Multi-fusion categories of Harish-Chandra bimodules

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Plan of the talk

1. Harish-Chandra (bi)modules.
2. Associated varieties and tensor product modulo “smaller size”.
3. Tensor categories and multi-fusion categories.
5. Sheaves.
Harish-Chandra modules

\( G_\mathbb{R} \) – real semi-simple Lie group, e.g. \( SL(n, \mathbb{R}) \)

Harish-Chandra (1953): many questions about continuous complex representations of \( G_\mathbb{R} \) can be reduced to pure algebra.

\( \mathfrak{g} = \text{Lie}(G_\mathbb{R}) \otimes_\mathbb{R} \mathbb{C} \), \( U(\mathfrak{g}) \) – universal enveloping algebra

\( K \subset G_\mathbb{R} \) – maximal compact subgroup, e.g. \( SO(n, \mathbb{R}) \subset SL(n, \mathbb{R}) \)

Definition

A \((\mathfrak{g}, K)\)–module (or Harish-Chandra module) is a space \( V \) with actions of \( \mathfrak{g} \) and \( K \) such that

1. \( V \) is **algebraic** \( K \)–module, i.e. \( V \) is a union of finite dimensional \( K \)–modules.
2. The actions are **compatible**: \( \mathfrak{g} \)–action is \( K \)–equivariant and the differential of \( K \)–action agrees with \( \text{Lie}(K) \subset \mathfrak{g} \)–action.
3. \( V \) is finitely generated \( U(\mathfrak{g}) \)–module.
Complex groups and bimodules

\( G_\mathbb{C} \) – complex simply connected semi-simple Lie group, e.g. \( SL(n, \mathbb{C}) \)
Let us consider \( G_\mathbb{C} \) as a real Lie group

\[ g = \text{Lie}(G_\mathbb{C}); \text{Lie}(G_\mathbb{C}) \otimes_\mathbb{R} \mathbb{C} = g \oplus g \]

representation of \( \text{Lie}(G_\mathbb{C}) \otimes_\mathbb{R} \mathbb{C} \) ⇔ module over \( \mathcal{U}(g \oplus g) = \mathcal{U}(g) \otimes_\mathbb{C} \mathcal{U}(g) \)
\( x \mapsto -x \) induces \( \mathcal{U}(g) \cong \mathcal{U}(g)^{\text{op}} \), so \( \mathcal{U}(g) \otimes_\mathbb{C} \mathcal{U}(g) \cong \mathcal{U}(g) \otimes_\mathbb{C} \mathcal{U}(g)^{\text{op}} \)
Thus \( \text{Lie}(G_\mathbb{C}) \otimes_\mathbb{R} \mathbb{C} \)–representation is the same as \( \mathcal{U}(g) \)–bimodule

We can choose \( K \subset G_\mathbb{C} \) such that \( \text{Lie}(K) \otimes_\mathbb{R} \mathbb{C} \subset \text{Lie}(G_\mathbb{C}) \otimes_\mathbb{R} \mathbb{C} \) is the diagonal \( \Delta g \subset g \oplus g \), e.g. \( K = SU(n) \subset SL(n, \mathbb{C}) \)
\( M \) – \( \mathcal{U}(g) \)–bimodule; adjoint action: \( \text{ad}(x)m := xm - mx \)
\( \mathcal{U}(g) \)–bimodule is algebraic if it is a union of finite dimensional \( g \)-modules with respect to the adjoint action.

Example

\( \mathcal{U}(g) \) is algebraic (use PBW filtration) and \( \mathcal{U}(g) \otimes_\mathbb{C} \mathcal{U}(g) \) is not.
Harish-Chandra bimodules

Definition

A Harish-Chandra bimodule over $\mathfrak{g}$ is a finitely generated $U(\mathfrak{g})$–bimodule which is algebraic.

Lemma

If $M$ and $N$ are Harish-Chandra bimodules then so is $M \otimes_{U(\mathfrak{g})} N$.

- The tensor product $\otimes_{U(\mathfrak{g})}$ is associative
- $U(\mathfrak{g})$ is the unit for this tensor product

Thus the category $\mathcal{H}$ of Harish-Chandra bimodules is a tensor category.

Remark. If $M$ is a Harish-Chandra bimodule over $\mathfrak{g}$ and $N$ is $(\mathfrak{g}, K)$–module then $M \otimes_{U(\mathfrak{g})} N$ is also $(\mathfrak{g}, K)$–module

Thus the category $\mathcal{H}$ acts on the category of $(\mathfrak{g}, K)$–modules.
Central characters and simple Harish-Chandra bimodules

\( Z(\mathfrak{g}) \subset U(\mathfrak{g}) \) center of the universal enveloping algebra
\( Z(\mathfrak{g}) \) acts on an irreducible \( \mathfrak{g} \)-module via central character \( \chi : Z(\mathfrak{g}) \to \mathbb{C} \)
\( \chi_1 \mathcal{H} \chi_2 \subset \mathcal{H} \) – full subcategory where the left \( Z(\mathfrak{g}) \)-action factors through \( \chi_1 \) and the right \( Z(\mathfrak{g}) \)-action factors through \( \chi_2 \)
Any irreducible Harish-Chandra bimodule is contained in a unique \( \chi_1 \mathcal{H} \chi_2 \subset \mathcal{H} \)

\( \chi_1 \mathcal{H} \chi_2 \otimes U(\mathfrak{g}) \chi_3 \mathcal{H} \chi_4 \subset \chi_1 \mathcal{H} \chi_4 \) and \( \chi_1 \mathcal{H} \chi_2 \otimes U(\mathfrak{g}) \chi_3 \mathcal{H} \chi_4 = 0 \) unless \( \chi_2 = \chi_3 \)
\( \mathcal{H}(\chi) := \chi \mathcal{H} \chi \) is tensor subcategory of \( \mathcal{H} \)
unit object: \( U(\mathfrak{g})\chi := U(\mathfrak{g})/\text{Ker}(\chi)U(\mathfrak{g}) \)

Convention: \( \chi \) is integral regular, e.g. \( \chi = \chi_0 \) trivial central character

Theorem (Bernstein-S. Gelfand, Enright, Joseph)
Irreducible bimodules in \( \mathcal{H}(\chi) \leftrightarrow \) elements of the Weyl group \( W \).

Proof uses Bernstein-Gelfand-Gelfand category \( \mathcal{O} \).
Associated varieties

\(M \in \mathcal{H}, \ M_0 \subset M\) finite dimensional subspace which generates \(M\) and which is invariant under the adjoint action

\[U(\mathfrak{g})_0 \subset U(\mathfrak{g})_1 \subset \cdots \subset U(\mathfrak{g})\) PBW filtration

\(M_n = U(\mathfrak{g})_n M_0 \Rightarrow\) filtration \(M_0 \subset M_1 \subset \cdots \subset M\)

Associated graded

\(\text{gr}M\) is a finitely generated module over \(\text{gr}U(\mathfrak{g}) = S^\bullet(\mathfrak{g})\)

Moreover, this module is equivariant with respect to \(G_\mathbb{C}\)–action

Let us identify \(\mathfrak{g}^* = \text{Spec}(S^\bullet(\mathfrak{g}))\) with \(\mathfrak{g}\) via the Killing form

Definition

The associated variety \(V(M)\) is the support of \(\text{gr}M\) in \(\mathfrak{g}\).

- \(V(M) = V(L) \cup V(K)\) for a s.e.s. \(0 \rightarrow L \rightarrow M \rightarrow K \rightarrow 0\)
- \(V(M \otimes U(\mathfrak{g}) N) \subset V(M) \cap V(N)\)
Nilpotent orbits

$x \in g$ is **nilpotent** if $ad(x) : g \to g$ is nilpotent

**Example.** $x \in sl(n, \mathbb{C})$ is nilpotent $\iff x^n = 0$

$\mathcal{N} \subset g$ is the nilpotent cone, i.e. the set of all nilpotent elements

Dynkin + Kostant: $\mathcal{N}$ consists of finitely many $G_{\mathbb{C}}$–orbits

**Example.** nilpotent orbits in $sl(n, \mathbb{C})$ $\leftrightarrow$ partitions of $n$

For $\mathcal{O} \subset \mathcal{N}$, $\bar{\mathcal{O}}$ is its closure; partial order: $\mathcal{O}' \leq \mathcal{O} \iff \mathcal{O}' \subset \bar{\mathcal{O}}$

- for $M \in \chi_1 \mathcal{H} \chi_2$ we have $V(M) \subset \mathcal{N}$. Moreover,

**Theorem (Borho-Brylinsky, Joseph)**

For irreducible $M \in \mathcal{H}$, $V(M)$ is irreducible, i.e. $V(M) = \bar{\mathcal{O}}$.

$\mathcal{H}(\chi)_{\leq \mathcal{O}}$ – full subcategory of $\mathcal{H}(\chi)$ consisting of $M$ with $V(M) \subset \bar{\mathcal{O}}$

$\mathcal{H}(\chi)_{< \mathcal{O}}$ – full subcategory of $\mathcal{H}(\chi)_{\leq \mathcal{O}}$ consisting of $M$ with $V(M) \neq \bar{\mathcal{O}}$

Both $\mathcal{H}(\chi)_{\leq \mathcal{O}}$ and $\mathcal{H}(\chi)_{< \mathcal{O}}$ are Serre subcategories

$\mathcal{H}(\chi)_{\leq \mathcal{O}}$ is closed under $\otimes_{U(g)}$; $\mathcal{H}(\chi)_{< \mathcal{O}}$ is “ideal” with respect to $\otimes_{U(g)}$
Cell categories

Serre quotients
We can form \( \tilde{\mathcal{H}}(\chi)_{\mathbb{O}} = \mathcal{H}(\chi)_{\leq \mathbb{O}} / \mathcal{H}(\chi)_{< \mathbb{O}} \).

Tensor products \( \otimes_{U(\mathfrak{g})} \) descends to \( \otimes : \tilde{\mathcal{H}}(\chi)_{\mathbb{O}} \times \tilde{\mathcal{H}}(\chi)_{\mathbb{O}} \rightarrow \tilde{\mathcal{H}}(\chi)_{\mathbb{O}} \).

- it is not clear whether \( \tilde{\mathcal{H}}(\chi)_{\mathbb{O}} \) has a unit object

\( \mathcal{H}(\chi)_{\mathbb{O}} \) – full subcategory of \( \tilde{\mathcal{H}}(\chi)_{\mathbb{O}} \) consisting of semisimple objects

Theorem (Joseph, Bezrukavnikov-Finkelberg-O, Losev)

\( \mathcal{H}(\chi)_{\mathbb{O}} \) is closed under \( \otimes \).

\( \mathcal{H}(\chi)_{\mathbb{O}} \) has a unit object: let \( Pr(\chi)_{\mathbb{O}} \) be the (finite) set of primitive ideals in \( U(\mathfrak{g})_\chi \) with \( V(U(\mathfrak{g})/I) = \bar{\mathcal{O}} \); then \( 1 = \bigoplus_{I \in Pr(\chi)_{\mathbb{O}}} U(\mathfrak{g})/I \)

Theorem (Bezrukavnikov-Finkelberg-O, Losev-O)

\( \mathcal{H}(\chi)_{\mathbb{O}} \) is a multi-fusion category.

We will call \( \mathcal{H}(\chi)_{\mathbb{O}} \) cell category associated with \( \mathbb{O} \)
Tensor (=monoidal) categories

Definition (MacLane)

Tensor category: quadruple $(\mathcal{C}, \otimes, a, \mathbf{1})$ where $\mathcal{C}$ is a category, $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, $a_{X,Y,Z}: (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ is an associativity constraint, $\mathbf{1}$ is the unit object.

1. Pentagon axiom: the following diagram commutes for all $W, X, Y, Z \in \mathcal{C}$:

\[
\begin{array}{c}
((W \otimes X) \otimes Y) \otimes Z \\
(W \otimes (X \otimes Y)) \otimes Z \\
W \otimes ((X \otimes Y) \otimes Z)
\end{array}
\begin{array}{c}
\rightarrow\\
\downarrow a_{W,X,Y,Z} \\
\rightarrow\\
\downarrow id_{W \otimes a_{X,Y,Z}} \\
\rightarrow
\end{array}
\begin{array}{c}
(W \otimes X) \otimes (Y \otimes Z) \\
W \otimes (X \otimes (Y \otimes Z))
\end{array}
\]

2. Unit axiom: both functors $\mathbf{1} \otimes ?$ and $? \otimes \mathbf{1}$ are isomorphic to the identity functor.
Rigidity

For \( X \in C \) its right dual is \( X^* \in C \) together with \( \text{ev}_X : X^* \otimes X \to 1 \) and \( \text{coev}_X : 1 \to X \otimes X^* \) such that the compositions equal the identities:

\[
X \overset{\text{coev}_X \otimes \text{id}_X}{\longrightarrow} (X \otimes X^*) \otimes X \overset{a_{X,X^*,X}}{\longrightarrow} X \otimes (X^* \otimes X) \overset{\text{id}_X \otimes \text{ev}_X}{\longrightarrow} X
\]

\[
X^* \overset{\text{id}_{X^*} \otimes \text{coev}_X}{\longrightarrow} X^* \otimes (X \otimes X^*) \overset{a_{X,X^*,X}^{-1}}{\longrightarrow} (X^* \otimes X) \otimes X^* \overset{\text{ev}_X \otimes \text{id}_{X^*}}{\longrightarrow} X^*
\]

**Definition**

\( C \) is rigid if any \( X \in C \) has right and left duals.

**Example (s)**

1. \( C = \text{Bimod}(R) \) bimodules over a ring \( R \): tensor product is \( \otimes_R \), \( 1 = R \). \( M \in C \) has right dual \( \iff \) \( M \) is f.g. projective as left \( R \)-module.
2. \( C = \text{End}(\mathcal{A}) \) functors from a category \( \mathcal{A} \) to itself; tensor product is composition, \( 1 = \text{Id} \). \( F \in C \) has a dual \( \iff \) adjoint of \( F \) exists.
3. \( C = \text{Mod}(R) \) modules over a commutative ring \( R \); e.g. vector spaces over a field. \( M \in C \) has right dual \( \iff \) \( M \) is f.g. projective \( \iff \) \( M \) has left dual.
4. (H. Sinh) Objects: elements of a group $A$; $\text{Hom}(g, h) = \emptyset$ if $g \neq h$, $\text{Hom}(g, g) = S$ where $S$ is an abelian group. $g \otimes h = gh$, $\alpha \otimes \beta = \alpha\beta$ for $g, h \in A$, $\alpha, \beta \in S$. Associativity constraint: $\omega_{g,h,k} \in S$ for any $g, h, k \in A$. Pentagon axiom $\Leftrightarrow$ $\partial \omega = 1$, i.e. $\omega$ is a 3-cocycle on $A$ with values in $S$. Tensor structures are parameterized by $H^3(A, S)$.

5. $R$ – algebra over $k$ with trivial center. Consider the category of invertible bimodules over $R$ (morphisms are isomorphisms of bimodules). This category is tensor equivalent to category from (4). $A = \text{Pic}(R)$ group of isomorphism classes of invertible bimodules (= non-commutattive Picard group of $R$); $S = k^{\times}$. Associator $\omega \in H^3(\text{Pic}(R), k^{\times})$.

5a. $\text{Pic}(R) \supset \text{Out}(R)$: $M_{\phi} = R$, $(a, b) \cdot c = ac\phi(b)$. Let $1 \neq \phi \in \mathbb{Z}/2\mathbb{Z} \subset \text{Out}(R)$, so $\phi^2 = \text{Ad}(g)$.

Exercise. (i) $\phi(g) = \pm g$; (ii) $\omega|_{\mathbb{Z}/2\mathbb{Z}} \neq 0 \Leftrightarrow \phi(g) = -g$; (iii) Let $\phi(g) = -g$. Then $M_{\phi} \not\cong M$ for any $M \in \text{Irr}(R)$.

$R = \mathbb{C}\langle g, x, y \rangle/(xy - yx - 1, g^2 - 1, gx + xg, gy + yg)$, $\phi(g) = -g, \phi(x) = -y, \phi(y) = x$. 

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Fusion of Harish-Chandra bimodules

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Multi-fusion categories

Definition (Etingof, Nikshych, O)

Tensor category $\mathcal{C}$ over $k$ is multi-fusion if it is rigid and semi-simple with finitely many simple objects. $\mathcal{C}$ is fusion if in addition $1$ is simple.

Example (char$(k)=0$)

0. Vec – finite dimensional vector spaces.
1. $\text{Rep}(A)$ – f.d. representations of finite group $A$.
2. $\text{Vec}_A$ – f.d. $A$–graded vector spaces. Thus simple objects are $k_a, a \in A$ and $k_a \otimes k_b = k_{ab}$. Generalization: $\text{Vec}_A^\omega$ – same as $\text{Vec}_A$ but $\omega \in H^3(A, k^\times)$ is used as the associator.
3. $\text{Bimod}(R)$ where $R$ is semisimple, e.g. $R = k \oplus k$. $1 = R$ is not simple.
4. $Y$ is a finite set with $A$–action. $\text{Coh}_A(Y \times Y) – A$–equivariant vector bundles (or coherent sheaves) on $Y \times Y$. Convolution product: $F_1 \ast F_2 = p_{13*}(p_{12*}(F_1) \otimes p_{23*}(F_2))$ where $p_{ij} : Y \times Y \times Y \to Y \times Y$.

Exercise. What is the number of simple summands in $1 \in \text{Coh}_A(Y \times Y)$?
Module categories

The categories \( \text{Coh}_A(Y \times Y) \) are not closed under the operation of taking full tensor subcategory. For example \( \text{Vec}_B^\omega \) with \( \omega \neq 0 \) is usually not of the form \( \text{Coh}_A(Y \times Y) \) but it can be found as a subcategory in a suitable \( \text{Coh}_A(Y \times Y) \).

Definition

Let \( C \) be a tensor category and \( M \) be a category. We say that \( M \) is a module category over \( C \) (or that \( C \) acts on \( M \)) if we have a tensor functor \( C \to \text{End}(M) \). Equivalently, we have a bifunctor \( C \times M \to M \) with associativity constraint satisfying suitable axioms.

Example

Consider \( C = \text{Vec}_A^\omega \). Let \( B \subset A \) and \( \psi \in Z^2(B, k^\times) \) be such that \( \partial \psi = \omega|_B \). Then \( R_B = \bigoplus_{b \in B} k_b \) acquires a structure of associative algebra in \( C \). Then \( \mathcal{M}(B, \psi) = \{ \text{right } R_B \text{-modules in } C \} \) is naturally a module category over \( C \). Simple objects of \( \mathcal{M}(B, \psi) \leftrightarrow A/B \).
Dual categories

**Convention:** If $C$ is a multi-fusion category then any module category is assumed to be semisimple with finitely many simple objects.

**Definition**

Let $\mathcal{M}$ be a module category over $C$. Then $C_{\mathcal{M}}^* := \text{End}_C(\mathcal{M})$ is called dual category of $C$ with respect to $\mathcal{M}$.

**Properties (Müger+Etingof, Nikshych, O)**

- $C_{\mathcal{M}}^*$ is multi-fusion category
- $C_{\mathcal{M}}^*$ is fusion $\Leftrightarrow \mathcal{M}$ is indecomposable module category over $C$
- $(C_{\mathcal{M}}^*)_{\mathcal{M}} \simeq C$
- $C \sim C_{\mathcal{M}}^*$ is an equivalence relation (2-Morita equivalence)
- $C \xrightarrow{F} \mathcal{D}$ tensor functor and $\mathcal{M}$ is module category over $\mathcal{D}$. Then we have $\mathcal{D}_{\mathcal{M}}^* \xrightarrow{F^*} C_{\mathcal{M}}^*$
- let us say that $F$ is injective if it is fully faithful and surjective if any object of $\mathcal{D}$ is a subquotient of $F(X)$. $F$ injective $\Leftrightarrow F^*$ surjective
Convolution with twists

Example

\[ \mathcal{C} = \text{Vec}_A \text{ and } \mathcal{M} = \bigoplus_i \mathcal{M}(B_i, 1). \]
Then \[ \mathcal{C}^\ast_{\mathcal{M}} = \text{Coh}_A(Y \times Y) \text{ where } Y = \bigsqcup_i A/B_i \text{ (so } \mathcal{M} = \text{Coh}(Y)). \]

Generalization

Let \[ \mathcal{C} = \text{Vec}_A^\omega \text{ and } \mathcal{M} = \bigoplus_i \mathcal{M}(B_i, \psi_i). \] We consider \[ \mathcal{C}^\ast_{\mathcal{M}} \] as cohomologically twisted version of \[ \text{Coh}_A(Y \times Y). \]

**Notation:** \[ \mathcal{C}^\ast_{\mathcal{M}} = \text{Coh}_{A,\omega}(Y \times Y). \] Note that the information about \( \psi_i \)'s is implicitly contained in \( Y \); \( Y \) is cohomologically twisted \( A \)–set.

Lemma

Let \( \mathcal{C} \subset \text{Coh}_A(Y \times Y) \) be a full multi-fusion subcategory such that \( \mathcal{M} = \text{Coh}(Y) \) is indecomposable over \( \mathcal{C} \). Then there exists a surjective functor \( F : \text{Vec}_A \rightarrow \text{Vec}_A^\omega \) such that the action of \( \text{Vec}_A \) on \( \mathcal{M} \) factors through \( F \) and such that \[ \mathcal{C} = \text{Coh}_{A,\omega}(Y \times Y) = (\text{Vec}_A^\omega)^\ast_{\mathcal{M}} \subset (\text{Vec}_A)^\ast_{\mathcal{M}}. \]
Whittaker modules

Let $e \in \mathfrak{g}$ be a nilpotent element.

**Jacobson-Morozov**: $\exists h, f \in \mathfrak{g}$ s.t. $[h, e] = 2e, [h, f] = -2f, [e, f] = h$.

$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}(n), \quad \mathfrak{g}(n) = \{x \in \mathfrak{g} | [h, x] = nx\}$.

E.g. $e \in \mathfrak{g}(2)$ and $f \in \mathfrak{g}(-2)$.

$x, y \mapsto (e, [x, y])$ non-degenerate skew-symmetric bilinear form on $\mathfrak{g}(-1)$.

Pick a lagrangian subspace $\ell \subset \mathfrak{g}(-1)$ and set $m = m_\ell = \ell \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i)$.

Then $\xi(x) = (x, e)$ is a Lie algebra homomorphism $m \to \mathbb{C}$.

$m_\xi :=$ Lie subalgebra of $U(\mathfrak{g})$ spanned by $x - \xi(x), x \in m$.

**Definition (Moeglin)**

We say that $\mathfrak{g}$–module is **Whittaker** if the action of $m_\xi$ on it is locally nilpotent. $\text{Wh}$ – full subcategory of Whittaker $\mathfrak{g}$–modules.

$\tilde{\text{Skr}} : \text{Wh} \to \text{Vect}, \quad M \mapsto \{v \in M | m_\xi v = 0\}$.

$U(\mathfrak{g}, e) = \text{End}(\tilde{\text{Skr}})$ – Premet’s finite $W$–algebra.

**Remark**: $U(\mathfrak{g}, e)$ does not depend on choice of $\ell \subset \mathfrak{g}(-1)$. 

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Lemma

For $M \in \mathcal{H}$ and $N \in \text{Wh}$, $M \otimes_{U(\mathfrak{g})} N \in \text{Wh}$. Thus $\mathcal{H}$ acts on $\text{Wh}$.

• Let $\chi\text{Wh}$ be the full subcategory of $M \in \text{Wh}$ such that $Z(\mathfrak{g})$—action factors through a central character $\chi$. Then $\mathcal{H}(\chi)$ acts on $\chi\text{Wh}$.

$\chi\text{Wh}^f$ — full subcategory of $\chi\text{Wh}$ consisting of semisimple $M$ such that $\text{Skr}(M)$ is finite dimensional ($\simeq$ semisimple f.d. $U(\mathfrak{g}, e)$—modules).

Theorem (Losev)

Let $\mathbb{O} = G_{C} e$. For $M \in \mathcal{H}(\chi)_{\leq \mathbb{O}}$ and $N \in \chi\text{Wh}^f$, $M \otimes_{U(\mathfrak{g})} N \in \chi\text{Wh}^f$.

For $M \in \mathcal{H}(\chi)_{< \mathbb{O}}$ and $N \in \chi\text{Wh}^f$, $M \otimes_{U(\mathfrak{g})} N = 0$. Thus the cell category $\mathcal{H}(\chi)_{\mathbb{O}}$ acts on $\chi\text{Wh}^f$.

Let $Q = Z_{G_{C}}(e, f, h)$. Then $Q$ acts on $U(\mathfrak{g}, e)$ and on $\chi\text{Wh}^f$.

$Q$—action on $\chi\text{Wh}^f$ commutes with $\mathcal{H}(\chi)_{\mathbb{O}}$—action.

$Q^0 \subset Q$ the unit component. The action of $Q^0$ on $\chi\text{Wh}^f$ is trivial.

**Warning:** this does not imply that $C(e) := Q/Q^0$ acts on $\chi\text{Wh}^f$. 
Irreducible finite dimensional $U(\mathfrak{g}, e)$-modules

Let us choose a finite subgroup $A \subset Q$ which surjects to $Q/Q^0$. Then $\chi \text{Wh}^f$ is a module category over $\text{Vec}_A$.

**Theorem (Losev, O)**

The functor $\mathcal{H}(\chi)_\emptyset \to (\text{Vec}_A)^*_{\chi \text{Wh}^f} = \text{End}_{\text{Vec}_A}(\chi \text{Wh}^f)$ is fully faithful.

**$\chi \text{Wh}^f$ as module category over $\text{Vec}_A$**

$Y$ – set of isomorphism classes of irreducible f.d. $U(\mathfrak{g}, e)$-modules. $A$ acts on $Y$; moreover we have data of cohomologically twisted $A$-set.

Thus $(\text{Vec}_A)^*_{\chi \text{Wh}^f} = \text{Coh}_A(Y \times Y)$ and $\mathcal{H}(\chi)_\emptyset \subset \text{Coh}_A(Y \times Y)$.

**Corollary**

There is a quotient $\bar{A}$ of $A$ and $\omega \in H^3(\bar{A}, \mathbb{C}^\times)$ such that the action of $\text{Vec}_A$ on $\chi \text{Wh}^f$ factors through tensor functor $\text{Vec}_A \to \text{Vec}_{\bar{A}}^\omega$ and the action on $\chi \text{Wh}^f$ induces tensor equivalence $\mathcal{H}(\chi)_\emptyset \simeq \text{Coh}_{\bar{A}, \omega}(Y \times Y)$. 
Complements

• \( \mathcal{H}(\chi)_{\mathbb{O}} \neq 0 \iff \) the nilpotent orbit \( \mathbb{O} \) is special in the sense of Lusztig.
• The quotient map \( A \to \tilde{A} \) factorizes through \( A \subset Q \to Q/Q^0 = C(e). \) \( \tilde{A} \) is Lusztig’s quotient of \( C(e) \) (defined for any special nilpotent orbit).
• Irr. summands of \( 1 \in \mathcal{H}(\chi)_{\mathbb{O}} \leftrightarrow \) primitive ideals \( I \) with \( V(U(\mathfrak{g})/I) = \bar{\mathbb{O}}. \) Irr. summands of \( 1 \in \text{Coh}_{\bar{A},\omega}(Y \times Y) \leftrightarrow \tilde{A} \)–orbits (\( = Q \)–orbits) in \( Y. \) Hence irreducible f.d. \( U(\mathfrak{g}, e)_{\chi} \)–modules which give rise to the same primitive ideal are \( Q \)–conjugated (Losev).
• Recall that irreducible objects of \( \mathcal{H}(\chi) \leftrightarrow W. \) It follows from Joseph’s irreducibility theorem that \( \text{Irr}(\mathcal{H}(\chi)) = \bigsqcup_{\mathbb{O}} \text{Irr}(\mathcal{H}(\chi)_{\mathbb{O}}). \) Hence we have a partition of \( W \) indexed by special nilpotent orbits. This is known to coincide with partition into Kazhdan-Lusztig two sided cells. Each two sided cell is in turn partitioned into left cells and into right cells. This corresponds to partitions \( \text{Irr}(C) = \bigsqcup_i 1_i \otimes \text{Irr}(C) = \bigsqcup_i \text{Irr}(C) \otimes 1_i \) where \( 1 = \bigoplus_i 1_i \) which holds for any multi-fusion category \( C. \)
• \( Y = \bigsqcup_i \tilde{A}/B_i \) where \( B_i \subset \tilde{A} \) is well-defined up to conjugacy. These are Lusztig’s subgroups attached to any left cell.
\[ \oplus_\Phi K(H(\chi)_\Phi) =: J \text{ is known to be asymptotic Hecke algebra (Lusztig).} \]

Lusztig’s isomorphism: \[ J \otimes_\mathbb{Z} \mathbb{Q} \simeq \mathbb{Q}[W] \]. Thus any \( \mathbb{Q} \)-module over \( K(H(\chi)_\Phi) \) gives rise to a \( W \)-module. For example \( K(H(\chi)_\Phi \otimes \mathbb{1}_I) \otimes \mathbb{Q} \) is constructible representation attached to a left cell.

Also \( K(\text{Coh}(Y)) \) is a module over \( K(\text{Coh}_{\bar{A},\omega}(Y \times Y)) \).

Dodd: there is \( W \times C(e) \)-equivariant embedding of \( K(\text{Coh}(Y)) \otimes \mathbb{Q} \) into Springer representation \( H^{top}(B_e) \).

- The 3-cocycle \( \omega \in H^3(\bar{A}, \mathbb{C}^\times) \) is almost always zero. \( \omega \neq 0 \) iff the corresponding two sided cell is exceptional. This happens only in types \( E_7 \) and \( E_8 \); in this case \( \bar{A} = \mathbb{Z}/2\mathbb{Z} \). Proof requires theory of character sheaves.

- Assume that \( \omega = 0 \). Then \( \chi \text{Wh}^f = \bigoplus_i \mathcal{M}(B_i, \psi_i) \). It can be shown that the cocycles \( \psi_i \) are all trivial.

- There is a conjectural description (Losev,O) of what happens in the case of \( \chi \) which is no longer integral. The calculations suggest that in this case nontrivial 2-cocycles show up often.

- Further results: Losev gave formulas for dimensions of irreducible modules in \( \chi \text{Wh}^f \) and proved that they coincide with Goldie ranks of quotients by primitive ideas.
Derived convolution

\( F \) – algebraically closed field (possibly of positive characteristic)
\( X \) – algebraic variety over \( F \)
Sheaves on \( X \) form a category over field \( k \):
(a) \( D \)-modules: \( \text{char}(F)=0, \ k=F \)
(b) perverse constructible sheaves in classical topology: \( F=\mathbb{C}, \ \text{any} \ k \)
(c) perverse constructible \( \ell \)-adic sheaves: \( \ell \neq 0 \) in \( F, \ k=\overline{\mathbb{Q}}_{\ell} \)

\( G \) – semisimple group over \( F \) of the same Dynkin type as \( \mathfrak{g} \)
\( B \) – flag variety of \( G \) (\( B = G/B \) where \( B \) is a Borel subgroup)
Simple \( G \)-equivariant sheaves on \( B \times B \leftrightarrow G \)-orbits on \( B \times B \leftrightarrow \text{Bruhat} W; \ w \leftrightarrow I_w \)
Convolution \( \ast \): \( F_1 \ast F_2 = p_{13 \ast}(p_{12}^*(F_1) \otimes p_{23}^*(F_2)) \) (use derived categories!)
\textbf{Decomposition Theorem} \hspace{1em} ({\text{Beilinson, Bernstein, Deligne and Gabber}}) \Rightarrow
\( I_u \ast I_v \simeq \bigoplus_{w,i} I_w[i]^{n_{u,v}(i)} \)
\( C_u C_v = \sum_{w,i} n_{u,v}^{w}(i)t^i C_w \) – Hecke algebra (over \( \mathbb{Z}[t, t^{-1}] \)) with Kazhdan-Lusztig basis
Asymptotic Hecke algebra and truncated convolution

\[ a(w) = \max\{i \mid n_{u,v}^w(i) \neq 0 \text{ for some } u, v\} - \text{Lusztig's } a-\text{function} \]

\[ t_u t_v = \sum_w n_{u,v}^w(a(w))t_w - \text{Lusztig's asymptotic Hecke algebra } J \text{ (over } \mathbb{Z}) \]

Lusztig: \( J \) is associative with unit; \( J \otimes \mathbb{Q} \cong \mathbb{Q}[W] \)

\[ J = \bigoplus_C J_C - \text{sum over two sided cells in } W; \ a|_C = \text{const } =: a(C) \]

Multi-fusion category \( J_C \): simple objects \( I_w, w \in C \)

truncated convolution: \( I_u \bullet I_v := \bigoplus_{w \in C} I_w^{n_{u,v}^w(a(C))} \)

Beilinson-Bernstein: \( D-\)modules on \( B \cong g-\)modules with central character \( \chi_0 \).

**Corollary:** \( G-\)equivariant \( D-\)modules on \( B \times B \cong H(\chi_0) \).

Beilinson-Ginzburg: we can change equivalence above and make it tensor

**Corollary** (Bezrukavnikov, Finkelberg, O): \( D-\)module version of \( J_C \cong H_\emptyset \).

**Theorem** (Bezrukavnikov, Finkelberg, O): \( J_C \cong \text{Coh}_{\overline{A},\omega}(Y \times Y) \) for any \( F \).
Character sheaves and Drinfeld center

$G$–equivariant sheaves on $B \times B = B \times B$–equivariant sheaves on $G$

Such sheaves are $\Delta(B)$–equivariant $= \text{Ad}(B)$–equivariant

$\Gamma^G_B : \text{Ad}(B)$–equivariant sheaves $\rightarrow \text{Ad}(G)$–equivariant sheaves

Simple constituents of $\Gamma^G_B(I_w) =: \text{(unipotent) character sheaves}$ (Lusztig)

$\mathcal{C}$ – tensor category $\Rightarrow$ Drinfeld center $\mathcal{Z}(\mathcal{C})$:

Objects of $\mathcal{Z}(\mathcal{C}) = \text{pairs } (X, \phi) \text{ where } \phi : X \otimes ? \simeq ? \otimes X$

Müger, O: $\mathcal{Z}(\mathcal{C}^*_M) \simeq \mathcal{Z}(\mathcal{C})$ for a multi-fusion category $\mathcal{C}$

Example: $\mathcal{Z}(\text{Coh}_{\tilde{A}, \omega}(Y \times Y)) \simeq \mathcal{Z}(\text{Vec}_{\tilde{A}}^\omega)$ – (twisted) Drinfeld double

Observation: the functor $\Gamma^G_B$ is formally similar to functor $I : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$

Bezrukavnikov, Finkelberg, O: using $D$–modules (so $\text{char}(F) = 0$)

Ben-Zvi, Nadler: in the setting of infinity categories

Lusztig: using mixed sheaves (for any $F$)

Corollary: unipotent character sheaves $\leftrightarrow \bigsqcup_{\text{Irr}} \text{Irr}(\mathcal{Z}(\text{Vec}_{\tilde{A}}))$. 
Coxeter groups

$W$ – finite crystallographic Coxeter group
What about more general Coxeter groups?

$W$ – affine Weyl group
Lusztig: two sided cells in $W \leftrightarrow$ nilpotent orbit in $\mathfrak{g}$
Bezrukavnikov, O: $\mathcal{J}_C \cong \text{Coh}_Q(Y \times Y)$ (recall $Q = Z_{G_C}(e, f, h)$)
Bezrukavnikov, Mirković: interpretation of the set $Y$ in terms of 
unrestricted representations of $\mathfrak{g}$ in positive characteristic

$W$ – infinite crystallographic group
Lusztig: category $\mathcal{J}_C$ makes sense; however
  • infinite number of simple objects
  • 1 might be “infinite direct sum”

Soergel+Elis, Williamson+Lusztig: $\mathcal{J}_C$ makes sense for any $W$!
  • rigidity is not known; usually $\mathcal{J}_C$ is not a convolution category
Thanks for listening!