $M_\infty \times_M EM \to BM$ is a homology fibration in this sense, and so is a homology fibration in the old sense (see [4] for details).

**Definition 7.9.** A map $p : E \to B$ is a homology fibration if whenever $b \in B$, then there exists arbitrarily small contractible neighbourhoods $U$ of $b$ such that the inclusion $p^{-1}(b) \hookrightarrow p^{-1}(U)$ is a homology equivalence for $b' \in U$.

**Proposition 7.10.** Let

$$
\begin{array}{ccc}
E_1 & \leftarrow & E_0 \rightarrow E_2 \\
\downarrow p_1 & & \downarrow p_0 & & \downarrow p_2 \\
B_1 & \leftarrow & B_0 \rightarrow B_2
\end{array}
$$

be a commutative diagram of spaces such that each $p_i$ is a homology fibration satisfy: if $b \in B_0$, then $p_0^{-1}(b) \cong p_1^{-1}(f_i(b))$ is a homology equivalence. Then the induced map of double cylinders

$$p : \text{cyl}(E_1 \leftarrow E_0 \to E_2) \to \text{cyl}(B_1 \leftarrow B_0 \to B_2)$$

is a homology fibration.

**Proof.** Let $b \in \text{cyl}(B_1 \leftarrow B_0 \to B_2)$. Then there exists arbitrarily small neighbourhoods $U$ in the form of mapping cylinders $V_0 \to V_i$ and $p^{-1}(U)$ is the mapping cylinder of $P_0^{-1}(V_0) \to p_i^{-1}(V_i)$.

**Proposition 7.11.** Let $p : E \to B$ be a map of simplicial spaces satisfying the following condition: if $k \in \mathbb{Z}_{\geq 0}$, the $E_k \to B_k$ is a homology fibration.

Assume further that we have a simplicial map $\theta : [k] \to [l]$ and if $b \in B_l$ then $p^{-1}(b) \to p^{-1}(\theta^* b)$ is a homology equivalence.

Then the map of realizations $|E| \to |B|$ is a homology fibration.

**Proof.** Observe that $|E|$ and $|B|$ are made up of skeletons. In particular $|B|_k$ is $\text{cyl}(|B|_k) \leftarrow \partial \Delta^k \times B_k \to \Delta^k \times B_k$. Similarly $|E|_k$ is a cylinder. We can then apply the previous proposition.
Note that $M_\infty \times_M EM$ and $BM$ are the realisations of simplicial spaces $E$ and $B$ with

$$E_k = M_\infty \times B_k$$

and

$$B_k = M^k$$

Now applying the proposition gives the result. Note that we have not yet checked the condition on the fibres that is required to make the proposition work. We leave this for another time (see McDuff, Segal).

### 7.6 Step 2: $hmfib \simeq_w \Omega BM$

We want to show that $M_\infty \times_M EM \simeq *$.

Consider $M_\infty = Tel(M \xrightarrow{m} M \xrightarrow{m} M \xrightarrow{m} \cdots)$. Then

$$M_\infty \times EM = Tel(EM \times M \xrightarrow{id \times m} EM \times M \xrightarrow{id \times m} \cdots)$$

and

$$M_\infty \times_M EM = Tel(EM \times_M M \to EM \times_M M \to \cdots) = Tel(EM \to EM \to \cdots) \simeq *$$

Here, the action of $M$ on $M_\infty$ is defined as follows. If $n \in M$, then $n$ acts on $M_k \times I$ by $(q,t).n = (qn,t)$. We need $M$ to act on the right since the telescope is given by $\bigsqcup M_k \times I/\sim$, where $(q,1)_k \sim (mq,0)_{k+1}$. In particular, we need $(q,1)_k.n \sim (mq,0)_{k+1}.n$ which works only if $M$ is acting on the right.

Given that $M_\infty \times_M EM \simeq *$, we get a diagram of fibre sequences

$$\begin{array}{ccc}
F_p & \xrightarrow{\simeq} & EM \times_M M_\infty \\
\xrightarrow{\simeq} & & \xrightarrow{\simeq} \\
F_q & \xrightarrow{\simeq} & * \\
\xrightarrow{q} & & \xrightarrow{q} \\
& & \xrightarrow{} B_2
\end{array}$$

Note that $F_q = \{(e,f : I \to BM)|f(0) = *, f(1) = q(e)\} = \{(*,f : I \to BM)|f(0) = *, f(1) = q(*) = *\} = \Omega BM$.

Therefore, $hmfib(p) = F_p \simeq_w F_q = \Omega BM$.

### 7.7 Step 3: $H_*M_\infty = H_*M[\pi_0^{-1}]$

Observe that

$$H_*M_\infty = \lim_{\to}(H_*M \xrightarrow{[m]} H_*M \xrightarrow{[m]} \cdots)$$

where $[m] : H_*M \to H_*M$ is multiplication by $[m] \in \pi_0 M = H_0 M$. If we use the Mayer-Vietoris sequence, draw a picture and use some magic (see [4] for details), we see that the limit is $H_*M[\pi_0^{-1}]$ which gives the desired result.

[insert picture of M-V bits]

### 8 Cobordism

Things we might be interested in:

- cobordism of manifolds
- cobordism of oriented manifolds
- cobordism of spin manifolds
- cobordism of framed manifolds
- cobordism of stably almost complex manifolds
Recall from differential geometry.

**Theorem 8.1** (Whitney Embedding). If $M^n$ is a smooth $n$-manifold, then there exists an embedding
\[ i : M^n \hookrightarrow \mathbb{R}^{N+n} \]
and $N$ may be taken to be $n$.

Since we can think of a $n$-manifold as sitting inside $\mathbb{R}^{N+n}$, we can consider its normal bundle.

**Definition 8.2.** The normal bundle of $i$ is $\nu \to M$, where
\[ \nu_x := di(T_x M)^\perp, \]
where $di : T_x M \to T_{i(x)} \mathbb{R}^{N+n} \cong \mathbb{R}^{N+n}$ is the derivative.

Observe that $\nu = \nu(i)$ is a $N$-plane bundle.

**Definition 8.3.** The Grassmannian of $N$ planes in $\mathbb{R}^{N+n}$ is
\[ Gr_N(\mathbb{R}^{N+n}) := \{ V \subset \mathbb{R}^{N+n} \mid \text{dim} V = N \} \]
\[ Gr_N(\mathbb{R}^{N+n}) = V_N(\mathbb{R}^{N+n})/O(N) \]

**Definition 8.4.** The Stiefel manifold of orthogonal $N$ frames in $\mathbb{R}^{N+n}$ is
\[ V_N(\mathbb{R}^{N+n}) := \{ (v_1, \ldots, v_N) \mid \{ v_i \} \text{ is orthonormal} \} \]

**Proposition 8.7.** $\pi_i(V_N(\mathbb{R}^{N+n})) = 0$, for $0 \leq i < n$.

**Proof.** Use induction and the fibre bundle
\[ F_{N-1}(\mathbb{R}^{N+n-1}) \to V_N(\mathbb{R}^{N+n}) \xrightarrow{p} S^{N+n-1}, \]
where $p((v_1, \ldots, v_N)) = v_N$. Complete the proof as an exercise. \qed

So we have
\[ V_N(\mathbb{R}^{N+n}) \xleftarrow{O(N)} V_N(\mathbb{R}^{N+n+1}) \xrightarrow{O(N)} \ldots \]
\[ Gr_N(\mathbb{R}^{N+n}) \xleftarrow{O(N)} Gr_N(\mathbb{R}^{N+n+1}) \xrightarrow{O(N)} \ldots \]
induced from $\mathbb{R}^{N+n} < \mathbb{R}^{N+n+1} < \ldots$
Taking \( \lim_{n \to \infty} \), and using the proposition, we have

\[
\begin{array}{c}
V_N(\mathbb{R}^\infty) \\ \downarrow O(N)
\end{array}
\quad \simeq \ast
\quad \begin{array}{c}
Gr_N(\mathbb{R}^\infty) \\ \downarrow O(N)
\end{array}
\simeq BO(N)
\]

Since at each stage, we have a principle \( O(N) \) bundle.

Assume that there exists a family of groups \( G(N) \) and representations \( \rho_n : G(N) \to O(N) \) and commutative diagrams

\[
G(N) \longrightarrow G(N + 1)
\quad \begin{array}{c}
\downarrow \rho_N \\ \downarrow \rho_{N+1}
\end{array}
\quad O(N) \longrightarrow O(N + 1)
\]

where \( i_N = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \).

Here are some examples that we should have in mind.

1. \( G(N) = O(N) \), \( \rho_n = id \) gives rise to bordism.
2. \( G(N) = SO(N) \), \( \rho_n = (SO(N) \hookrightarrow O(N) \) gives rise to oriented bordism.
3. \( G(2N) = U(N) \), gives rise to complex bordism.
4. \( G(N) = 0 \), gives rise to framed bordism.
5. \( G(N) = Spin(N) \), \( \rho_n = (Spin(N) \to SO(N) \hookrightarrow O(N) \) gives rise to spin bordism.
6. \( G(N) = String(N) \), gives rise to string bordism.

**Definition 8.8.** \( \nu \) admits a stable \( G(N) \)-structure if there exists a lifting:

\[
\begin{array}{c}
\nu \quad \longrightarrow \quad \longrightarrow
\end{array}
\quad \begin{array}{c}
BG(N) \\ \downarrow B_{\nu_N}
\end{array}
\quad \begin{array}{c}
M \\ \longrightarrow
\end{array}
\quad \begin{array}{c}
\nu \quad \longrightarrow \quad \longrightarrow
\end{array}
\quad \begin{array}{c}
G_N(\mathbb{R}^{N+m}) \\ \downarrow k
\end{array}
\quad \begin{array}{c}
Gr_N(\mathbb{R}^\infty) \\ \longrightarrow
\end{array}
\quad \begin{array}{c}
BO(N) \\ \longrightarrow
\end{array}
\quad \begin{array}{c}
\nu \quad \longrightarrow \quad \longrightarrow
\end{array}
\]

**Definition 8.9.** The \( n \)th stable \( G \)-bordism group is

\( (MG)_n := \{ M^n \ text{ compact without } \partial \ s.t \ there \ exists \ i : M \hookrightarrow \mathbb{R}^{N+n} \ along \ with \ a \ choice \ of \ lift \ f_\nu \ of k \circ f_\nu \}/ \sim \)

where \( (M,f_\nu \sim (N,g_\nu) ) \) if there exists \( l : W^{n+1} \hookrightarrow \mathbb{R}^{N'+n+1} \) with \( G(N') \) structure on \( \nu(l) \) such that

1. \( \partial W^{n+1} = M \cup N \)
2. \( \tilde{h}_\nu |_{M \cup N} = i_N^* \ldots i_N^* f_\nu \cup i_N^* \ldots i_N^* g_\nu \). Here, \( \tilde{h}_\nu \) is the lift of \( h_\nu : W \to BO(N') \) to \( BG(N') \) obtained from the \( G(N') \) structure on \( V(l) \).

**Example 8.10.**
1. If \( G(N) = O(N) \), then the second assumption is vacuous. In this case, \( MO_n = \{ \text{ compact } n\text{-manifolds } \}/ \text{ bordism } \).
2. If \( G(N) = SO(N) \), \( MSO_n = \{ \text{ compact oriented } n\text{-manifolds } \}/ \text{ oriented bordism } \).
Claim: $M$ is oriented $\iff TM$ is oriented $\iff \nu$ is oriented for all $i : N \to \mathbb{R}^{N+n}$.

[insert picture of craig demonstrating orientation with arms and legs]

**Definition 8.11.** Let $\xi \to X$ be a vector bundle with a metric on $\xi$ (ie. a fibre wise dot product). The disk bundle on $\xi$ is

$$D(\xi) := \{ v \in \xi : \|v\| \leq 1 \}.$$  

The sphere bundle is

$$S(\xi) := \{ v \in \xi : \|v\| = 1 \}.$$  

The Thom Space is

$$X^\xi := D(\xi)/S(\xi).$$

Claim: If $X$ is paracompact, we can always define a metric on $\xi$. Moreover, any two $\xi$ give homeomorphic Thom spaces.

**Proposition 8.12.**

$$X^{\xi \oplus 1} = \Sigma X^\xi,$$

where $1 = X \times \mathbb{R}$.

For examples,

$$\text{Thom}(1) = \frac{X \times [-1,1]}{X \times \{-1,1\}} = \Sigma X.$$  

**Proposition 8.13.** Given a continuous map $f : Y \to X$, there is an induced map $f^\xi : Y^f \xi \to X^\xi$.

Exercise.

We want to study

$$\text{Bi}_N(\gamma_{N+1}) = \gamma_N \oplus 1 \xrightarrow{\gamma_N} \gamma_{N+1}$$

$$\text{BO}(N) \xrightarrow{\text{Bi}_N} \text{BO}(N+1)$$

$$\text{Gr}_N(\mathbb{R}^\infty) \xrightarrow{f} \text{Gr}_{N+1}(\mathbb{R}^\infty).$$

Here, $f$ maps $V \subset \mathbb{R}^\infty$ to $V \oplus \mathbb{R} \subset \mathbb{R}^\infty \oplus \mathbb{R} = \mathbb{R}^{\infty+1}$.

So we get maps

$$\Sigma \text{BO}(N)^{\gamma_N} = \text{BO}(N)^{\gamma_{N+1}} = \text{BO}(N)^{\text{Bi}_N(\gamma_{N+1})} \to \text{BO}(N+1)^{\gamma_{N+1}}$$

Moreover, we have a map $\Sigma \text{BG}(N)^{B_p \gamma_N} \to \text{BG}(N+1)^{B_p \gamma_{N+1}}$ from the commutative diagram

$$\text{BG}(N) \xrightarrow{B_p} \text{BG}(N+1)$$

**Definition 8.14.** Let $MG$ be the spectrum associated to the prespectrum $\text{BG}(N)^{B_p \gamma_N}$, ie

$$MG_k = \lim_{n \to \infty} \Omega^N \text{BG}(N+k)^{B_p \gamma_{N+k}}$$

\[\text{As a somewhat related but off topic note, the reader may want to look up the phrase Eilenberg swindle.}\]
Theorem 8.15 (Thom).

\((MG)_n \cong \pi_n MG\)

What has been computed:

1. \(G = O\) - Thom (not too hard);
2. \(G = U\) - Novikov/Milnor;
3. \(G = SO\) - Stong (messy, nice away from \(p = 2\));
4. \(G = \{1\}\) - \(G(n) = \{1\}, BG(n) = \ast\), \(BG(\gamma_n) = D^n/S^{n-1} = S^n\), so \(MG_k = \lim_{n \to \infty} \Omega^n S^{n+k}\) (which is hard).

Moral: \(n\)-dimensional framed bordism = \(\pi_{n}^{s}(S^{0}) = \lim_{k \to \infty} \pi_{n+k}(S^{k})\), the stable homotopy groups of spheres is hard to compute.

8.1 Sketch of proof

We briefly summarise the main idea of the previous section and give a sketch of Theorem 8.16, which is a restatement of the theorem earlier.

**Setup:** If \(G(n) \xrightarrow{\rho_n} O(n)\) is a sequence of groups and group homomorphisms with commuting diagrams

\[
\begin{array}{ccc}
G(n) & \xrightarrow{\rho_n} & G(n+1) \\
\downarrow & & \downarrow \\
O(n) & \xrightarrow{\rho_{n+1}} & O(n+1),
\end{array}
\]

then we define the \(G\)-bordism groups to be the bordism of manifolds whose stable normal bundle admits the structure of a \(G\)-bundle. ie. there exists a morphism \(X \to BG(n)\) that makes the following commute.

\[
\begin{array}{ccc}
BG(n) & \xrightarrow{B\rho(n)} & BO(n) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f'} & BO(n)
\end{array}
\]

**Theorem 8.16.** The \(n\)th such bordism group := \((MG)_n \cong \pi_n(MG)\), where \(MG\) is the spectrum associated to the prespectrum \(\{BG(n)^{\gamma_n}\}\).

We will use the following two facts.

1. Let \(Y_i\) be a sequence of spaces such that \(Y_i \subset Y_{i+1} \subset \ldots\) and let \(X\) be compact. If \(f : X \to \bigcup_{i=1}^{\infty} Y_i\), then \(\text{im}(f) \subset Y_j\) for some \(j\). Note that here, we endow \(U_i Y_i\) with the topology of the union. [exercise: prove this using the usual compactness type arguments.]

2. (Smooth approximation theorem). If \(M, N\) are smooth manifolds and \(f : M \to N\) is continuous, then there exists \(f' \simeq f\) such that \(f' : M \to N\) is smooth. ie. \(C^\infty(M)\) is dense in \(C^0(M)\).

3. Let \(Z \subset N\) be a smooth submanifold. There exists \(f'' : M \to N\) which is smooth and transverse to \(Z\). ie if \(x \in (f'')^{-1}(Z)\), then

\[T_{f''(x)} Z + df''_{x} T_{x} M = T_{f''(x)} N.\]

Then Sard’s Theorem implies that \((f'')^{-1}(Z)\) is a submanifold of \(M\) and \(\text{codim}_M((f'')^{-1}(Z)) = \text{codim}_N(Z)\). [See Guillemin and Pollack for a reference]
Sketch of Theorem 8.16. We want to construct maps $T : \pi_n MG \to (MG)_n$ and $P : (MG)_n \to \pi_n MG$ such that $TP = id$ and $PT = id$.

We construct $T$ as follows. Let $\alpha \in \pi_n MG = \pi_n \lim_{k \to \infty} \Omega^k BG^*_k$. Then we can use the above fact to find an $\tilde{\alpha}$ and a $k$ such that the following commutes.

\[
\begin{array}{c}
S^n \xrightarrow{\alpha} \lim_{k \to \infty} \Omega^k BG^*_k \\
\downarrow \tilde{\alpha} \quad \quad \quad \quad \quad \quad \downarrow \\
\Omega^k BG^*_k
\end{array}
\]

Then via the loops suspension adjunction, we get and

\[
\bar{\alpha} : S^{n+k} \to BG^*_k.
\]

For simplicity, assume $G = O$. In particular, we have

\[
\bar{\alpha} : S^{n+k} \to BO(k)^\gamma_k.
\]

Moreover, $BO(k) = \lim_{k \to \infty} Gr_k(\mathbb{R}^{k+p})$. So again, using fact 1, we can find an $\hat{\alpha}$ and a $p$ such that the following commutes.

\[
\begin{array}{c}
S^{n+k} \xrightarrow{\pi} BO(k)^\gamma_k \\
\downarrow \hat{\alpha} \quad \quad \quad \quad \quad \quad \downarrow \\
Gr_k(\mathbb{R}^{k+p})^\gamma_k
\end{array}
\]

Observe that $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\}$ and $Gr_k(\mathbb{R}^{k+p})^\gamma_k = \gamma_k \cup \{\infty\}$ and since we are dealing with based maps, we have $\hat{\alpha}(\infty) = \infty$. In particular, we can think of $\hat{\alpha}$ as a map

\[
\hat{\alpha} : \mathbb{R}^{n+k} \to \gamma_k,
\]

and $Gr_k(\mathbb{R}^{n+k})$ sits inside $\gamma_k$ as the zero section. By fact 3, there is a $\hat{\alpha} \simeq \hat{\alpha}$ which is smooth and transverse to $Gr_k(\mathbb{R}^{n+k})$. Define

\[
T(\alpha) := \hat{\alpha}^{-1}(Gr_k(\mathbb{R}^{n+k})).
\]

Note in defining $T$, we made several choice:

1. Representative of homotopy class of $\alpha$.
2. $k$
3. $p$
4. Homotopical replacement $\hat{\alpha}$ for $\tilde{\alpha}$.

**Homework:** Think about why $T$ is well defined. Moreover why is $T(\alpha) \in (MG)_n$?

We now construct the inverse map $P : (MG)_n \to \pi_n MG$. Let $X$ be an $n$-dimensional manifold. Then by the Whitney embedding theorem, there is a $k$ such that there is an embedding $i : X \hookrightarrow \mathbb{R}^{n+k}$. Then $\nu(i)$ is a $G(k)$-bundle.

Recall that the classifying map is $f^\nu : X \to BO(k)$. We have $f^*\gamma_k = \nu(i)$, and so the following diagram commutes.
We now use the tubular neighbourhood theorem from differential topology:

**Theorem 8.17.** Let $X \subset \mathbb{R}^{n+k}$ be a compact $n$-manifold. Then $X$ has an open neighbourhood homeomorphic to $\nu$.

If $n = 1$, then the theorem corresponds exactly to drawing a tube around our $X$. Let $U$ be the tubular neighbourhood as in the theorem, i.e. $\varphi : U \to \nu$ is a homeomorphism.

We construct a map $P(X) : S^{n+k} \to \gamma_k \cup \{\infty\}$ (again $\infty \mapsto \infty$) by

$$P(X)(y) = \begin{cases} \infty & y \notin U, \\ F\varphi(y) & y \in U. \end{cases}$$

Observe that the map is continuous since if we are near the end of the tubular neighbourhood, then $\varphi$ maps $U$ to larger and larger values.

Again, we have made some choices in defining $P$, but one can check that this does not make a difference. We can observe that $TP(X) = X$ since $TP(X) = P(X)^{-1}BG(k) = X$. Craig intimidates us into thinking that the rest of the proof works. That is, $PT = id_{\pi_nMG}$. \hfill \qed

Note again that in constructing $P$ we made the following choices.

1. A representative $X \in (MG)_n$.
2. A choice of embedding $X \hookrightarrow \mathbb{R}^{n+k}$ and choice of $k$.
3. Different radii of tubular neighbourhoods.

Different choices will give homotopic maps.

### 9 Moduli Spaces and Mapping Class Groups

#### 9.1 Mapping Class Group

Denote by $S_{g,r}$ the topological surface of genus $g$ with $r$ labelled boundary components.

**Definition 9.1.** The mapping class group of $S_{g,r}$ is

$$\Gamma_{g,r} := \frac{\text{Homeo}^+(S_{g,r}, \partial)}{\text{Homeo}_0^+(S_{g,r}, \partial)},$$

where $\text{Homeo}^+(S_{g,r})$ is the group of orientation preserving homeomorphisms of $S_{g,r}$ that fix the boundary, and $\text{Homeo}_0^+(S_{g,r}, \partial)$ the homeomorphisms that are also isotopic to the identity.

We also define a related object,

$$\Gamma_g := \frac{\text{Homeo}^+(S_{g,r})}{\text{Homeo}_0^+(S_{g,r})},$$
which consists also of homeomorphisms that do not fix the boundary. While we will not talk about this other object too much, we note that there is a short exact sequence

\[ 1 \rightarrow \mathbb{Z}^r \rightarrow \Gamma_{g,r} \rightarrow \Gamma_g \rightarrow 1. \]

There are several reasons to care about mapping class groups.

1. Homeo\(^+\)(S, \partial) is cool.
2. Diff\(^+\)(S, \partial) is cool. Moreover it turns out that
   \[ \Gamma_{g,r} \cong \frac{\text{Diff}^+(S_{g,r}, \partial)}{\text{Diff}^+_0(S_{g,r}, \partial)}. \]
3. Characteristic classes are invariants of a F-bundle with respect to F-bundle isomorphism. In particular, we are interested in \( S_g \)-bundles.
   
   Fact: There is a bijection
   \[ \{ S_g \rightarrow E \rightarrow X \}/\text{iso} \leftrightarrow \text{Map}(X, B\text{Diff}^+(S_g))/\text{homotopy equivalence} \]
   between isomorphism classes of \( S_g \)-bundles over \( X \) and homotopy maps from \( X \) to \( B\text{Diff}^+(S_g) \).

   This is analogous to the situation we saw in the previous sections where rank \( n \) vector bundles over \( X \) were classified by homotopy maps from \( X \) to \( BU_n \).

9.2 How do we get characteristic classes?

Characteristic classes of surface bundles are given by \( H^*(B\text{Diff}^+(S_g)) \). To do this, we take a cohomology class \([\alpha]\in H^i(B\text{Diff}^+(S_g))\). Then given an \( S_g \)-bundle

\[ \pi : E \rightarrow X \]

we can go via our bijection to get a homotopy map

\[ [f] : X \rightarrow B\text{Diff}^+(S_g). \]

By applying the induced map on cohomology to \([\alpha]\), we get a cohomology class \( f^*([\alpha]) \in H^i(X) \). In particular, \( f^* [\alpha] \) is an invariant of \( S_g \)-bundle isomorphisms. WINNAR.

Thus, if we are interested in characteristic classes of surface bundles, we want to study \( B\text{Diff}^+(S_g) \).

A theorem of Earl and Eels tells us that, for \( g \geq 2 \), \( \text{Diff}^+(S_g) \simeq_{h.e.} \Gamma_g \). In particular, \( \text{Diff}^+_0(S_g) \simeq \{ \ast \} \).

Therefore, rather than studying the rather complicated infinite dimensional space \( B\text{Diff}^+(S_g) \), we can look at the much nicer \( B\Gamma_g \).

To get \( B\Gamma_g \), we want to find \( E\Gamma_g \) which is contractible and has a free \( \Gamma_g \) action. Then \( B\Gamma_g = E\Gamma_g/\Gamma_g \).

This leads us to the study of moduli spaces.

9.3 Moduli Spaces

Key Idea: A moduli space is a space that parametrizes a class of objects we care about.

Example 9.2. Here are some examples that may be familiar.

1. \( \mathbb{RP}^1 = S^1/\sim = -x \) parametrizes lines through the origin in \( \mathbb{R}^2 \).
2. \( \mathbb{RP}^n \) parametrizes lines through the origin in \( \mathbb{R}^{n+1} \).
3. \( \text{Gr}_n(\mathbb{R}^{n+m}) \) parametrizes \( n \)-dimensional vector subspaces of \( \mathbb{R}^{n+m} \).
In our construction, both our models for $E\Gamma_g$ and $B\Gamma_g$ will be moduli spaces.

Our model for $E\Gamma_g$ will be the moduli space of complete hyperbolic metrics on $\hat{S}_{g,r}$ up to isotopy. It is called Teichmüller space, and can be written as

$$\mathcal{T}_g^r := \left\{ (S,f) \mid \begin{array}{l} f : \hat{S}_{g,r} \to S, \\
 f \text{ is a homeo,} \\
 S \text{ is a complete hyperbolic surface with labelled cusps and;} \\
 f \text{ preserves labelling} \end{array} \right\} / \sim$$

where $(S_1,f_1) \sim (S_2,f_2) \iff f_2 \circ f_1^{-1} : S_1 \to S_2$ is isotopic to an isometry.

**Example 9.3.** $\mathcal{T}_0^3 \cong \{ * \}$.

Fact: Specifying the length $(l_i \in \mathbb{R}_{\geq 0},$ for $i = 1, 2, 3)$ of the boundary components of $S_{0,3}$ fixes a unique hyperbolic metric on it.

Question: What about $\mathcal{T}_4^0$?

Fix a curve $\alpha$ on $\hat{S}_{0,4}$. Then given a complete hyperbolic metric on $S_{0,4}$, there is a unique geodesic in the same homotopy class of $\alpha$. If we cut along $\alpha$, do a small twist and stick the surfaces back together along $\alpha$, we get a different metric on $S_{0,4}$.

The idea then is that choosing a length in $\mathbb{R}_{>0}$ and a twisiting angle in $\mathbb{R}$ specifies a metric on $S_{0,4}$. Thus we have a heuristic argument that $\mathcal{T}_0^3 \cong \mathbb{R}_{>0} \times \mathbb{R}$.

More generally

$$\mathcal{T}_g^r \cong \mathbb{R}^{3g-3+r} \times \mathbb{R}^{3g-3+r}.$$ 

To see this, one uses similar arguments to the $S_{0,4}$ case. The $3g-3+r$ comes from the fact that there are $3g-3+r$ non-intersecting essential closed curves on $S_{g,r}$.

Observe that $\mathcal{T}_g^r$ is contractible. Moreover, $\Gamma_g^r$ acts on $\mathcal{T}_g^r$ by

$$[\varphi].[S,f] = [S,f \circ \varphi].$$

Here, $\varphi:S_{g,r} \to S_{g,r}$, so $f \circ \varphi:S_{g,r} \to S$ and $[S,f \circ \varphi] \in \mathcal{T}_g^r$. Unfortunately, the actions is not free.

Recall that a group action $G \times X \to X$ is free if $g.x = x$ only when $g = id$. To see an example of the mapping class group not acting freely, then imagine a surface with a symmetry, say it has a $\pi$ rotation symmetry, call it $\iota$. Then $\iota$ is a homeomorphism of $S$, so $\iota \in \Gamma_g^r$. But because of the symmetry, $[\iota].[S,f] = [S,f]$ for $[S,f] \in \mathcal{T}_g^r$. Surfaces with this type of symmetry are called degenerate surfaces.

Thus $\Gamma_g^r$ does not act freely on $\mathcal{T}_g^r$. However it is not far off since in most non-degenerate cases the action the action is free. Thus

$$B\Gamma_g^r \approx \mathcal{T}_g^r / \Gamma_g^r,$$

where $\approx$ means almost equal to.

**Definition 9.4.** The moduli space of complete hyperbolic surfaces of genus $g$ with $r$ labelled cusps is

$$\mathcal{M}_g^r := \mathcal{T}_g^r / \Gamma_g^r.$$ 

**Example 9.5.** $\mathcal{M}_0^3 = \{ * \}$, since $\mathcal{T}_0^3 = \{ * \}$.

**Example 9.6.** $\mathcal{M}_1^4 \cong S^2 - \{ 3 \text{ points.} \}$. 

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While it is not true that $B\Gamma_g$ equals $\mathcal{M}_g$ because of the non-free action, the two are still very closely related. In particular, it is true that their rational cohomologies agree. That is $H^*(B\Gamma_g, \mathbb{Q}) \simeq H^*(\mathcal{M}_g, \mathbb{Q})$.

Thus, we are interested in $H^*(\mathcal{M}_g, \mathbb{Q})$. The Mumford conjecture states that $H^*(\mathcal{M}_\infty, \mathbb{Q}) \cong \mathbb{Q}[k_1, k_2, \ldots]$, where $deg(k_i) = 2i$. A solution to the Mumford Conjectures comes in the form the the Madsen-Weiss Theorem.

Philosophically, the idea is that the homology of $\Gamma_{g,r}$ in a stable range is the homology of an infinite loop space. In later lectures, we will look in more detail at the Madsen-Weiss theorem. For now we give a big outline of the proof. There are three main steps.

1. $Diff^+(S_{g,r})$ has contractible components for $g \geq 2$.
2. Harer stability: $H_i(\Gamma_{g,r})$ is independent of $g$ and $r$ for $i \ll g$.
3. Group completion theorem.

We have already encountered step 1. Step 3 will allow us to show that the classifying space of compactly supported diffeomorphisms of an infinite genera surface will be homology equivalent to some infinite loop space. Step 1 relates the former with $B\Gamma_\infty$, which we can hope to understand because of Harer stability.

10 Harer Stability

Recall the key players from the previous section: $S_{g,r}, \Gamma_g, T_g^r$ and $\mathcal{M}_g$. We also had maps

$$BDiff(S_g) \to B\Gamma_g \to \mathcal{M}_g.$$ 

The first map is a homotopy equivalence, while the second is a rational homology equivalence. We also gave a rough statement of Harer stability which said the following.

**Theorem 10.1 (Harer Stability).** $H_i(\Gamma_{g,r}) := H_i(B\Gamma_{g,r})$ is independent of $g$ and $r$ for $i \ll g$.

In this section, we will describe the proof of Harer stability. We will be following closely [8].

### 10.1 General nonsense on homological stability

Let $G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots$ be a sequence of groups and group inclusions. Let $X^*_n \subset X^*_2 \subset \ldots \subset X^*_n \subset \ldots$ be a sequence of simplicial complexes with a simplicial $G_n$ action such that

1. The $X^*_n$ are highly connected. i.e. $\pi_k(X^*_n) = 0$ for some $k \leq f(n)$, where $f$ is some increasing function of $n$.
2. The stabilizer of a $p$-simplex is $Stab_{X^*_n}(\sigma_p) \cong G_{n-p-1}$.
3. $G_n$ acts transitively on $X^*_n$ (or at least as transitively on the vertices).

Then one might hope for: $H_*(G_n)$ is independent of $n$ for $n \ll *$. Here are some examples.

1. The symmetric group $S_n$. There are inclusions $S_n \subset S_{n-1}$. $S_n$ acts on the standard ($p-1$)-simplex $\Delta^p$.
2. Braid groups.
3. Unordered configuration spaces.
10.2 The maps

What are the inclusion maps (also called stabilization maps)? Let

\[ \alpha : S_{g,r+1} \hookrightarrow S_{g+1,r} \]

be the map the obtained from gluing two boundaries of a pair of pants to two boundary components. Then this induces a map

\[ \alpha_g : \Gamma_{g,r+1} \hookrightarrow \Gamma_{g+1,r} \]

obtained by sending a map \( f \) to \( f \cup id \). Similarly from the map

\[ \beta : S_{g,r} \hookrightarrow S_{g,r+1} \]

obtained from gluing one boundary component of a pair of pants, we get the map

\[ \Gamma_{g,r+1} \hookrightarrow \Gamma_{g,r+1} \]

Lastly, let

\[ \delta : S_{g,r+1} \hookrightarrow S_{g,r} \]

be the map obtained by gluing a disk to a boundary component and let

\[ \delta_g : \Gamma_{g,r+1} \hookrightarrow \Gamma_{g,r} \]

be the corresponding map of mapping class groups. Note that \( \delta \) is the left inverse of \( \beta \).

Let \( H_*(\alpha_g), H_*(\beta_g) \) and \( H_*(\gamma_g) \) be the induced maps on homology. We can now state the theorem more precisely as it appears in [8].

**Theorem 10.2** (Harer Stability 1). Let \( g \geq 0, r \geq 1 \).

1. \( H_*(\alpha_g) : H_*(\Gamma_{g,r+1}) \rightarrow H_*(\Gamma_{g+1,r}) \)
   
is surjective for \( * \leq \frac{2}{3}g + \frac{1}{3} \) and is an isomorphism for \( * \leq \frac{2}{3}g - \frac{2}{3} \).
2. \( H_*(\beta_g) : H_*(\Gamma_{g,r}) \rightarrow H_*(\Gamma_{g,r+1}) \)
   
is always injective and is an isomorphism for \( * \leq \frac{2}{3}g \).

For the statement for closed surfaces, consider \( \delta_g : \Gamma_{g,1} \rightarrow \Gamma_{g,0} \).

**Theorem 10.3** (Harer Stability 2).

\( H_*(\delta_g) : H_*(\Gamma_{g,1}) \rightarrow H_*(\Gamma_{g,0}) \)

is surjective for \( * \leq \frac{2}{3}g + 1 \) and is an isomorphism for \( * \leq \frac{2}{3}g \).

In particular, if we combine the second theorem with the first theorem, then we get homological stability for closed surfaces.
10.3 The Arc Complexes

Let $S$ be a connected, oriented surface with boundary and let $b_0, b_1 \in \partial S$.

An arc on $S$ is an embedded path in $S$ such that

1. the path intersects $\partial S$ only at the endpoints; and
2. the intersection is transverse.

A collection $\{a_0, \ldots, a_n\}$ of arcs is non-separating if $S - \{a_0, \ldots, a_n\}$ is connected.

**Definition 10.4.** $\mathcal{O}(S, b_0, b_1)$ is the simplicial complex with

- vertices: isotopy classes of non-separating arcs with boundary $\{b_0, b_1\}$.
- simplices: A $p$-simplex of $\mathcal{O}(S, b_0, b_1)$ is a collection of $p + 1$ distinct isotopy classes of arcs $(a_0, \ldots, a_p)$ such that
  1. There exists representative $a_0, \ldots, a_p$ with disjoint interiors,
  2. $\{a_0, \ldots, a_p\}$ are non-separating, and
  3. The anticlockwise ordering of $a_0, \ldots, a_p$ at $b_0$ agree with the clockwise ordering at $b_1$.

[Insert Picture]

We will denote by $\mathcal{O}^1(S, b_0, b_1)$ to be the arc complex $\mathcal{O}(S, b_0, b_1)$ where $b_0$ and $b_1$ are on the same boundary component. Similarly, $\mathcal{O}^2(S, b_0, b_1)$ will be the arc complex where $b_0$ and $b_1$ are on different boundary components.

Note that $\Gamma$ acts on $\mathcal{O}(S, b_0, b_1)$ by $[f](a_0, \ldots, a_p) = (f(a_0), \ldots, f(a_p))$. There are four properties of $\mathcal{O}(S, b_0, b_1)$ that we will use in the proof of 10.2. Essentially, they will relate to the idea of want a “transitive action, the stabilizer condition and high connectivity” which we described in the general nonsense section. We will also describe relationships between the $\mathcal{O}^1$ and $\mathcal{O}^2$. These are the four ingredients the Wahl refers to in [8, Section 2].

10.4 Four properties of $\mathcal{O}$

**Proposition 10.5.** 1. $\Gamma_{g,r}$ acts transitively on the $p$-simplices of $\mathcal{O}(S_{g,r}, b_0, b_1)$ for each $p$.

2. There exist isomorphisms

$$St_{\mathcal{O}^1}(\sigma_p) \xrightarrow{s_1} \Gamma_{g-p-1, r+p+1}$$

$$St_{\mathcal{O}^2}(\sigma_p) \xrightarrow{s_2} \Gamma_{g-p, r+p-1}$$

where $\sigma_p$ is a $p$-simplex of $\mathcal{O}^i(S_{g,r}, b_0, b_1)$ and $St_{\mathcal{O}^i}(\sigma_p)$ is the stabilizer of a $p$-simplex in $\mathcal{O}^i$ under the action of $\Gamma_{g,r}$.

**Proof.** (Sketch) Let $\sigma_p = (a_0, \ldots, a_p)$. Consider $S - \sigma_p$. Since $\sigma_p$ is non-separating, $S - \sigma_p \cong S_{g_\sigma, r_\sigma}$. In particular, we can show that $g_\sigma$ and $r_\sigma$ depend only on $p$ (by an argument of fattening up arcs and removing them from our original surface). So for any other $p$-simplex $\tau_p$, $S - \tau_p \cong S_{g_\tau, r_\tau}$. By the classification of surfaces, there is an isomorphism $f : S_{g_\tau, r_\tau} \cong S_{g_\sigma, r_\sigma}$. Then looking at the lift of $f$ to our original surface, we see that $f$ is a homeomorphism of $S$ that sends $\sigma$ to $\tau$, hence the action is transitive.

For the second part, we similar consider the cut surface and need to argue that all homeomorphisms that fix the boundary of the cut surface are stabilizers of $\sigma_p$.

**Proposition 10.6.** Let $\gamma : \mathcal{O}^2(S_{g,r+1}) \to \mathcal{O}^1(S_{g+1,r})$ be the map that connects the two components of $S$ that contains $b_0, b_1$ via a strip.

[PICTURE]
Let $\beta : O^1(S_{g,r}) \to O^2(S_{g,r+1})$ be the map that separates the component of $S$ that contains $b_0,b_1$ via a strip.

[PICTURE]

Given a $p$-simplex $\sigma_p$ in $O^2(S_{g,r+1})$, we have the following diagram which commutes.

\[
\begin{array}{ccc}
\Gamma_{g,r+1} & \xleftarrow{\beta_{g,r}} & St_{O^2(S_{g,r+1})}(\sigma_p) \\
\alpha 
\downarrow & & \downarrow \alpha \\
\Gamma_{g+1,r} & \xleftarrow{\beta_{g+1,r}} & St_{O^1(S_{g,r})}(\alpha(\sigma_p))
\end{array}
\]

Similarly, given a $p$-simplex in $O^2(S_{g,r})$ we have

\[
\begin{array}{ccc}
\Gamma_{g,r} & \xleftarrow{\beta_{g,r}} & St_{O^2(S_{g,r})}(\sigma_p) \\
\beta 
\downarrow & & \downarrow \beta \\
\Gamma_{g,r+1} & \xleftarrow{\beta_{g,r+1}} & St_{O^2(S_{g,r+1})}(\beta(\sigma_p))
\end{array}
\]

Proof. See [8, Proposition 2.3]

The intuition though is that the right square is given by the previous proposition. The inclusions to the left are just including a stabilizer of a group into the whole group. The square then reads, $\alpha_g$ on $\Gamma$ induces $\beta_g$ on the stabilizers (and vice-versa for the second diagram).

**Proposition 10.7.** Let $S_{\alpha}$ and $S_{\beta}$ denote the surface $S$ union a strip glued via $\alpha$ and $\beta$ respectively. The maps

\[
\begin{align*}
\alpha &: \Gamma(S) \to \Gamma(S_{\alpha}) \\
\beta &: \Gamma(S) \to \Gamma(S_{\beta})
\end{align*}
\]

are injective. Moreover, for any vertex $\sigma_0$ of $O^i$, there are conjugations $c_\alpha$ and $c_\beta$ such that the following diagrams commute.

\[
\begin{array}{ccc}
St_{O^2}(\sigma_0) & \xleftarrow{c_\alpha} & St_{O^1}(\alpha(\sigma_0)) \\
\downarrow & & \downarrow \\
\Gamma(S) & \xleftarrow{c_\beta} & \Gamma(S_{\alpha})
\end{array}
\quad
\begin{array}{ccc}
St_{O^1}(\sigma_0) & \xleftarrow{c_\beta} & St_{O^2}(\beta(\sigma_0)) \\
\downarrow & & \downarrow \\
\Gamma(S) & \xleftarrow{c_\alpha} & \Gamma(S_{\beta})
\end{array}
\]

Proof. For a proof, see [8, Proposition 2.4].

In terms of utility of the proposition, the injectivity statements as allowing us to consider the relative homology groups $H_*(St_{O^1}(\alpha(\sigma_0)), St_{O^2}(\sigma_0))$ and $H_*(St_{O^2}(\beta(\sigma_0)), St_{O^1}(\sigma_0))$. Then if we want to map these to $H_*(\Gamma(S_{\alpha}), \Gamma(S))$ and $H_*(\Gamma(S_{\beta}), \Gamma(S))$ respectively, this map will be trivial, since the commutating diagrams tell us the map goes through $\Gamma(S)$.

**Proposition 10.8.** $O^i(S_{g,r},b_0,b_1)$ is $g - 2$ connected.

This is the high connectivity statement that we would like to hold for our arc complexes. The proof is technical and is the contents of section 4 of [8]. We will not say anything about it here.
10.5 Spectral sequence argument

We will use what follows as a black box.

**Setup:** Let $G$ and $H$ be groups. Let $X$ and $Y$ be simplicial complexes with $G$ acting on $X$ and $H$ acting on $Y$ (simplicially). Let $\phi : G \to H$ be a homomorphism and let $f : X \to Y$ be an $\phi$-equivariant map. If $G$ and $H$ act transitively on the $p$-simplices for each $p$, $X$ is $g - 2$ connected and $Y$ is $g - 1$ connected, then there exists a spectral sequence with

$$E_{1}^{p,q} = H_{q}(St_{Y}(\sigma_{p}), St_{X}(\sigma_{p})) \implies 0$$

for $p + q \leq g - 1$.

We will now set ourselves up to prove Theorem 10.2.

Recall the maps $\alpha : \Gamma_{g+1,r+} \to \Gamma_{g,r+1}$ and $\beta : \Gamma_{g,r} \to \Gamma_{g,r+1}$. We can then form the relative homology groups

$$H_{*}(\alpha) := H_{*}(\Gamma_{g+1,r}, \Gamma_{g,r+1})$$

$$H_{*}(\beta) := H_{*}(\Gamma_{g,r+1}, \Gamma_{g,r})$$

We can restate Theorem 10.2 as follows.

**Theorem 10.9.** Using the above notation, we have

1. $H_{i}(\alpha) = 0$ for $i \leq \frac{2g+1}{3}$; and
2. $H_{i}(\beta) = 0$ for $i \leq \frac{2g}{3}$.

**Proof.** We will do double induction on $g$ and $r$ by proving the following propositions.

(1): $H_{i}(\alpha) = 0$ for $i \leq \frac{2g+1}{3}$.

(2): $H_{i}(\beta) = 0$ for $i \leq \frac{2g}{3}$.

If $g = 0$ for (1) or $g = 0, 1$ for (2), then both statements are talking about $H_{0}$. But the spaces we are talking about are classifying spaces of groups, and so are connected. Thus, stability holds for these cases.

**Inductive steps:** We will show the following:

1. For $g \geq 1$, (2) $\implies$ (1).
2. For $g \geq 2$, (1) $\implies$ (2).

**Step 1.** Using the above spectral sequence setup, take $G = \Gamma_{g+1,r}, H = \Gamma_{g+1,r+1}, X = O^{2}(S_{g}, r + 1), Y = O^{1}(S_{g+1,r})$. Also, the maps $\phi : G \to H$ and $f : X \to Y$, both are induced by $\alpha : S_{g,r+1} \to S_{g+1,r}$. Proposition 10.5 gives us the transitive conditions, while Proposition 10.8 gives us the connectivity conditions.

Thus there is a spectral sequence with

$$E_{1}^{p,q} = H_{q}(St_{Y}(\sigma_{p}), St_{X}(\sigma_{p})) \implies 0$$

for $p + q \leq g - 1$.

Now, when $p = -1$, we can interpret $\sigma_{-1}$ as an empty simplex, thus the stabilizers are the whole groups. So

$$E_{-1,q}^{1} = H_{q}(\Gamma_{g+1,r+1}, \Gamma_{g+1}) = H_{q}(\alpha_{g}).$$

When $p \geq 0$, we can use Proposition 10.5 to recognise the stabilizers as mapping class groups of cut surfaces to get

$$E_{p,q}^{1} = H_{q}(\Gamma_{g-p,r+1}, \Gamma_{g-p}) = H_{q}(\beta_{g-p}).$$

In particular, we are now in a position to use induction on $H_{q}(\beta_{g-p})$.

We want to show $E_{-1,q}^{1} = H_{q}(\alpha_{g}) = 0$ for $q \leq \frac{2g+1}{3}$ to complete step 1. We will do this by showing three claims:
1. \( E_{1,q}^\infty = 0 \) for \( q \leq \frac{2g+1}{3} \).

2. \( E_{p,q}^1 = 0 \) (or equivalently \( E_{p,q-p}^1 = 0 \)) for \( q \leq \frac{2(g-p)}{3} \).

3. \( d^1 : E_{0,q}^1 \to E_{-1,q}^1 \) is the 0 map.

Given these three claims, the result then follows since

- Step 1 tells us that \( E_{-1,q} \) eventually dies in our desired range.
- Step 2 and 3 tell us that there can be no differentials into \( E_{-1,q} \) in our desired range, so nothing can kill it.
- Thus, it had to be dead to begin with. i.e. \( E_{-1,q}^1 = 0 \) in our desired range.

Proof of claim:

1. Since \( E_{p,q}^\infty = 0 \) for \( p+q \leq g-1 \), we have \( E_{1,q}^\infty = 0 \) for \( q \leq g \) and for \( g \geq 1 \), \( (2g+1)/3 \leq g \).

2. By induction, \( E_{p,q}^1 = H_q(\beta_{g-p}) = 0 \) for \( q \leq 2(g-p)/3 \).

3. Lastly, \( d^1 : E_{0,q}^1 \to E_{-1,q}^1 \) is a map \( H_q(\text{St}_{\mathcal{O}^1}(\alpha(\sigma_0)), \mathcal{O}^2(\sigma_0)) \to H_q(\Gamma_{g+r,1}, \Gamma_{g,r+1}) \). By Proposition 10.7, this map is 0.

To do step 2, we need to show that for \( g \geq 2 \), \( \{1,g\} \) implies \( \{2,g\} \). To do this, do a similar argument with \( G = \Gamma_{g,r}, H = \Gamma_{g,r+1}, X = \mathcal{O}^1(S_{g,r}), Y = \mathcal{O}^2(S_{g,r+1}) \) and \( \phi, f \) induced by \( \beta : S_{g,r} \to S_{g,r+1} \).

\[ \square \]

## 11 Madsen-Weiss Theorem (Wonderful Copenhagen)

Recall the Mumford conjecture.

**Theorem 11.1.**

\[
H^*(\mathcal{M}_g, \mathbb{Q}) \cong \text{stably } \mathbb{Q}[x_1, \ldots, x_n, \ldots],
\]

where \( |x_i| = 2t \).

Stably means that the isomorphism holds for \( * \leq f(g) \) for some increasing \( f \). In this case \( f \) is linear. This was answered by the Madsen-Weiss theorem, but is still an active area of research. Some ongoing questions are: What happens unstably? What happens in higher dimensions?

Our strategy will be to look for a string of isomorphisms that look like the following:

\[
H^*(\mathcal{M}_g; \mathbb{Q}) \cong H^*(\mathcal{B}_1); \mathbb{Q}) \cong H^*(\text{BDiff}(S_g); \mathbb{Q}) \cong \text{stably } H^*(\Omega_0^\infty \text{MTSO}(2); \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n].
\]

We have already seen many of these isomorphisms in the previous sections. In this section, we focus on the stable isomorphism, which is essentially the contents of the Madsen-Weiss theorem.

We are after the following zig-zag of equivalences:

\[
\Omega^\infty \text{MTSO}(2) \to \Omega^\infty \Psi \leftarrow \bigsqcup_{g \geq 0} \text{BDiff}(S_g),
\]

which will specialise to

\[
\Omega^\infty_0 \text{MTSO}(2) \to w.e \Omega^\infty_{S_g} \Psi \leftarrow \text{BDiff}(S_g),
\]

where the right map is a homology isomorphism for \( * \leq \frac{2}{3}g \). We will now try to describe what \( \Omega^\infty \Psi \) is.

We will use the convention that a subset \( W \subset \mathbb{R}^n \) is topologically closed if \( W \) is a closed subset. In particular, we do not want to confuse this notion with a “closed” manifold.

**Definition 11.2.** Let

\[
\Psi_k(\mathbb{R}^n) = \{(W, w) \subset \mathbb{R}^n \mid W \text{ is a topologically closed } k\text{-manifold and } w \text{ is an orientation}\}.
\]

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11.1 The topology on $\Psi_k(\mathbb{R}^n)$

While we did not say what that topology on $\Psi_k(\mathbb{R}^n)$ was explicitly, we noted that a neighbourhood basis around $\emptyset \in \Psi_k(\mathbb{R}^n)$ is given by

$$U(\{K\}) = \{W \in \Psi_k(\mathbb{R}^n) \mid W \cap K = \emptyset\}.$$ 

In particular, this is meant to capture the notion of being “close to infinity”. This topology is meant to feel a bit strange. For example, if $\{K\}$ is a family of compact subspaces of $\mathbb{R}^n$, the map

$$f : \mathbb{R} \to \Psi_k(\mathbb{R}^n)$$

given by $f(t) = \{1/t\} \times \mathbb{R}^2$ for $t \neq 0$ and $f(t) = \emptyset$ when $t = 0$ is a continuous map, given the topology on $\Psi_k(\mathbb{R}^n)$.

**Definition 11.3.** Let

$$B_n := \{W \in \Psi_k(\mathbb{R}^n) \mid W \subset (0,1)^n\}.$$ 

In particular, if $X$ is a $k$-manifold and $f : X \to B_n$, then the graph of $f$,

$$\Gamma_f \subset X \times (0,1)^n$$

has the projection $\Gamma_f \to X$ as a fibre bundle with fibre a compact surface of a fixed diffeomorphism type.

On the other hand, if $f : E \to X$ is a fibre bundle with fibre a surface of a fixed diffeomorphism type, then $E$ is a $k+2$ manifold and we can use the Whitney embedding theorem to have an embedding

$$j : E \to X \times (0,1)^n$$

for $n > 2(k+2) = 2k + 4$.

Thus we can define $f : X \to B_n$ by

$$f(x) = j(E_X) = \{x\} \times i(F_X).$$

Homotopies of maps $X \to B_n$ correspond to isomorphism classes of surface bundles.

**Proposition 11.4.** Let $X$ be a $k$-dimensional manifold. $[X,B_n]$ is isomorphic to isomorphism classes of surface bundles over $X$ whenever $n > 2k + 4$.

If we let $n \to \infty$, we have the following.

**Proposition 11.5.** $B_\infty$ classifies surface bundles over $X$.

Not that we have a homeomorphism

$$B_n = \bigsqcup_{g \geq 0} \text{Emb}(S_g,(0,1)^n)/\text{Diff}(S_g).$$

If we let $n \to \infty$, the Whitney embedding theorem tells us that $\text{Emb}(S_g,(0,1)^\infty) \cong_{w.e.}^\infty$. The idea is that

$$\pi_n(\text{Emb}(S_g,(0,1)^\infty)) \cong \pi_0(\text{Emb}(S_g \times S^k,(0,1)^\infty)).$$

So $B_n = B\text{Diff}(S_g)$ and $B_\infty = \bigsqcup_{g \geq 0} B\text{Diff}(S_g)$.

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11.2 Defining the maps

Let \( \emptyset \) be the basepoint of \( B_n \). Let \( S^n = \mathbb{R}^n \cup \{ \infty \} \). Define

\[ \alpha : B_n \rightarrow \Omega^n \Psi_2(\mathbb{R}^n) \]

by

\[ \alpha(W)(v) = \begin{cases} W + v, & \text{if } v \in \mathbb{R}^n \\ \emptyset, & \text{if } v = \infty. \end{cases} \]

This gives a map

\[ B_\infty \rightarrow \text{colim}_{n \rightarrow \infty} \Omega^n \Psi_2(\mathbb{R}^n) =: \Omega^\infty \Psi. \]

Madsen-Weiss says that \( H^*(\alpha) \) is an isomorphism for \( \alpha \) restricted to a specific diffeomorphism type \( S_g \) when \( * \leq \frac{3}{4} g \).

For the grassmannian \( Gr_2(\mathbb{R}^n) \), we have the canonical bundle

\[ \gamma_n \rightarrow Gr_2(\mathbb{R}^n). \]

We also have

\[ \gamma_n^\perp \rightarrow Gr_2(\mathbb{R}^n), \]

where \( \gamma_n^\perp \) consists of \( (V, w) \), where \( V \in Gr_2(\mathbb{R}^n) \) is a 2-plane and \( w \in V^\perp \).

\( \gamma_n^\perp \) has Thom Spaces \( Th(\gamma_n^\perp) \). We can define a map \( q : Th(\gamma_n^\perp) \rightarrow \Psi_2(\mathbb{R}^n) \) by

\[ q(V, w) = \begin{cases} V + w, & \text{if } (v, w) = \infty \\ \emptyset, & \text{if } (V, w) = \infty. \end{cases} \]

Taking the \( \Omega^\infty \) functor, we get

\[ Q : \Omega^\infty MTSO(2) \rightarrow \Omega^\infty \Psi, \]

where

\[ \Omega^\infty MTSO(2) := \text{colim}_{n \rightarrow \infty} \Omega^\infty Th(\gamma_n^\perp) \]

and

\[ \Omega^\infty \Psi := \text{colim}_{n \rightarrow \infty} \Omega^n \Psi_2(\mathbb{R}^n). \]

\( Q \) is the first of the maps we wanted to defined at the start of the section.

**Proposition 11.6.** \( Q \) is a weak equivalence.

11.3 \( \alpha_n \) is a homology equivalence

Last time, we looked at the maps

\[ \alpha_n : B_n \rightarrow \Omega^n \Psi(\mathbb{R}^n) \]

where \( B_n \) was the space of 2-manifolds in \((0, 1)^n\) and \( \Psi(\mathbb{R}^n) \) was the space of topologically closed 2-manifolds in \( \mathbb{R}^n \). \( (k = 2 \text{ if we look back at our original definition}) \). The map \( \alpha_n \) was given by the formula

\[ \alpha(W)(v) = \begin{cases} W + v, & \text{if } v \in \mathbb{R}^n \\ \emptyset, & \text{if } v = \infty, \end{cases} \]

where \( S^n = \mathbb{R}^n \cup \{ \infty \} \).

We would like to show that \( \alpha_n \) is a homology equivalence. Our strategy will be to decompose \( \alpha_n \) by looking at one direction at a time.
Definition 11.7. Define
\[ \Psi(n, k) := \{ W \in \Psi(\mathbb{R}^n) \mid W \subset \mathbb{R}^k \times (0,1)^{n-k} \} \].

Define \( \tilde{\alpha}_i : \Psi(n, i) \to \Omega \Psi(n, i + 1) \), where
\[ \tilde{\alpha}(W)(t) = \begin{cases} W + te_{i+1}, & \text{if } t \in \mathbb{R} \\ \emptyset, & \text{if } t = \infty, \end{cases} \]
where \( S^1 = \mathbb{R} \cup \{ \infty \} \).

We can decompose \( \alpha_n \) into the following sequence.
\[ B_n = \Psi(n, 0) \xrightarrow{\tilde{\alpha}_0} \Omega \Psi(n, 1) \xrightarrow{\Omega \tilde{\alpha}_1} \Omega^2 \Psi(n, 2) \rightarrow \ldots \xrightarrow{\Omega^{n-1} \tilde{\alpha}_{n-1}} \Omega^n \Psi(n, n) = \Omega^n \Psi(\mathbb{R}^n) \]

We want to show
1. \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \) are weak equivalences. This almost uses the group completion theorem.
2. \( \tilde{\alpha}_0 \) is a homology equivalence in a range. This uses the group completion theorem and Harer stability.

Firstly, we would like to describe another model for the classifying space of a monoid.

11.4 Another way to look at classifying spaces

For \( M \) a topological monoid, we can think of \( BM \) as pairs \((A, f)\) where \( A \subset \mathbb{R} \) is a finite subset, and \( f : A \to M \).

We allow the \( a_i \) and \( a_j \) to collide, giving rise to a multiplication \( f(a_i), f(a_j) \in BM \). [I don’t quite understand this interpretation]

We will also be using the following theorem.

Theorem 11.8. For \( M \) a topological monoid, the map Define
\[ \beta : M \rightarrow \Omega BM, \]
by
\[ \beta(m)(t) = \begin{cases} \{ t \}, t \mapsto m & \text{if } t \in \mathbb{R} \\ \emptyset, & \text{if } t = \infty. \end{cases} \]
Then \( \beta \) is a weak equivalence if and only if \( M \) has a homotopy unit, and \( \pi_0(M) \) is a group.

11.5 The monoid

For \( i \geq 1 \), we now want to define a monoid \( M \) that fits into the diagram
\[ \begin{array}{ccc} M & \xrightarrow{\beta} & \Omega BM \\ \simeq_w & & \simeq_w & \downarrow^{\simeq_w} \\ \Psi(n, i) & \xrightarrow{\tilde{\alpha}_i} & \Omega \Psi(n, i + 1), \end{array} \]

since if we can show that \( \beta \) is a weak equivalence using the previous theorem, this we give that \( \tilde{\alpha}_i \) is a weak equivalence, which is what we wanted to show.

Here is the monoid \( M \).
Definition 11.9.

\[ M := \{(t, W) \in (0, \infty) \times \Psi(n, k + 1) \mid W \subset \mathbb{R}^k \times (0, t) \times (0, 1)^{n-k-1}\}. \]

Define the multiplication in \( M \) to be

\[(t, W), (t', W') = (t + t', W \cup (W' + te_{k+1})).\]

The way to think about \( M \) is that \( M \) is a space of subsets with one component that can extend off arbitrarily far (up to \( t \)). This, multiplication is to put the two subspaces next to each other, where the \( +te_{k+1} \) term is to make the two subspaces disjoint.

There is a map \( \Psi(n, i) \to M \) defined by \( M \mapsto (1, M) \) which is a homeomorphism onto its image. There is also a map \( BM \to \Psi(n, k + 1) \) defined by putting the labels on points in \( \mathbb{R} \) next to each other on the \((k + 1)\)st coordinate. Looping this map gives a map from \( \Omega BM \to \Omega \Psi(n, k + 1) \). The also claim (without proof - is there a quick reason why???) that this map is a weak equivalence.

We now want to show that \( M \) satisfies the monoid conditions so that \( \beta : M \to \Omega BM \) is a weak equivalence. In particular we need the following theorem.

**Theorem 11.10.** \( M \) is grouplike. That is \( M \) has a homotopy unit and \( \pi_0(M) \) is a group.

**Proof.** The homotopy unit of \( M \) is \( \emptyset \). We want to show that \( mm' \) and \( m'm \) are in the same path component as \( \emptyset \). (I think we mean, for \( m \in M \), there exists \( m' \in M \) such that \( mm' \) and \( m'm \) are in the same path component.)

[INSERT PICTURE PROOF - \( m' \) is a copy of \( m \) but upside down. Then do some pushing of the pictures]

What’s left to show is that \( o_0 \) is a homology equivalence. We cannot do the previous trick since things in the \( \Psi(n, 0) \) do not stretch to infinity.

**Definition 11.11.** Let \( L_t = [0, t] \times [0, 1] \subset \mathbb{R}^2 \). Let \( M'_n \) consist of pairs \( (t, W) \) where \( t \in (0, \infty) \) and \( W \subset L_t \times (-1, 1)^{n-2} \) such that \( W \) agrees with \( L_t \times \{0\} \) near \( \partial L_t \times \mathbb{R}^{n-2} \). We can define multiplication on \( M'_n \) by

\[(t, W), (t', W') = (t + t', W \cup \{W' + te_1\}).\]

Recall that \( B_{\infty} \) classifies surface bundles. \( M'_{\infty} \) classifies surface bundles with one boundary component. So

\[ M'_{\infty} \simeq \bigsqcup_{g > 0} BDiff(\Sigma_{g, 1}, \emptyset). \]

As before, we want to construct a diagram

\[
\begin{array}{ccc}
M'_n & \xrightarrow{\beta} & \Omega BM'_n \\
\approx_{w,e} & & \approx_{w,e} \\
\Psi(n, 0) & \xrightarrow{\tilde{o}_0} & \Omega \Psi(n, 1).
\end{array}
\]

Due to time, we did not say what the maps were.

We now want to study the map \( \beta \). Since \( M'_n \) is not group like. eg. \( \pi_0(M'_n) = \mathbb{N} \) when \( n \) is sufficiently large (\( \geq 5 \)).

But we can apply the group completion theorem using

\[ M \xrightarrow{m_0} M, \]

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where \( m_0 \) is multiplying by a surface of genus 1. In the group completion theorem, we consider the telescope of

\[
M'_\infty \xrightarrow{m_0} \ldots
\]

In our case, \( M'_\infty = \bigsqcup_{g \geq 0} \text{BDiff}(\Sigma_{g,1}, \partial) \). By the group completion theorem, we have

\[
H_*(\text{Tel}M'_\infty) = H_*(M'_\infty) \left[ m_0^{-1} \right] = H_*(\mathbb{Z} \times \text{BDiff}_\infty),
\]

where \( \text{BDiff}_\infty \) is the limit \( \text{BDiff}(\Sigma_{g,1}) \to \text{BDiff}(\Sigma_{g+1}, 1) \to \ldots \) and \( H_*(M'_\infty) = \bigoplus_{g \geq 0} H_*(\text{BDiff}(\Sigma_{g,1})) \).

For \( g \to \infty \), we have isomorphisms. But Harer stability gives that this is an isomorphism for \( \star \leq \frac{3}{2}g \).

12 Algebraic K-Theory: Introduction and examples

Let \( R \) be a ring. An \( R \)-module \( M \) is projective if it satisfies the following universal lifting property: Given a diagram of \( R \)-modules and \( R \)-module homomorphisms as follows,

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{F} & \mathcal{M}
\end{array}
\]

there exists an \( \mathcal{F} \) making the diagram commute. Equivalently, \( M \) is a summand of a free module (exercise).

Let

\[
P(R) := \{ \text{finitely generate projective } R\text{-modules} \}/\text{iso}.
\]

Note that \( P(R) \) is a commutative monoid. The unit is 0 and addition is \( \oplus \).

**Definition 12.1.** Let \( M \) be a commutative monoide. The group completion or Grothendieck group of \( M \) is

\[
\text{Groth}(M) := M \times M/\sim
\]

where \( (a, b) \sim (a + c, b + c) \) for all \( a, b, c \in M \).

**Definition 12.2.** The K-theory of a ring is

\[
K(R) = K_0(R) := \text{Groth}(P(R)).
\]

Here are some examples of why one might care about K-theory. The idea is that we can study \( R \) via studying its category of \( R \)-modules.

1. Let \( R \) be a field. An \( R \)-module is a vector space which is free and thus is projective. So

\[
P(R) = \{ \text{finite dim } R\text{-vector spaces} \}/\cong.
\]

As a monoid under \( \oplus \), \( P(R) \cong \mathbb{Z}_{\geq 0} \) by looking at the dimension. We thus have

\[
K(R) = \mathbb{Z}.
\]

2. Let \( R \) be a Dedekind domain, i.e. every ideal of \( R \) admits a unique factorisation into a product of prime ideals. There is a theorem by Steinitz (?) that states the following.

**Theorem 12.3.** Every finitely generated projective \( R \)-module is isomorphic to \( R^n \oplus I \) where \( I \) is a fractional ideal, i.e. \( I \) is an \( R \)-submodule of the field of fractions of \( R \) such that there exists \( \alpha \in R \) such that \( \alpha I \subset R \).
Thus
\[ P(R) \cong \mathbb{Z}_{\geq 0} \oplus \{ \text{fraction ideals} \} \]
and
\[ K(R) = \mathbb{Z} \oplus Cl(R), \]
where \( Cl(R) \) is the ideal class group, or sometimes called the Picard group \( Pic(R) \). Fact: \( Cl(R) = 0 \iff R \) admits a unique factorisation of elements.

3. If \( G \) is a discrete group, then consider \( K_0(\mathbb{Z}[G]) \), where \( \mathbb{Z}[G] \) is the group ring. There is a map
\[ \mathbb{Z} = K_0(\mathbb{Z}) \rightarrow K_0(\mathbb{Z}[G]) \]
induced by the map \( \mathbb{Z} \rightarrow \mathbb{Z}[G] \).

Aside: If \( R \rightarrow S \) is a ring homomorphism, the map \( R\text{-mod} \rightarrow S\text{-mod} \) that send \( M \mapsto M \otimes_R S \) defines a map from \( P(R) \rightarrow P(S) \). This then induces a map \( K(R) \rightarrow K(S) \), which is what we are using above.

**Definition 12.4.** The Whitehead group of \( G \) is
\[ Wh_0(G) := K_0(\mathbb{Z}[G]) / \mathbb{Z}. \]

**Definition 12.5.** A topological space \( X \) is finitely dominated if there exists a finite CW complex \( K \) and a map \( f : K \rightarrow X \) which admits right homotopy inverse, i.e., there exists a \( h : X \rightarrow K \) such that \( fh \simeq id_X \).

**Theorem 12.6** (Wall). Assume \( X \) is finitely dominated. Then there exists an invariant \( w(X) \in Wh_0(\pi_1 X) \) which is \( 0 \iff X \) is homotopy equivalent to a finite CW complex.

While example 1 was rather simple, examples 2 and 3 are two good reasons as to why we might care about \( K \)-theory. As another wishy washy example, orders and ranks of \( K\text{-theory} \) should also be related to special values of \( L\)-functions. For example, if \( R \) is the ring of integers of a number field, then the Dedekind zeta functions is
\[ \zeta_R(S) = \sum_{I \subseteq R} \frac{1}{\text{nm}(I)^S}. \]
Then the analytic class number formula relates the residues of \( \zeta \) with the size of the \( Cl(R) \).

**12.1 K\text{-theory of a symmetric monoidal category**

Let \( C \) be a small symmetric monoidal category, i.e., there is a map \( \otimes : C \times C \rightarrow C \) satisfying . . .

Let \( BC \) be the classifying space of \( C \). \( B \) is a functor and \( B(C \times D) \cong B(C) \times B(D) \). Thus we get a map
\[ B(\otimes) : BC \times BC \rightarrow BC \]

One might be worried that this might not be a strictly unital, associative multiplication, but it turns out to be \( E_\infty \) [WHATEVER THAT MEANS]

Define
\[ K(C) = \Omega B(BC), \]
where the second \( B \) is the classifying space of \( BC \) with respect to the monoidal structure defined above.

**Definition 12.7.** The algebraic \( K\text{-theory groups of } C \) are
\[ K_i(C) := \pi_i(K(C)). \]

Here are some example.
1. Let $\mathcal{C}$ be the category with

$$\text{Ob}(\mathcal{C}) = \text{finitely generated projective } R\text{-modules}$$

$$\text{Mor}(\mathcal{C}) = \text{isomorphisms of } R\text{-modules}$$

Then

$$K_0(\mathcal{C}) = \pi_0(\Omega B(\mathcal{C}))$$

$$= \pi_0(\mathcal{C})[\pi_0^{-1}] \quad \text{via group completion theorem}$$

$$= \text{Groth}(P(R))$$

Which is the $K$-theory of a ring.

2. Let $\mathcal{C}$ be the category with

$$\text{Ob}(\mathcal{C}) = \text{vector bundles over } X$$

$$\text{Mor}(\mathcal{C}) = \text{isomorphisms of vector bundles over } X$$

Then similarly to before,

$$K_0(\mathcal{C}) = \text{Groth}(\text{Ob}(\mathcal{C})/ \cong)$$

$$= \text{Groth}(\text{Vect}(X))$$

$$=: K(X)$$

which is topological $K$-theory.

As a slightly more in depth example, Let $X$ be a point. Then in this case, $\mathcal{C} = \{ \text{C-vector spaces, isos} \} = \{ 0, \mathbb{C}, \mathbb{C}^2 \ldots \}$. The morphisms are $\mathbb{C}^n \to \mathbb{C}^n$ given by an element of $GL_n(\mathbb{C})$. Thus

$$BC = * \sqcup BGL_1\mathbb{C} \sqcup BGL_2\mathbb{C} \sqcup \ldots$$

So

$$\Omega B(BC) = BGL_\infty \mathbb{C} \times \mathbb{Z}$$

Here again, get the last line from the group completion theorem since

$$H_*(\Omega B(BC)) = H_*(BC)[\pi_0^{-1}]$$

$$= \mathbb{Z}[\mathbb{Z}] \otimes \lim H_*(BGL_n \mathbb{C})$$

$$= \mathbb{Z}[\mathbb{Z}] \otimes H_*(BGL_\infty \mathbb{C})$$

$$= H_*(\mathbb{Z} \times BGL_\infty \mathbb{C}).$$

Lastly, we can get a homotopy equivalence using the Whitehead theorem.

3. Let $\mathcal{C}$ be the category with

$$\text{Ob}(\mathcal{C}) = \text{closed, smooth, oriented topological surfaces}$$

$$\text{Mor}(\mathcal{C}) = \text{diffeomorphisms}$$

We have

$$\mathcal{C} \cong \{ \Sigma_g | g = 0, 1 \ldots \}.$$ 

So

$$BC = \bigsqcup_{g=0}^\infty B\text{Diff}(\Sigma_g)$$

and

$$K_i(\mathcal{C}) = \pi_i \Omega B(BC) = \pi_i(\Omega^\infty MTSO(2)).$$
4. Let $K$ be a field. A *central simple algebra* (CSA) over $K$ is an associative, finite dimensional $K$-algebra $A$ which is simple, with centre $K$. Eg. if $K = \mathbb{R}$, then $\mathbb{R}, \mathbb{H}$ and $M_n(\mathbb{R})$ are CSAs, but $\mathbb{C}$ is not a CSA.

Let $C$ be the category with

- $\text{Ob}(C) = \text{CSAs over } K$
- $\text{Mor}(C) = \text{isomorphisms}$

This is symmetric monoidal since $A \otimes_K B$ is central simple if $A$ and $B$ are. The *Brauer group* of $K$ is

$$\text{Br}(K) := K_0C = \pi_0\Omega B(BC).$$

13 *$K$-theory of general linear groups over a finite field*

This section describes Quillen’s paper [5]. We have also used Daniela Egas Santander’s master’s project [6] as a guide.

Here is the setup: Let $q = p^d$ for some prime $p$. We will show that for $j \geq 0$,

$$K_{2j-1}(\mathbb{F}_q) = \mathbb{Z}/q^{2j} - 1$$
$$K_{2j}(\mathbb{F}_q) = 0$$

Combining this with our previous calculation of $K_0(\mathbb{F}) = \mathbb{Z}$ for any field $\mathbb{F}$, this means we have a complete description of the $K$-theory of finite fields.

In order to do this, we will define $K$-theory in terms of Quillen’s plus construction

$$K_i(R) := \pi_i(BGL(R)^+).$$

We will then define a space $F\psi^q$, whose homotopy groups we can compute, and show that

$$F\psi^q \xrightarrow{h.c.} BGL(\mathbb{F}_q)^+.$$

13.1 *Topological $K$-theory*

1. $K^0(X) = K(X) = [X, BU \times \mathbb{Z}]$.
2. $K^1(X) = [X, U]$.
3. $\tilde{K}(X) = [X, BU]$ for $X$ compact and connected.

[WHAT COMMENTS SHOULD I ADD?]

13.2 *Adams operations*

**Proposition 13.1.** Let $B$ be a compact, Hausdorff space. Then for all $l \geq 0$, there exists ring homomorphisms $\psi^k : K(B) \rightarrow K(B)$ (called Adam’s operations) with the following properties.

1. If $f : B \rightarrow B'$ induces $f^* : K(B') \rightarrow K(B)$ then $\psi^k f^* = f^* \psi^k : K(B') \rightarrow K(B)$;
2. If $L$ a line bundle, then $\psi^k(L) = L^k$;
3. $\psi^k \circ \psi^l = \psi^{kl}$

**Proof.** Omitted.  

\[\square\]
Example 13.2. $\tilde{K}(S^2) \cong \mathbb{Z}$ is generated by $H - 1$, with $(H - 1)^2 = 0$, where $H$ is the canonical line bundle over $\mathbb{C}P^1$ and $1$ is the trivial bundle over $S^2$. Then $\psi^k : \tilde{K}(S^2) \to \tilde{K}(S^2)$ is a map $\mathbb{Z} \to \mathbb{Z}$. Then

$$\psi^k (H - 1) = H^k - 1 = (H - 1 + 1)^k - 1 = k(H - 1),$$

using the fact that $(H - 1)^2 = 0$. Thus, $\psi^k$ is multiplication by $k$.

More generally, we can use induction and the fact (didn’t prove this) that there is an isomorphism $\tilde{K}(S^2) \otimes \tilde{K}(S^2) \to \tilde{K}(S^2)$, we have

$$\psi^k (\alpha \beta) = \psi^k (\alpha) \psi^k (\beta) = (k \alpha)(k^{-1} \beta) = k^n \alpha \beta.$$ 

Thus $\psi^k : \tilde{K}(S^2) \to \tilde{K}(S^2)$ is multiplication by $k^n$.

For a more detailed account of this example, see Hatcher’s Vector Bundles and $K$-theory book in progress.

13.3 Fixed points and homotopy fixed points

We fix define fixed points in terms of a pullback diagram. We do this so that when we define homotopy fixed points, they will look analogous.

Let $\Delta : X \to X \times X$ be given by $(x \mapsto (x, x))$. And let $\phi : X \to X$ be any self map.

Definition 13.3. The fixed points of $\phi$ is the pullback $X^\phi$ of the following diagram.

$$\xymatrix{ X^\phi \ar[r] \ar[d] & X \ar[d]^\Delta \\ X \ar[r]_{(id, \phi)} & X \times X.}$$

In particular, $X^\phi = \{(x, y) \mid \Delta(x) = (y, \phi(y))\}$, which agrees with our intuition that a fixed point is a point such that $\phi(x) = x$.

Now let $I = [0, 1]$ and let $\tilde{\Delta} : X \times X \to X \times X$ be given by $\gamma \mapsto (\gamma(0), \gamma(1))$. That is, $\tilde{\Delta}$ takes a path to its endpoints. Again, let $\varphi : X \to X$ be any self map.

Definition 13.4. The homotopy fixed points of $\varphi$ is the pullback $X^{h\varphi}$ of the following diagram.

$$\xymatrix{ X^{h\varphi} \ar[r] \ar[d] & X^I \ar[d]^\tilde{\Delta} \\ X \ar[r]_{(id, \varphi)} & X \times X.}$$

Note that $\tilde{\Delta} : X^I \to X \times X$ is a fibration, and the fibre is $\Omega X$.

Definition 13.5. We say that $(X, x)$ has additive structure if it has maps

1. (addition) $+$ : $X \times X \to X$, that is associative and commutative up to homotopy.
2. (additive inverse) $\epsilon : X \to X$, which is an additive inverse up to homotopy.

The map $d : X \times X \xrightarrow{(1, \epsilon)} X \times X \xrightarrow{+} X$ is called subtraction.

Lemma 13.6. If $X$ has an additive structure, then $X^{h\varphi}$ is the homotopy fibre of $1 - \varphi$.
Proof. We have the following commutative diagram.

\[
\begin{array}{ccc}
X^{h\varphi} & \to & X^I \\
\downarrow & & \downarrow \\
X & \to & X \times X
\end{array}
\]

\[
\begin{array}{ccc}
& & PX \\
\downarrow & & \downarrow \\
X & \to & X \\
\end{array}
\]

The map \(\eta : PX \to X\) is the map in the path space loop space fibration, that sends a path to its endpoint. The map \(\lambda : X^I \to PX\) is given by

\[
\lambda([t \mapsto w(t)]) = [t \mapsto d(w(t), w(0))]
\]

which changes the starting point of the path to the origin (up to homotopy at least). Thus \(X^{h\varphi}\) is the homotopy fibre of \(1 - q = d(1, \varphi)\) since it is the pullback of

\[
\begin{array}{ccc}
X^{h\varphi} & \to & PX \\
\downarrow & & \downarrow \\
X & \to & X
\end{array}
\]

We now list some properties of a space with additive structure, but without proof.

**Lemma 13.7.** If \(X\) has an additive structure, then so does \(X^{h\varphi}\).

**Lemma 13.8.** Let \(X\) be a space with additive structure and \(\varphi : X \to X\) be a self map. Let \(Y\) be a space such that \(Y \to \Omega X\) is nullhomotopic. Let \([Y, X]^r\) be the fixed points of \([X, Y]\) under composition with \(\varphi\). Then

\[
[Y, X^{h\varphi}] = [Y, X]^r.
\]

Recall that the Adams operations were maps \(\psi^q : BU \to BU\). Let \(F\psi^q\) be the homotopy fixed point of \(\psi^q\). That is, it is the pullback

\[
\begin{array}{ccc}
F\psi^q & \to & BU \\
\downarrow & & \downarrow \\
BU & \to & BU \times BU
\end{array}
\]

**Theorem 13.9.** For \(j \geq 0\), the homotopy groups of \(F\psi^q\) are

\[
\begin{align*}
\pi_{2j-1}(F\psi^q) &= \mathbb{Z}/(q^{2j} - 1) \\
\pi_{2j}(F\psi^q) &= 0
\end{align*}
\]

Proof. Observe that \(F\psi^q\) is the homotopy fibre of \(1 - \psi^q\). The long exact sequence in homotopy then gives

\[
\cdots \to \pi_j(BU) \xrightarrow{(1-\psi^q)} \pi_j(BU) \to \pi_{j-1}(F\psi^q) \to \pi_{j-1}(BU) \to \cdots
\]

But \(\pi_j(BU) = [S^j, BU] = K(S^j) = \begin{cases} \mathbb{Z} & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases}\)

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From our example, we know that $\psi^q$ acts on $\tilde{K}(S^2)$ by multiplication by $q$ and on $\tilde{K}(S^j)$ by multiplication by $q^j$. Thus the long exact sequence breaks off into short exact sequences of the form

$$0 \to \mathbb{Z} \xrightarrow{(q^j - 1)} \mathbb{Z} \to \pi_{2j - 1}(F\psi^q) \to 0$$

$$0 \to \pi_{2j}(F\psi^q) \to \mathbb{Z} \xrightarrow{(q^j)} \mathbb{Z} \to 0$$

from which we get the result of the theorem.

Lemma 13.10. Let $R(F_qG)$ be the representation ring. There is a map $R(F_qG) \to [BG, F\psi^q]$.

Proof. For $M$ a representation of $G$, there is a map

$$EG \times_G M \to EG \times_G \{pt\} = BG$$

ie. $EG \times_G M$ is a complex vector bundle over $BG$. Thus, given a complex representation of $G$, we get an element of $K(BG)$.

Consider the following commutative diagram.

$$
\begin{array}{ccc}
R(CG) & \longrightarrow & \tilde{K}^0(BG) \\
\psi^q \downarrow & & \downarrow \psi^q \\
R(CG) & \longrightarrow & \tilde{K}^0(BG).
\end{array}
$$

Here, the left map $\psi^q : R(CG) \to R(CG)$ is an Adams operations defined on representation rings, which are analogous to the Adams operations we have been looking at.

Assuming this, then a map $R(CG) \to \tilde{K}^0(BG) = [BG, BU]$ gives a map $R(CG)^{\psi^q} \to [BG, BU]^{\psi^k}$.

The Atiyah-Segal completion theorem states that

$$0 \simeq K^1(BG) = [BG, U] \simeq [BG, \Omega BU].$$

By Lemma 13.8, we have that

$$[BG, BU]^{\psi^q} \simeq [BG, BU^{h\psi^q}].$$

Then the Brauer lift [a fact we’ve taken without explaining], gives $R(F_qG) \simeq R(CG)^{\psi^q}$. Altogether we have maps

$$R(F_qG) \simeq R(CG)^{\psi^q} \to [BG, BU]^{\psi^q} \to [BG, F\psi^q].$$

If we consider modular representations of $GL_{n+1}(F_q)$, this gives us a map

$$BGL_{n+1}(F_q) \to F\psi^q.$$ Composing with $BGL_n \to BGL_{n+1}$ and taking a limit gives a map

$$\theta : BGL(F_q) \to F\psi^q$$

that is well defined up to homotopy.

Theorem 13.11. $\theta$ induces a homology isomorphism.

Proof. Omitted. (This is a big theorem)
13.4 Quillen Plus construction

The idea of the Quillen Plus construction is to define a new space that changes the fundamental group, but not homology.

**Theorem 13.12.** Let $X$ be a CW-complex and $H$ be a normal subgroup of $\pi_1 X$. i.e. $[H,H] = H$. Then there is a space $X^+$ with an inclusion

$$i : X \to X^+$$

satisfying the following properties.

1. $X^+$ is obtained from $X$ by attaching 2 and 3 cells.
2. $i_* : \pi_1(C) \to \pi_1(X^+)$ is surjective with kernel $H$.
3. Let $f : (X,x_0) \to (Z,z_0)$ be a map of connected spaces with $\ker(f_*) = H$. Then there is a map $f' : (X^+,x_0) \to (Z,z_0)$ such that the following diagram commutes.

```
X+ \quad f' \downarrow \\
\quad X \quad f \\
\quad \downarrow \\
\quad Z
```

4. $H_1(X) = H_n(X^+)$ for all $n$.

*Proof.* Omitted.

**Theorem 13.13.** $BGL^+$ and $F\psi^q$ are homotopy equivalent.

*Proof.* Omitted.

13.5 More about the plus construction

This section aims to clear up the relationship between the plus construction and group completion. I think at some point, this section would be better if it were amalgamated into the previous section.

The overall moral of this story is that their is a way to constuct the plus construction - and just like with singular homology, we really don’t care. More important are the properties this space has, which is the main focus in the following few pages.

Let $X$ be a connected topological space. Let $G \leq \pi_1 X$ be a normal, perfect\(^2\) subgroup. The plus construction $X^+$ (with respect to $G$) is a space which is obtained from $X$ by attaching 2-cells to kill $G \leq \pi_1 X$, and 3-cells so that the inclusion

$$H \hookrightarrow X^+$$

is a homology isomorphism.

Note that we can do the first bit (attach 2-cells) since $\pi_1 X$ is normal. In particular,

$$H_1(X) = (\pi_1 X)^{ab} = \frac{\pi_1 X}{[\pi_1 X, \pi_1 X]}$$

Since we kill off $G$ in $X^+$, this means

$$H_1(X^+) = (\pi_1 X^+)^{ab} = (\pi_1 X/G)^{ab} = \frac{\pi_1 X/G}{[\pi_1 X/G, \pi_1 X/G]}.$$  

Since $G = [G : G] \leq \pi_1 X, \pi_1 X$, we see that

$$H_1(X) \cong H_1(X^+).$$

Perfection can be used to show that $X \hookrightarrow X^+$ is a homology isomorphism.

\(^2\) $G = [G,G]$
Proposition 13.14. If $R$ is a ring and $s \in R$, then
\[ R[s^{-1}] \cong \lim_{\to} (R \xrightarrow{\cdot s} R \xrightarrow{\cdot s} R \to \ldots) \]

Proof. The map from the direct system to $R[s^{-1}]$ is as in the diagram below.

Now given any $A$ and maps $f_n$ that make the following diagram commute,

we can define $f : R[s^{-1}] \to A$ by $f(r/s^k) = f_k(r)$ which will ensure that $R[s^{-1}]$ satisfies the universal property of direct limits.

Now let $M$ be a topological monoid with $\pi_0 M = \bigoplus N$. Our main example is $M = \bigsqcup_k BGL_k(R)$, from which we get $K$-theory defined as $K_i R = \pi_i \Omega \Omega \Omega \Omega \Omega (\bigwedge_k R)$.

Remark: $\pi_0 (\Omega \Omega \Omega \Omega \Omega (R)) \cong \mathbb{Z}$.

Question: Why is the group completion of $\pi_0 M = \pi_0 (\Omega \Omega \Omega \Omega \Omega (M))$?

Denote by $\Omega_k M$ the component of $\Omega \Omega \Omega \Omega \Omega (M)$ corresponding to $k \in \mathbb{Z}$. For example, $\Omega_0 M$ are the null homotopic loops in $BM$. There exists a natural map
\[ i : M \to \Omega \Omega \Omega \Omega \Omega (BM) \]
given by $i(m) = \text{loop in } BM$ indexed by $m^3$. This induces a map
\[ I_* : \pi_0 M \to \pi_0 \Omega \Omega \Omega \Omega \Omega (BM) = \pi_1 BM. \]

Observe that the left hand side is a monoid, while the right hand side is a group. We study $H_0(M) = \mathbb{Z}[\pi_0]$. We have
\[ \mathbb{Z}[\pi_0 \Omega \Omega \Omega \Omega \Omega (BM)] = H_0(\Omega \Omega \Omega \Omega \Omega (BM)) = H_0(M)[\pi_0^{-1}] \]
\[ = \mathbb{Z}[\pi_0 M][\pi_0^{-1}] \]
\[ = \mathbb{Z}[\text{Groth}(\pi_0 M)] \]

So $\pi_0 \Omega \Omega \Omega \Omega \Omega (BM) = \text{Groth}(\pi_0 M)$. Thus for $M$ as above, $\pi_0 \Omega \Omega \Omega \Omega \Omega (BM) = \mathbb{Z}$.

Now we have $H_*(\Omega \Omega \Omega \Omega \Omega (BM)) = H_*(M)[\pi_0^{-1}]$. Pick $s \in M_1$, where $M = \bigsqcup_{k \in \mathbb{N}} M_k$ so that $[s]$ is a generator of $\pi_0 M$. Then
\[ H_*(\Omega \Omega \Omega \Omega \Omega (BM)) = H_*(M)[\pi_0^{-1}] \]
\[ = H_*(M)[s^{-1}] \]
\[ = \lim_{\to} (H_* M \xrightarrow{\cdot s} H_* M \to \ldots) \]

\[ ^{3}\text{recall that the 1-skeleton of } BM \text{ is a wedge of loops indexed by } M \]
We also have
\[ H_*(\Omega_0BM) \cong \lim_{\leftarrow} (H_*M_1 \xrightarrow{s} H_*M_2 \xrightarrow{s} H_*M_3 \xrightarrow{s} \ldots) \]
\[ \cong H_*M_{\infty} \]
where \( M_{\infty} = \lim_{\rightarrow} M_n \) (the telescope).
For our example, we have \( \Omega_0B(\sqcup_k BGL_k(R)) \cong_{H_*} BGL_{\infty}(R) \).

**Lemma 13.15.** For all spaces \( S \), every component of \( \Omega X \) is homotopy equivalent.

**Proof.** We show that \( \pi_0(\Omega X) \simeq \pi_1 X \). Pick \( \tilde{g} \in \Omega g X \), consider the map \( \Omega_0X \to \Omega g X \)
given by \( f \mapsto f \ast \tilde{g} \).
It has a homotopy inverse given by \( h \mapsto h \ast [\tilde{g}]^{-1} \).

**Theorem 13.16.** If \( \pi_0M = \mathbb{N} \) and \( [\pi_1M_{\infty}, \pi_1M_{\infty}] \) is perfect, then
\[ \Omega BM \simeq \mathbb{Z} \times M_{\infty}^+ \]

**Proof.** We have the following diagram.

\[
\begin{array}{c}
M_1 \xrightarrow{s} M_2 \xrightarrow{s} M_3 \xrightarrow{s} \ldots \\
\downarrow i \quad \downarrow i \quad \downarrow i \\
\Omega_1BM \xrightarrow{s} \Omega_2BM \xrightarrow{s} \Omega_3BM \xrightarrow{s} \\
\end{array}
\]

This gives a map \( i : M \to \Omega BM \). Thus there is a map \( I : \lim M_n \to \lim \Omega_nBM \cong_{\text{lemma}} \Omega_0BM \).

By the previous calculation, \( I \) is a homology isomorphism. By the universal property of the plus construction, \( I \) factors as

\[
\begin{array}{c}
M_{\infty} \xrightarrow{i} M_{\infty}^+ \\
\downarrow \quad \downarrow \quad \downarrow \\
\Omega_0BM \xrightarrow{I} \Omega_0BM \\
\end{array}
\]

So we have
\[ \pi_1M_{\infty}^+ \]
\[ = \frac{\pi_1M_{\infty}}{[\pi_1M_{\infty}, \pi_1M_{\infty}]^{ab}} \]
\[ = H_1M_{\infty} \]
\[ \cong H_1\Omega_0BM \]
\[ = \pi_1(\Omega_0BM)^{ab} \]
\[ = \pi_1\Omega_0BM \]

since \( \pi_1\Omega BM = \pi_1\Omega BM = \pi_2BM \) is abelian.

So \( I \) is an isomorphism in \( H_* \) and \( \pi_1 \), so is a weak equivalence by the Whitehead theorem.

**Example:** \( K_i(R) = \pi_i[\Omega B(\sqcup_k BGL_k(R))] = \pi_i[(\mathbb{Z} \times BGL_{\infty}(R)^+) \]
14 Brauer Groups

Let \( k \) be a field, and \( \mathcal{C} \) be the category of finite dimensional central simple algebras (CSA’s) over \( k \), where the morphisms are isomorphisms. This is a symmetric monoidal category since the tensor product of a CSA is a CSA. Previously, we defined the Brauer group of \( k \) as

\[
Br(k) := K_0(\mathcal{C})
\]

This section will go into more detail on Brauer groups.

**Definition 14.1.** A central simple algebra over \( k \) is a simple \( k \)-algebra \( A \) such that the centre, \( Z(A) = k \).

A simple algebra is an algebra with no non-trivial two sided ideals. Examples of simple algebras are fields and division algebras.

**Example 14.2 (CSA’s).** Examples of central simple algebras are

1. \( H = \mathbb{R}[i,j,k]/(i^2 = j^2 = k^2 = -1, ij = -ji = k) \) is a central simple algebra over \( \mathbb{R} \);
2. \( M_n(k) \) is a central simple algebra over \( k \); and
3. If \( D \) is a division algebra over \( k \), then \( D/Z(D) \) is a central simple algebra over \( k \).

**Theorem 14.3 (Artin-Wedderburn).** The following are equivalent:

1. \( A \) is a finite dimensional central simple algebra over \( k \)
2. There is a division algebra \( D \) over \( k \) such that \( A \cong M_n(D) \cong D \otimes_k M_n(k) \)

**Sketch of proof.** We use Schur’s Lemma which states

**Lemma 14.4 (Schur’s Lemma).** If \( M \) is a simple \( A \)-module, then \( \text{End}_A M \) is a division ring.

We also use a result due to Rieffel.

**Lemma 14.5.** If \( A \) is a finite dimensional algebra over \( k \), and \( L < A \) is a non-zero left ideal, then the following holds: If \( D = \text{End}_A L \), then \( A \cong \text{End}_D L \).

We now proceed to prove the Artin-Wedderburn theorem.

If \( A \) is finite dimensional over \( k \), then \( A \) is Artinian. This means there exists a minimal non-trivial left ideal \( L < A \). Thus \( L \) is simple. By Schur’s lemma, \( D := \text{End}_A L \) is a division ring. By Rieffel, \( A \cong \text{End}_D L \), and the latter is the matrix ring \( M_n(D) \).

14.1 Some computations

**Proposition 14.6.** For \( \overline{k} \) an algebraically closed field, \( Br(\overline{k}) = 0 \).

**Proof.** By Artin-Wedderburn, if \( A \) is a central simple algebra over \( \overline{k} \), then \( A \cong M_n(D) \), for some (finite dimensional) division algebra \( D \). We will show that \( D = \overline{k} \), so that \( A \cong M_n(\overline{k}) \), which means the class of \( A \) is zero in \( Br(\overline{k}) \).

Let \( D \) be a finite dimensional division algebra over \( \overline{k} \). Pick \( d \in D \setminus \{0\} \). Then for some \( n \), the set \( \{1, d, d^2, \ldots, d^n\} \) is becomes linearly dependent. Thus there exists a polynomial \( f \in \overline{k}[X] \setminus 0 \) such that \( f(d) = 0 \). Thus, there are \( a, b \in \overline{k} \) such that \( ad + b = 0 \) and so \( d \in \overline{k} \). Thus \( D \subset \overline{k} \). To see that \( \overline{k} \subset D \), note that \( 1 \in D \) and so \( \overline{k} = \overline{k}-\text{span}(1) \subset D \).

**Proposition 14.7.** If \( A \) is a central simple algebra over \( k \) and \( B \) is a simple algebra over \( k \), then

1. The center \( Z(A \otimes_k B) = Z(B) \); and

---

*4A quick sketch: If \( f : M \to M \) then \( \text{ker}(f) \) is a submodule of \( M \), which means that \( \text{ker}(f) = 0 \)
2. $A \otimes_k B$ is simple; and

Proof. 1. Let $z \in Z(A \otimes_k B)$. It is enough to show that $z \in 1 \otimes_k B$, since this will mean the map $b \mapsto 1 \otimes_k b$ is a bijection between the two centres.

Let $\{b_i\}$ be a basis for $B$ as a $k$-module. Then $\{1 \otimes_k b_i\}$ is a basis for $A \otimes_k B$ as an $A$ module. Thus we can write $z$ uniquely as

$$z = a_1 \otimes b_1 + \cdots + a_n \otimes b_n$$

Now,

$$aa_1 \otimes b_1 + \ldots + aa_n \otimes b_n = (a \otimes_k 1)z$$

$$= z(a \otimes_k 1) = a_1 a \otimes_k b_1 + \ldots + a_n a \otimes_k b_n$$

So, if $a \in A$, then $aa_i = a_i a$. Thus $a_i \in Z(A) = k$. Thus

$$z = 1 \otimes a_1 b_1 + \ldots + 1 \otimes a_n b_n = 1 \otimes_k b$$

2. Let $I$ be a non-trivial 2-sided ideal of $A \otimes_k B$. We want to show that $I = A \otimes_k B$. Since clearly $I \subset A \otimes_k B$, we need to show that $A \otimes_k B \subset I$.

We want to find $z \in I$ such that

$$z = 1 \otimes_k b \in I.$$ 

Then $(1 \otimes_k B)(1 \otimes_k b)(1 \otimes_k B) = 1 \otimes_k B B b B = 1 \otimes_k B \subset I$. So $(A \otimes 1)(1 \otimes B) \subset I$ and we would be done.

To find the element $z$, pick a basis $\{b_i\}$ as before. Pick $0 \neq \zeta \in I$, so that

$$\zeta = a_1 \otimes_k b_1 + \ldots + a_n \otimes_k b_n$$

with $n$ minimal. Since $\zeta \neq 0$ we know that $a_1 \neq 0$. We have $Aa_1 A = A$. Thus there are elements $a', a'' \in A$ so that

$$a'a_1 a'' = 1.$$ 

Then, let

$$z = (a' \otimes_k 1)\zeta(a'' \otimes_k 1) = 1 \otimes_k b_1 + a'a_2 a'' \otimes_k b_2 + \ldots + a'a_n a'' \otimes_k b_n$$

$$= 1 \otimes_k b_1 + a_2'' \otimes_k b_2 + \ldots + a_n'' \otimes_k b_n$$

is in $I$.

Now for $a \in A$, consider

$$az - za = (a a'' - a'' a) \otimes_k b_2 + \ldots + (a a'' - a'' a) \otimes_k b_n = 0$$

by minimality of $n$. Thus $aa'' = a'' a$, so $a_i \in Z(A) = k$. Thus

$$z = 1 \otimes b$$

for some $b \in B$, which is what we want. \qed

Corollary 14.8. 1. If $A$ and $B$ are central simple algebras over $k$, then $A \otimes_k B$ is a central simple algebra over $k$.

2. If $K$ is a field extension of $k$, then $K \otimes_k A$ is a central simple algebra over $K$. Moreover, we get a map $CSA's/k \rightarrow CSA/K$ by $A \mapsto K \otimes_k A$. 

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**Definition 14.9.** Let \((\mathcal{C}, \otimes)\) be a symmetric monoidal category. A full subcategory \(T\) of \(\mathcal{C}\) is cofinal if it satisfies the following conditions.

1. \((T, \otimes)\) is a symmetric monoidal category; and
2. If \(A \in \text{Ob}(\mathcal{C})\), then there exists \(B \in \mathcal{C}\) such that \(A \otimes B \in T\).

**Example 14.10.** For \(R\) a ring, we defined \(K_0(R) := \text{Groth}(P(R))\), where \(P(R)\) is the symmetric monoidal category of finite dimensional projective modules over \(R\), with isomorphisms. Recall that \(P\) is projective if and only if there exists a module \(Q\) such that \(P \oplus Q\) is a free module. Thus the free modules are cofinal in the category \(P(R)\).

**Proposition 14.11.** The matrix algebras \(\{M_n(k) \mid n \in \mathbb{Z}_{\geq 0}\}\) are cofinal in the category of central simple algebras over \(k\), with isomorphisms.

**Proof.** We check the two conditions.

1. \(M_n(k) \otimes M_m(k) \cong (M_n(M_n(k)) \cong M_{nm}(k)\). Essentially, it is the idea that an matrix of matrices is just a big matrix if we forget the parentheses inside our big matrix.

2. If \(R\) is a ring, define the ring \(R^{op}\) to be the set \(R\), with addition \(a +_{R^{op}} b = a +_R b\), and \(a \times_{R^{op}} b = b \times_R a\).

**Claim:** Let \(R\) and \(S\) be rings, with inclusions \(k \hookrightarrow R\) and \(k \hookrightarrow S\) turning them into \(k\)-algebras. Whenever we have the following commutative diagram of \(k\)-algebras such that the images of \(R\) and the images of \(S\) commute, then the dotted arrow exists, and is unique.

![Diagram](image)

In our case, Let \(A\) be a central simple algebra over \(k\). Let

\[ R = \{L_a \in \text{End}_k A \mid L_a(x) = ax\} \]

and let

\[ S = \{T_b \in \text{End}_k A \mid T_b(x) = xb\}. \]

Observe that \(R \cong A\) and \(S \cong A^{op}\) and both embed into \(\text{End}_k(A)\). We have

\[ L_a(T_b(x)) = axb = T_b(L_a(x)). \]

Thus \(L_aT_b = T_bL_a\), so the images of \(R\) and \(S\) commute so we have

![Diagram](image)

So there is a unique map

\[ A \otimes_k A^{op} \cong R \otimes_k S \to \text{End}_k(A) \cong M_n(k) \]
In particular, we have done the necessary work to show that  

\[ Br(k) = \{ \text{finite dim CSA's over } k \text{ /isomorphisms} \}/(A \otimes M_n(k) = A) \]

is a group.

We now list some Brauer groups that have been calculated.

**Example 14.12** (Wedderburn). *Every finite division algebra is a field. Hence \( Br(F_q) = 0 \).*  

[Is this because of Artin-Wedderburn?]

**Theorem 14.13** (Frobenius). *Two calculations.*

1. \( Br(R) = \{ R, H \} \cong \mathbb{Z}/2\mathbb{Z}; \) and
2. \( H \otimes_R H \cong M_n(R). \)

**Theorem 14.14.** If \( K \) is a local field, then \( Br(k) = Q/Z. \)

An example of a local field is \( Q_p. \)

### 14.2 Some Galois theory

We saw earlier that if \( A \) is a central simple algebra over \( k \), and \( K \) is a field extension of \( k \), then \( A \otimes_k K \) is a central simple algebra over \( k \). This induces a map from  

\[ Br(k) \to Br(K) \]

**Definition 14.15.** The relative Brauer group is  

\[ Br(K/k) := \ker(\text{Br}(k) \to \text{Br}(K)) \]

**Definition 14.16.** \( K \) is a splitting field of \( A \) if \( A \otimes_k K \cong M_n(K). \) We also say that \( K \) splits \( A \).

**Theorem 14.17** (Noether). *Every central simple algebra over \( k \) has a finite Galois splitting field.*

**Corollary 14.18.** \( Br(k) = \bigcup_{K \subset K^s} Br(K/k) \), where \( k^s \) is the separable closure of \( k \).  

There is a result that relates the Brauer group to Galois cohomology. It says the following.

\[ Br(K) \cong H^1(K^s/k, PGL_\infty(k^s)) \]

\[ \cong_{\text{Hilbert90}} H^2(K^s/k, (k^s)\times) \]

As a result of this, we have the following. If \( G \) is a finite group and \( G \) acts on \( A \), which is abelian, then \( H^n(G, A) \) are torsion, so \( Br(k) \) is torsion. If \( K \) is a local field,  

\[ Br(K) = H^1(G_{k^s/k}, (k^s)\times) \]

\[ \cong H^2(G_{k^{un}/k}, (k^{un})\times), \]

where \( u_m \) is the largest unramified separable extension.
15 Waldhausen $K$-theory

15.1 Exact categories

Definition 15.1 (Exact category). An exact category is a pair $(C, E)$, where $C$ is an additive category and $E$ is a family of sequences in $C$ of the form

$$0 \to B \overset{i}{\to} C \overset{j}{\to} D \to 0,$$

satisfying the following condition: there is an embedding of $C$ as a full subcategory of an abelian category $A$ so that

1. $E$ is the class of all sequences $(†)$ in $C$ which are exact in $A$;
2. $C$ is closed under extensions in $A$ in the sense that if $(†)$ is an exact sequence in $A$, with $B, D \in C$, then $C$ is isomorphic to an object in $C$.

Definition 15.2. $C$ is closed under kernels of surjections in $A$, if it satisfies the following property: If $f : B \to C$ in $C$ is a surjection in $A$, then $\ker(f) \in C$.

The condition of being closed under kernels of surjections is sometimes taken as part of the definition of exact category.

Definition 15.3 ($K_0$ of an exact category). Let $C$ be a small exact category. $K_0(C)$ is the abelian group with one generator $[C]$ for each object $C \in C$, and relations $[C] = [B] + [D]$ for every short exact sequence

$$0 \to B \to C \to D \to 0$$
in $C$.

Example 15.4. The category $\text{P}(R)$ of finitely generated projective $R$-modules is exact since every exact sequence of projective modules splits. We have $K_0(\text{P}(R)) = K_0(R)$.

Here are some simple properties of $K_0(A)$.

1. $[0] = 0$ since $0 \to A \overset{\cong}{\to} A \to 0 \to 0$ is a short exact sequence.
2. If $A \cong A'$ then $[A] = [A']$.
3. $[A' \oplus A''] = [A'] + [A'']$

Any additive category is a symmetric monoidal category under $\oplus$. In general, by the above $K_0(C)$ is a quotient of the group $K_0^\oplus(C)$ which we defined in previous sections, but they need not be equal.

Theorem 15.5 (Localization theorem). Let $A$ be a small abelian category, and $B$ a Serre subcategory of $A$. Then the following sequence is exact:

$$K_0(B) \to K_0(A) \xrightarrow{\text{loc}} L_0(A/B) \to 0$$

Proof. Omitted. We didn’t talk about this in the lecture.

Theorem 15.6 (Resolution theorem). Let $\mathcal{P} \subset C \subset A$ be an inclusion of additive categories with $A$ abelian. Assume the following.

1. Every object $C$ of $C$ has finite $\mathcal{P}$-dimension; and
2. $C$ is closed under kernels of surjections in $A$.

Then the inclusion $\mathcal{P} \subset C$ induces and isomorphism $K_0(\mathcal{P}) \cong K_0(C)$.
Proof. We claim the map \( \psi : K_0(\mathcal{P}) \to K_0(\mathcal{C}) \) given by \([P] \mapsto [P]\) is an isomorphism.

To show surjectivity, let \([C] \in K_0(\mathcal{C})\). Let \( P_\bullet \to C \) be a finite \( P \)-resolution. The exact sequence

\[
0 \to P_n \to \ldots \to P_0 \to C \to 0
\]

has \( \chi = 0 \), where \( \chi|[C_\bullet] = \sum (-1)^i[C_i] \) (the sum is in \( K_0(\mathcal{C}) \)). In particular, \( [C] = \sum (-1)^i[p] = \chi(P_\bullet) \), so \( K_0(\mathcal{P}) \) surjects onto \( K_0(\mathcal{C}) \).

The map is injective, since the map \( \chi : K_0(\mathcal{C}) \to K_0(\mathcal{P}) \) defined by \( \chi(C) = \chi(P_\bullet) \) is a left inverse for \( \psi \).

\[\Box\]

Example 15.7. Let \( R \) be a ring. Let \( \mathbf{H}(R) \) be the category of all \( R \)-modules \( M \) with a finite projective resolution of finitely generated projective modules, and let \( \mathbf{H}_n(R) \) be the subcategory in which the resolutions have length \( \leq n \). It can be shown that these \( \mathbf{H}_n(R) \) and \( \mathbf{H}(R) \) satisfy the conditions of the resolution theorem. Thus we have

\[
K_0(R) \cong K_0 \mathbf{H}(R) \cong K_0 \mathbf{H}_n(R)
\]

for \( n \geq 1 \). In particular, \( \mathbf{H}_1(R) = \mathbf{P}(R) \). This is one reason why it is ok to define the \( K \)-theory of a ring just in terms of its finitely generated projective modules.

15.2 Waldhausen Categories

Intuitively, a Waldhausen category is a category where the notions of cofibrations and weak equivalences make sense. More precisely, we have the following definitions, from Weibel’s \( K \)-book.

Definition 15.8. Let \( \mathcal{C} \) be a category equipped with a subcategory \( \text{co} = \text{co}(\mathcal{C}) \) of morphisms in a category \( \mathcal{C} \), called “cofibrations” (indicated by \( \rightarrow \)). The pair \( (\mathcal{C}, \text{co}) \) is called a category with cofibrations if the following axioms are satisfied:

\[ W0 \] Every isomorphism in \( \mathcal{C} \) is a cofibration;

\[ W1 \] There is a distinguished zero object ‘0’ in \( \mathcal{C} \), and the unique map \( 0 \rightarrow A \) in \( \mathcal{C} \) is a cofibration for every \( A \) in \( \mathcal{C} \);

\[ W2 \] If \( A \rightarrow B \) is a cofibration, and \( A \rightarrow C \) is any morphism in \( \mathcal{C} \), then the pushout \( B \cup_A C \) of these two maps exists in \( \mathcal{C} \), and moreover the map \( C \rightarrow B \cup_A C \) is a cofibration.

Definition 15.9. A Waldhausen category is a category \( \mathcal{C} \) with cofibrations, together with a family \( w(\mathcal{C}) \) of morphisms in \( \mathcal{C} \) called “weak equivalences” (abbreviated ‘w.e’, \( \sim \)). Every isomorphism in \( \mathcal{C} \) is to be a weak equivalence, and weak equivalences are to be closed under composition. Thus \( w(\mathcal{C}) \) can be regarded as a subcategory of \( \mathcal{C} \). In addition, the following “Glueing axiom” must be satisfied:

\[ W3 \] Glueing for weak equivalences. For every commutative diagram of the form

\[
\;
\]

the induced map

\[
B \cup_A C \to B' \cup_{A'} C'
\]

is also a weak equivalence.

A Waldhausen category is a triple \( (\mathcal{C}, \text{co}, w) \), but we often just write \( \mathcal{C} \). Usually, for a given category \( \mathcal{C} \), the cofibrations are given and there is not much choice in what they are. However, there is often a choice on the type of weak equivalences \( w \) are. Thus, we sometimes write \( w\mathcal{C} \) instead of \( \mathcal{C} \) to emphasise the \( w \) we are using.
Definition 15.10. Let $C$ be a Waldhausen category. $K_0(C)$ is the abelian group with one generator $[C]$ for each object $C \in C$, with relations

1. If $C \xrightarrow{\sim} C$, then $[C] = [C']$; and
2. If $B \to C \to C/B$ is a cofibration sequence, then $[C] = [B] + [C/B]$.

Example 15.11. An exact category is a Waldhausen category, with cofibrations being the admissible monomorphisms and weak equivalences being isomorphisms. By definition, the Waldhausen definition of $K_0$ agrees with the exact category definition of $K_0$ in this case.

Example 15.12. This is an example of a Waldhausen category that is not additive. Let $R(\ast)$ be the category whose objects are CW-complexes with countably many cells and morphisms are cellular maps. This is a Waldhausen category. The cofibrations are cellular inclusions, and weak equivalences are weak homotopy equivalences.

Instead of direct sum, since this is not an additive category, the coproduct is the wedge product $A \vee B$, which is the disjoint union of $A$ and $B$ with basepoints identified. The Eilenberg Swindle shows that $K_0 R = 0$. This is because $C^\infty = C \vee C^\infty$, hence the cofibration sequence

$$C \to C \vee C^\infty \to C$$

shows that $[C] = 0$.

Example 15.13. Let $R_f(\ast) \subset R(\ast)$ be the full subcategory of finite based CW complexes. This is a Waldhausen category with the same cofibrations and weak equivalences. In this case, we have

$$K_0 R_f \cong \mathbb{Z}.$$

Proof. We have

$$S^{n-1} \to D^n \to S^n,$$

so

$$[S^{n-1}] + [S^n] = [D^n] = 0.$$

Hence, $[S^n] = (-1)^n [S^0]$. If $C$ is obtained from $B$ by attaching an $n$-cell, then $C/B \cong S^n$ and $[C] = [B] + [S^n]$. Hence $K_0 R_f$ is generated by $[S^0]$.

Moreover, there is a surjective homomorphism $\chi : K_0 R_f \to \mathbb{Z}$, given by

$$\chi(C) = \sum (-1)^i \dim \tilde{H}^i(X, \mathbb{Q}).$$

Hence $[S^0] \mapsto 1$ is an isomorphism.

Theorem 15.14 (Cofinality). Let $B$ be a Waldhausen subcategory of $C$ closed under extensions. If $B$ is cofinal in $C$, then $K_0(B)$ is a subgroup of $K_0(C)$.

Proof. The proof is the same as for exact categories. If we consider $B$ and $C$ as symmetric monoidal categories with product $\coprod$, we have $K_0^{\coprod}(B) \subset K_0^{\coprod}(C)$. Given a cofibration sequence

$$C_0 \to C_1 \to C_2$$

in $C$, choose $C_0'$ and $C_2'$ in $C$ so that $B_0 = C_0 \coprod C_0'$ and $B_2 = C_2 \coprod C_2'$ are in $B$. Setting $B_1 = C_1 \coprod C_0' \coprod C_2'$, we have a cofibration sequence

$$B_0 \to B_1 \to B_2$$

in $C$. Since $B$ is closed under extensions in $C$, $B_1 \in B$. Therefore, in $K_0^{\coprod}(C)$, we have

$$[C_1] - [C_0] - [C_2] = [B_1] - [B_0] - [B_2].$$

Thus the kernel of $K_0^{\coprod} \to K_0(C)$ is the kernel of $K_0^{\coprod}(B) \to K_0(B)$, which means $K_0(B) \to K_0(C)$ is injective.

\footnote{Here, cofinal means: for all $C \in C$, there is a $C' \in C$ so that $C \coprod C' \in B$.}
Note: The proof shows that $K_0(C)/K_0(B) \cong K^H_1(B)/K^H_1(C)$. Moreover, every element of $K_0(C)$ has the form $[C] - [B]$ for some $B \in \mathcal{B}$ and $C \in \mathcal{C}$.

**Theorem 15.15 (Approximation).** Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between two Waldhausen categories. Suppose also that $F$ satisfies the following conditions:

1. A morphism $f$ in $\mathcal{A}$ is a weak equivalence if and only if $F(f)$ is a weak equivalence in $\mathcal{B}$.
2. Given any map $b : F(A) \to B$ in $\mathcal{B}$, there is a cofibration $a : A \to A'$ in $\mathcal{A}$ and a weak equivalence $b' : F(A') \to B$ in $\mathcal{B}$ so that $b = b' \circ F(a)$.
3. If $b$ is a weak equivalence, we may choose $a$ to be a weak equivalence in $\mathcal{A}$.

Then $F$ induces an isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{B})$.

**Proof.**

**Example 15.16.** Let $\mathcal{R}_{hf}(\ast)$ be the Waldhausen subcategory of $\mathcal{R}(\ast)$ of all based CW-complexes homotopic to a finite CW-complex. The approximation theorem applies to the inclusion $\mathcal{R}_f(\ast)$ into $\mathcal{R}_{hf}(\ast)^6$. Hence

$$K_0\mathcal{R}_{hf}(\ast) \cong K_0\mathcal{R}_f(\ast) \cong \mathbb{Z}$$

### 16 Higher $K$-theory

#### 16.1 $S_\ast$-construction ($wS_\ast$)

Let $\mathcal{C}$ be a category with cofibrations. We will define a simplicial category $S_\ast \mathcal{C}$.

$S_0 \mathcal{C}$ is the zero category. $S_1 \mathcal{C}$ is the category $\mathcal{C}$ but whose objects are thought of as cofibrations $0 \hookrightarrow A$.

More generally $S_n \mathcal{C}$ is the category whose objects are sequences of $n$ cofibrations in $\mathcal{C}$:

$$A_\bullet : 0 = A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_n$$

together with a choice of sub quotients $A_{ij} = A_j/A_i$ that are compatible in the sense that some diagram commutes.

A morphism $A_\bullet \to B_\bullet$ in $S_n \mathcal{C}$ is a natural transformation of sequences.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & \ldots & \longrightarrow & A_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & \ldots & \longrightarrow & B_n
\end{array}
$$

Note: If $\mathcal{C}$ is a Waldhausen category, then so too is $S_n \mathcal{C}$

A weak equivalence in $S_n \mathcal{X}$ is a map $A_\bullet \to B_\bullet$ such that each $A_i \xrightarrow{B_{ij}} A_j$ are weak equivalences in $\mathcal{C}$.

A map $A_\bullet \to B_\bullet$ is a cofibration if for every $0 \leq i < j < k \leq n$, the map of cofibrations sequences

$$
\begin{array}{ccccccc}
A_{ij} & \longrightarrow & A_{ik} & \longrightarrow & A_{jk} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_{ij} & \longrightarrow & B_{ik} & \longrightarrow & B_{jk}
\end{array}
$$

is a cofibration in $S_2 \mathcal{C}$.

---

$^6$By the Whitehead theorem
16.2 How to think of $S_2C$

The objects of $S_2C$ can be thought of as cofibration sequences $A_1 \rightarrow A_2 \rightarrow A_{12}$ in $C$.

A cofibration is a commutative diagram

$$
\begin{array}{ccc}
A_1 & \rightarrow & A_2 \\
\downarrow^{u_1} & & \downarrow^{u_2} \\
B_1 & \rightarrow & B_2 \\
\end{array}
$$

such that the maps $A_1 \rightarrow B_1$, $A_{12} \rightarrow B_{12}$ and $B_1 \cup_{A_1} A_2 \rightarrow B_2$ are cofibrations. These conditions ensure that the Waldhausen category axioms are satisfied, turning the $S_nC$’s into Waldhausen categories.

**Proposition 16.1.** $S_* C$ is a simplicial (Waldhausen) category.

*Proof.* The simplicial boundary maps are $\partial _i : S_n C \rightarrow S_{n-1} C$ by forgetting the $i^{th}$ term. $\partial _0$ is slightly different, in that we quotient the sequence by the first term so that our sequence starts at 0.

The simplicial degeneracy maps $s_i : S_n C \rightarrow S_{n+1} C$ are defined by repeating the $i^{th}$ term. We can then check that the simplicial axioms are satisfied. Thus the $S_nC$ fit together to form a simplicial Waldhausen category $S_* C$. The subcategories $wS_n C$ where all morphisms are weak equivalences fit together to form a simplicial category $wS_* C$.

16.2.1 Geometric realization

We can do the classifying space construction to $wS_n C$ for each $n$ to form a simplicial space $BwS_* C$. We write $|wS_* C|$ for the geometric realisation of $BwS_* C$. Since $S_0 C = 0$ is trivial, the space $|wS_* C|$ is connected.

Note that secretly, we have a bi-simplicial complex - in one simplicial direction, the maps are the cofibrations, while in the other simplicial direction the maps are the homotopy equivalences of sequences. $|wS_* C|$ is just the geometric realisation of this.

**Definition 16.2.** Let $C$ be a small Waldhausen category. The algebraic $K$-theory space of $C$ is

$$K(C) := \Omega |wS_* C|$$

The $K$-groups of $C$ are defined to be the homotopy groups

$$K_i(C) := \pi_i K(C) = \pi_{i+1} |wS_* C|$$

**Proposition 16.3.** If $C$ is a Waldhausen category then $\pi_1 |wS_* C| \cong K_0(C)$, where $K_0(C)$ is $K_0$ of a Waldhausen category we defined last time.

*Proof.* If $X_*$ is any simplicial space and $|X_0|$ is a point, then $|X_*|$ is connected and $\pi_1 |X_*|$ is the free group on $\pi_0(X_1)$ modulo the relation $\partial _1(x) = \partial _2(x)\partial _0(x)$ for every $x \in \pi_0(X_2)$. In our case, we have the following:

1. $X_* = BwS_* C$;
2. $\pi_0|BwS_1 C| = \{ \text{objects in } C \} / \sim$;
3. $\pi_0|BwS_2 C| = \{ \text{cofibration sequences in } C \} / \sim$;
4. The maps $\partial _i : S_2 C \rightarrow S_1 C$ map the cofibration sequence $A \rightarrow B \rightarrow B/A$ to $B/A$, $B$ and $A$ respectively.

We can then match things up. \qed
17 K-theory spectrum

We can iterate the $S_\bullet$ construction to form the iterates

$$wS^n\mathcal{C} = wS\ldots S\mathcal{C}.$$  

One can show moreover that there are natural homotopy equivalences

$$|wS\mathcal{C}| \simeq \Omega|wS\ldots S\mathcal{C}|.$$  

So we have a chain of homotopy equivalences

$$|w\mathcal{C}| \to \Omega|wS\mathcal{C}| \to \Omega^2|wS\ldots S\mathcal{C}| \to \ldots$$  

Note that the first map is not a homotopy equivalence [check this]. As suspension is adjoint to the loops functor, we have maps

$$\Sigma|wS^n\mathcal{C}| \to |wS^{n+1}\mathcal{C}|.$$  

If we let $E_n = |wS^n\mathcal{C}|$, we can show that $E_\bullet$ is a $\Omega$-spectrum. This spectrum is usually denoted by $KC$, and is called the K-theory spectrum of $\mathcal{C}$. Moreover, the coproduct on $\mathcal{C}$ induces a product on $KC$, so $KC$ is a ring spectrum.

17.1 examples

Example 17.1 (Exact categories). Let $\mathcal{A}$ be an exact category. Recall that it is a Waldhausen category where the cofibrations are the admissible monica and the weak equivalences are isomorphisms. If $i(\mathcal{C})$ denotes the category of isomorphisms of $\mathcal{C}$, then

$$K(\mathcal{A}) = \Omega|iS\mathcal{A}|.$$  

Waldhausen proved that there a homotopy equivalence between $|iS\mathcal{A}|$ and $BQA$, where $BQA$ is the K-theory space of an exact category gotten via the $Q$-construction.

Example 17.2 (Algebraic K theory of a point). Let $R_f(*)$ be the Waldhausen category of based CW complexes. We write $A(*)$ for $K(R_f(*)) = \Omega|hS\mathcal{R}_f|$. From last time we have

$$A_0(*) = K_0R_f(*) = \mathbb{Z}.$$  

Example 17.3 (K-theory of spaces). Let $X$ be a CW complex. Let $R(X)$ be the category whose objects are CW complexes $Y$ obtained from $X$ by attaching cells, and having $X$ as a retract. The morphisms are the cellular maps. $R(X)$ is a Waldhausen category; the cofibrations are the cellular inclusions and the weak equivalences are weak homotopy equivalences.

Let $R_f(X)$ be the subcategory of $R(X)$ where $Y$ is obtained from $X$ by attaching only finitely many cells. We write

$$A(X) = K(R_f(X)) = \Omega|hS\mathcal{R}_f(X)|.$$  

$A(X)$ is called the algebraic K-theory space of $X$.

Exercise: Show that $A_0(X) = \mathbb{Z}$.

17.2 Waldhausen’s Splitting Theorem

One of the motivating reasons for studying Waldhausen categories was to prove a result known as Waldhausen’s splitting theorem. We state it here out of interest, but do not attempt to prove it. A proof can be found in [REFERENCE]. Waldhausen’s splitting theorem roughly says the following.

Theorem 17.4. For any finite CW complex $X$,

$$A(X) \simeq Wh(X) \times Q(X_+),$$

Where $Wh(X)$ is the Whitehead space of $X$ (which can also be thought of as the moduli space of h-cobordisms of $X$) and $Q(X_+) = \Omega^\infty\Sigma^\infty(X_+)$ is the stable homotopy space of $X$.
References


