ON THE DERIVED DG FUNCTORS

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Abstract. Assume that abelian categories \( A, B \) over a field admit countable direct limits and that these limits are exact. Let \( F : D^{+}_{dg}(A) \to D^{+}_{dg}(B) \) be a DG quasi-functor such that the functor \( Ho(F) : D^{+}(A) \to D^{+}(B) \) carries \( D^{\geq 0}(A) \) to \( D^{\geq 0}(B) \) and such that, for every \( i > 0 \), the functor \( H^iF : A \to B \) is effaceable. We prove that \( F \) is canonically isomorphic to the right derived DG functor \( RH^0(F) \). We also prove a similar result for bounded derived DG categories and a formula that expresses Hochschild cohomology of the categories \( D_{dg}^b(A), D_{dg}^+(A) \) as the \( Ext \) groups in the abelian category of left exact functors \( A \to \text{Ind}A \). The proofs are based on a description of Drinfeld’s category of quasi-functors as the derived category of a category of sheaves.

1. Main results

Let \( A \) and \( B \) be abelian categories, and let

\[
RF_{tri} : D^{+}(A) \to D^{+}(B)
\]

be the right derived functor of some left exact functor \( F : A \to B \). Then, the corresponding cohomological \( \delta \)-functor \( R^*F = H^*RF_{tri} : A \to B \) has the following property: the functor \( H^iRF_{tri} \) is 0 for \( i < 0 \), effaceable for \( i > 0 \), and \( H^0RF_{tri} \) is isomorphic to \( F \). Conversely, according to a result of Grothendieck ([G]) every cohomological \( \delta \)-functor \( T^* : A \to B \) satisfying the above property is canonically isomorphic to the right derived functor \( R^*F \). The purpose of this paper is to extend this extremely useful characterization of \( R^*F \) to the derived category level. Unfortunately, Verdier’s notion of triangulated functor seems too poor to allow such a simple characterization of the derived functors. In order to get a meaningful statement one has to consider triangulated functors with some kind of enrichment. Arguably the most useful notion here is the one of DG quasi-functor (or essentially equivalent notion of \( A_\infty \)-functor). Indeed, works of Keller and Drinfeld ([K2], [Dri]) provide a canonical DG enhancement \( D_{dg}^+(A) \) of Verdier’s triangulated derived category. Roughly, a DG quasi-functor \( F : D_{dg}^b(A) \to D_{dg}^b(B) \) is a diagram of the form

\[
(1.1) \quad D_{dg}^+(A) \xrightarrow{S} \mathcal{C} \xrightarrow{G} D_{dg}^+(B).
\]

where \( \mathcal{C} \) is a DG category, \( S \) and \( G \) are DG functors, and, in addition, \( S \) is a homotopy equivalence. Every quasi-functor (1.1) yields a triangulated functor \( Ho(F) : D^{+}(A) \to D^{+}(B) \), but the converse is not true in general. Nevertheless, many of the natural triangulated functors come together with a DG enhancement. For example, the triangulated derived functor \( RF \) can be canonically promoted to a DG quasi-functor ([Dri] §5). The main result of this paper states that under certain mild assumptions on abelian categories \( A \) and \( B \) the DG quasi-functors isomorphic

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to the DG derived ones are precisely the DG quasi-functors satisfying Grothendieck’s condition above. To state the result we need to introduce a bit of notation.

Let $k$ be a commutative ring. Denote by $\text{Mod}(k)$ the category of $k$-modules. We shall say that a $k$-linear category is $k$-flat if, for every two objects $X, Y$, the $k$-module $\text{Hom}(X, Y)$ is flat. Given a $k$-linear exact category $\mathcal{A}$ we denote by $D^b_d(\mathcal{A})$ the corresponding bounded derived DG category over $k$. This is the DG quotient ([Dri]) of the DG category $C^b_d(\mathcal{A})$ of bounded complexes by the subcategory of acyclic ones ([N], §1). The homotopy category of $D^b_d(\mathcal{A})$ is the triangulated derived category $D^b(\mathcal{A})$ as defined in ([N]). Let $\mathcal{B}$ be another $k$-linear abelian category, $D^b_d(\mathcal{B})$ the corresponding bounded derived DG category, and let $\mathcal{T}(D^b_d(\mathcal{A}), D^b_d(\mathcal{B}))$ be the triangulated category of DG quasi-functors $\mathcal{F} : D^b_d(\mathcal{A}) \to D^b_d(\mathcal{B})$ ([Dri], §16.1). Given such $\mathcal{F}$ and an integer $i$ we denote by $H^i\mathcal{F} : \mathcal{A} \to \mathcal{B}$ the composition

$$\mathcal{A} \to D^b_d(\mathcal{A}) \xrightarrow{\mathcal{F}} D^b_d(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}.$$  

**Theorem 1.** Let $\mathcal{A}$ be a small $k$-flat exact idempotent complete category and $\mathcal{B}$ a small abelian $k$-linear category.

1. Assume that a DG quasi-functor

$$\mathcal{F} : D^b_d(\mathcal{A}) \to D^b_d(\mathcal{B})$$

has the following property:

(P) The functor $H^i\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is 0 for every $i < 0$ and effaceable for every $i > 0$.

Then the functor $F := H^0\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is left exact, has a right derived DG quasi-functor ([Dri] §5)

$$RF : D^b_d(\mathcal{A}) \to D^b_d(\mathcal{B}),$$

and there is a unique isomorphism $\mathcal{F} \simeq RF$ such that the induced automorphism $F = H^0(\mathcal{F}) \simeq H^0(RF) = F$ equals $\text{Id}$. Conversely, the right derived DG quasi-functor of any left exact functor $F : \mathcal{A} \to \mathcal{B}$ satisfies property (P).

2. For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}(D^b_d(\mathcal{A}), D^b_d(\mathcal{B}))$ satisfying property (P) and every $i < 0$, we have

$$\text{Hom}_{\mathcal{T}(D^b_d(\mathcal{A}), D^b_d(\mathcal{B}))}(\mathcal{F}, \mathcal{G}[i]) = 0,$$

$$\text{Hom}_{\mathcal{T}(D^b_d(\mathcal{A}), D^b_d(\mathcal{B}))}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\text{Fct}(\mathcal{A}, \mathcal{B})}(H^0\mathcal{F}, H^0\mathcal{G}).$$

Here $\text{Fct}(\mathcal{A}, \mathcal{B})$ denotes the category of all $k$-linear functors $\mathcal{A} \to \mathcal{B}$.

**Remark 1.1.** I do not know if the analogous statement holds for merely triangulated functors.

**Remark 1.2.** It is likely that the $k$-flatness assumption on $\mathcal{A}$ is unnecessary. However, I can not prove this.

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1 i.e., a category enriched over $\text{Mod}(k)$.

2 An additive category is called idempotent complete if any its morphism $p : X \to X$ such that $p \circ p = p$ is the projection on a direct summand of a decomposition $X \simeq Y \oplus Z$.

3 That is, for every object $X \in \mathcal{A}$, there exists an admissible monomorphism $X \hookrightarrow Y$ such that the induced morphism $H^0\mathcal{F}(X) \to H^0\mathcal{F}(Y)$ is 0.
We have a similar result for bounded from below derived DG categories. If $\mathcal{A}$ is a $k$-linear abelian category we will write $D^+_{dg}(\mathcal{A})$ for the bounded from below derived DG category of $\mathcal{A}$ and $D^+_{dg}(\mathcal{A})$ for the corresponding triangulated category. Let $D^{\geq n}(\mathcal{A})$ be the full subcategory of $D^+_{dg}(\mathcal{A})$ that consists of complexes with trivial cohomology in degrees less then $n$. We say that a DG quasi-functor
$$\mathcal{F}: D^+_{dg}(\mathcal{A}) \to D^+_{dg}(\mathcal{B})$$
has property $(P')$ if
$(P')$ The functor $H_0(\mathcal{F})$ takes every object of the category $D^{\geq 0}(\mathcal{A})$ to an object of $D^{\geq 0}(\mathcal{B})$ and, for every $i > 0$, the functor $H^i(\mathcal{F}): \mathcal{A} \to \mathcal{B}$ is effaceable.

**Theorem 2.** Let $k$ be a field and let $\mathcal{A}, \mathcal{B}$ be small abelian $k$-linear categories. Assume that both categories are closed under countable direct limits and that these limits are exact.

1. Let $\mathcal{F} \in T(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ be a DG quasi-functor satisfying property $(P')$ and $F := H^0(\mathcal{F}): \mathcal{A} \to \mathcal{B}$. The functor $F$ admits a right derived DG quasi-functor $RF: D^+_{dg}(\mathcal{A}) \to D^+_{dg}(\mathcal{B})$ and there is a unique isomorphism $\mathcal{F} \cong RF$ such that the induced automorphism $F = H^0(\mathcal{F}) \cong H^0(RF) = F$ equals $Id$. Conversely, a right derived DG quasi-functor of any left exact functor $F: \mathcal{A} \to \mathcal{B}$ satisfies property $(P')$.

2. For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in T(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ satisfying property $(P')$ and every $i < 0$, we have
$$Hom_{T(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))}(\mathcal{F}, \mathcal{G}[i]) = 0,$$
$$Hom_{T(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))}(\mathcal{F}, \mathcal{G}) = Hom_{T(\mathcal{A}, \mathcal{B})}(H^0(\mathcal{F}), H^0(\mathcal{G})).$$

The main ingredient of the proof of Theorem 2 is the following construction. Let $Sh(\mathcal{A}^\circ \otimes_k \mathcal{B})$ be the category of $k$-linear contravariant functors $\mathcal{A}^\circ \otimes_k \mathcal{B} \to Mod(k)$ that are left exact with respect to both arguments. Every $k$-linear left exact functor $F: \mathcal{A} \to \mathcal{B}$ yields $s(F) \in Sh(\mathcal{A}^\circ \otimes_k \mathcal{B})$:
$$s(F)(X \times X') = Hom_B(X', F(X)).$$

Let $\mathcal{T}^+ \subset T(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors $\mathcal{F}$ such that $Ho(\mathcal{F})(D^{\geq 0}(\mathcal{A})) \subset D^{\geq n}(\mathcal{B})$ for some $n$. Using key Lemma 2.1 we construct a fully faithful embedding
$$(1.2) \quad \mathcal{T}^+ \hookrightarrow D(Sh(\mathcal{A}^\circ \otimes_k \mathcal{B}))$$
that carries every DG quasi-functor $\mathcal{F}$ satisfying property $(P')$ to $s(F) \in Sh(\mathcal{A}^\circ \otimes_k \mathcal{B}) \subset D(Sh(\mathcal{A}^\circ \otimes_k \mathcal{B})).$

**Remark 1.3.** In ([T], Th. 8.9), Toën gave an analogous description of the category of quasi-functors between the derived DG categories of (quasi)-coherent sheaves.

As another application of (1.2) we compute the Hochschild cohomology of a derived DG category. Recall (see, e.g. [K1], §5.4, [T], §8.1) that the Hochschild cohomology of a DG category $\mathcal{C}$ can be interpreted as
$$HH^i(\mathcal{C}, \mathcal{C}) = Hom_{\mathcal{T}(\mathcal{C}, \mathcal{C})}(Id\mathcal{C}, Id\mathcal{C}[i]).$$

The composition in $\mathcal{C}$ makes $HH^*(\mathcal{C}, \mathcal{C})$ a graded commutative algebra over $k$.
Theorem 3. Let \( k \) be a field, and let \( A \) be a small abelian \( k \)-linear category. There is an isomorphism of algebras
\[
(1.4) \quad HH^*(D^b_{dg}(A), D^b_{dg}(A)) \simeq Ext^*_\mathcal{S}(A^\circ \otimes_k A)(s(Id_A), s(Id_A)).
\]
If, in addition, \( A \) is closed under countable direct limits and that these limits are exact, we have
\[
(1.5) \quad HH^*(D^+_{dg}(A), D^+_{dg}(A)) \simeq Ext^*_\mathcal{S}(A^\circ \otimes_k A)(s(Id_A), s(Id_A)).
\]

Remark 1.4. This is a remarkable phenomenon the Hochschild cohomology does not change we “enlarge” the DG category. A similar result, that the Hochschild cohomology of a small DG category coincides with the Hochschild cohomology of its DG ind-completion, is due to Toën ([T], §8). An analogous statement for Grothendieck abelian categories was proved by Lowen and Van den Bergh ([LV]).

Remark 1.5. The category \( Sh(A^\circ \otimes_k A) \) has a tensor structure that extends the tensor structure on the category of left exact endofunctors \( A \rightarrow A \) given by the composition. This can be used to promote (1.4), (1.5) to isomorphisms of Gerstenhaber algebras (see, e.g. [K1], §5.4).

Notation. Given a category \( C \) we denote by \( C^\circ \) the opposite category. If \( C \) is a DG category we will write \( HoC \) for the corresponding homotopy category ([Dri], §2.7). For example, \( HoC(\text{Mod}(k)) \) denotes the homotopy category of complexes of \( k \)-modules. The derived category of right DG modules over a DG category \( C \) will be denoted by \( \mathbb{D}(C) \) ([Dri], §2.3) \(^4\). We will write \( C^{\mathsf{dg}} \) for the DG category of semi-free right DG modules over \( C \) ([BV], 1.6.1). We have a canonical equivalence of triangulated categories \( HoC^{\mathsf{dg}} \rightarrow \mathbb{D}(C) \) ([BV], 1.6.4). For DG categories \( C, C' \) we denote by \( T(C, C') \) the category of DG quasi-functors ([Dri], §16.1). If \( C' \) is a pretriangulated ([Dri], §2.4) \( T(C, C') \) has a canonical structure of triangulated category. If \( F \in T(C, C') \) we will write \( Ho(F) \) for the corresponding functor between the homotopy categories. The expression “direct limit” always means “filtrant direct limit” ([KS], §3).

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2. Proofs

Proof of theorem 1. Let \( T^+ \subset T := T(D^b_{dg}(A), D^b_{dg}(B)) \) be the full triangulated subcategory whose objects are quasi-functors \( F \) such that \( H^iF = 0 \) for sufficiently small \( i \). To prove the Theorem, we shall construct (in Lemma 2.1 below) a fully faithful embedding of \( T^+ \) into the derived category of a certain abelian category \( Sh(A^\circ \otimes_k B) \) that takes every functor \( F \in T^+ \) satisfying property (P) to an object of the heart \( Sh(A^\circ \otimes_k B) \subset D(Sh(A^\circ \otimes_k B)) \).

Under our flatness assumption on \( A \), the category \( T \) is a full subcategory of the derived category \( \mathbb{D}(D^b_{dg}(A)^\circ \otimes_k D^b_{dg}(B)) \) of right DG modules over \( D^b_{dg}(A)^\circ \otimes_k D^b_{dg}(B) \)

\(^4\)Drinfeld’s notation for this category is \( D(C) \). We use a different notation to avoid a possible confusion with Verdier’s derived category of an abelian category \( C \) that is denoted by \( D(C) \).
that consists of all $M \in \mathbb{D}(\mathcal{D}_{dg}^b(A)^{\circ} \otimes_k \mathcal{D}_{dg}^b(B))$ such that, for every $X \in \mathcal{D}_{dg}^b(A)^{\circ}$, the module $M(X) \in \mathbb{D}(\mathcal{D}_{dg}^b(B))$ belongs to the essential image of the Yoneda embedding $\mathcal{D}_{dg}^b(B) \to \mathbb{D}(\mathcal{D}_{dg}^b(B))$ ([Dri], §16.1).

Consider the restriction functor
$$\mathbb{D}(\mathcal{D}_{dg}^b(A)^{\circ} \otimes_k \mathcal{D}_{dg}^b(B)) \xrightarrow{\beta} \mathbb{D}(A^{\circ} \otimes_k B)$$

induced by the DG quasi-functor $A^{\circ} \otimes_k B \to \mathcal{D}_{dg}^b(A)^{\circ} \otimes_k \mathcal{D}_{dg}^b(B)$. By definition, the triangulated category $\mathbb{D}(A^{\circ} \otimes_k B)$ is the derived category of the abelian category $PSh := PSh(A^{\circ} \otimes_k B)$ of $k$-linear presheaves i.e., the category of $k$-linear contravariant functors $A^{\circ} \otimes_k B \to Mod(k)$. Consider a Grothendieck topology on $A^{\circ} \otimes_k B$ whose covers are maps of the form $f \otimes g : Y \otimes Y' \to X \otimes X'$, where $X, Y \in A^{\circ}$, $X', Y' \in B$, and $f : Y \to X$, $g : Y' \to X'$ are admissible epimorphisms 5 i.e., a sieve $\mathcal{C}$ over $X \otimes X'$ is a covering sieve if there exist $f : Y \to X$, $g : Y' \to X'$ as above such that $Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X' \in \mathcal{C}$. The axioms of Grothendieck topology (see, e.g. [KS], §16.1) are immediate except for the one which is the following statement: for every cover $Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X'$ and every morphism $Z \otimes Z' \xrightarrow{\phi} X \otimes X'$ there exists a cover $T \otimes T' \xrightarrow{\psi} Z \otimes Z'$ and a morphism $T \otimes T' \xrightarrow{\psi} Y \otimes Y'$ such that $(f \otimes g) \circ \psi = \phi \circ (p \otimes q)$, which is a consequence of the base change axiom of exact category ([Q], §2). Let $Sh := Sh(A^{\circ} \otimes_k B)$ be the subcategory of $PSh$ that consists of objects satisfying the sheaf property. Explicitly, objects of the category $Sh(A^{\circ} \otimes_k B)$ are contravariant functors $A^{\circ} \otimes_k B \to Mod(k)$ that are left exact with respect to both arguments. The embedding $Sh \to PSh$ has a left adjoint functor (sheafification)
$$\gamma : PSh \to Sh,$$

which is exact ([KS], §17.4). We denote by $\gamma : D(PSh) \to D(Sh)$ the induced functor between the derived categories. The composition
$$\mathbb{D}(\mathcal{D}_{dg}^b(A)^{\circ} \otimes_k \mathcal{D}_{dg}^b(B)) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh)$$

is not fully faithful in general, however, we have the following result.

**Lemma 2.1.** (cf. [T], Th. 8.9) Let $D^+ \subset \mathbb{D}(\mathcal{D}_{dg}^b(A)^{\circ} \otimes_k \mathcal{D}_{dg}^b(B))$ be the full subcategory whose objects are DG modules $M$ such that $\beta(M)$ is bounded from below. Then the functor
$$S : D^+ \xrightarrow{\beta} D^+(PSh) \xrightarrow{\gamma} D^+(Sh)$$

is an equivalence of categories.

**Proof.** The category $\mathcal{D}_{dg}^b(A)^{\circ} \otimes_k \mathcal{D}_{dg}^b(B)$ is the DG quotient of the category $C_{dg}^b(A)^{\circ} \otimes_k C_{dg}^b(B)$ by the full subcategory whose objects are of the form $X \otimes X'$, where either $X$ or $X'$ is acyclic. It then follows from ([Dri], Theorem 1.6.2) that the functor
$$\beta : \mathbb{D}(\mathcal{D}_{dg}^b(A)^{\circ} \otimes_k \mathcal{D}_{dg}^b(B)) \to \mathbb{D}(C_{dg}^b(A)^{\circ} \otimes_k C_{dg}^b(B)) = D(PSh)$$

is fully faithful and that its essential image consists of all DG-modules $M \in \mathbb{D}(C_{dg}^b(A)^{\circ} \otimes_k C_{dg}^b(B))$ that carry every $X \otimes X'$ with the above property to an acyclic complex. Identifying the category $\mathbb{D}(C_{dg}^b(A)^{\circ} \otimes_k C_{dg}^b(B))$ with $D(PSh)$ and observing that the subcategories of acyclic complexes in the homotopy categories $HoC_{dg}^b(A)$, $HoC_{dg}^b(B)$

5By definition, admissible epimorphisms $Y \to X$ in $A^{\circ}$ are admissible monomorphisms $X \to Y$ in $A$. 

are generated by short exact sequences ([N], §1) we exhibit $\mathbb{D}(D^b_d(A) \otimes_k D^b_d(B))$ as a full subcategory $\mathcal{R} \subset D(PSh)$ whose objects are complexes $F'$ of presheaves satisfying the following two conditions:

- For any exact sequence $0 \to Z \to Y \to X \to 0$ in $A^o$ and any $X' \in B$ the total complex of

$$F'(X \otimes X') \to F'(Y \otimes X') \to F'(Z \otimes X')$$

is acyclic.

- For any $X \in A^o$ and any exact sequence $0 \to Z' \to Y' \to X' \to 0$ in $B$ the total complex of

$$F'(X \otimes X') \to F'(X \otimes Y') \to F'(X \otimes Z')$$

is acyclic.

Observe that, for every $F \in \mathcal{R}$ and an exact sequence $0 \to Z \to Y \to X \to 0$ in $A^o$, we have a long exact sequence of $k$-modules

$$\cdots H^{m-1}(F(Z \otimes X')) \to H^m(F(X \otimes X')) \to H^m(F(Y \otimes X')) \to H^m(F(Z \otimes X')) \to \cdots$$

The equivalence of categories

$$\beta : \mathbb{D}(D^b_d(A)^o \otimes_k D^b_d(B)) \sim \mathcal{R} \subset D(PSh)$$

carries $D^+$ to the subcategory $\mathcal{R}^+$ of $\mathcal{R}$ that consists of bounded from below complexes.

The derived category of sheaves $D(Sh)$ is the quotient of the derived category of presheaves by the subcategory $\mathcal{I}_{lac} \subset D(PSh)$ of locally (for our Grothendieck topology on $A^o \otimes_k B$) acyclic complexes ([BV], §1.11). We shall prove that

$$\mathcal{R}^+ \subset \mathcal{I}_{lac}^\perp,$$

where $\mathcal{I}_{lac}^\perp$ denotes the right orthogonal complement to $\mathcal{I}_{lac}$ in $D(PSh)$ ([BV] §1.1); i.e.

$$\text{Hom}_{D(PSh)}(G, F') = 0.$$~(2.3)

for every $G \in \mathcal{I}_{lac}$ and $F' \in \mathcal{R}^+$. Without loss of generality we may assume that $F'$ has trivial cohomology in negative degrees: $F' = F^0 \to F^1 \to \cdots$. Let $\tilde{F}' = F^0 \to F^1 \to \cdots$ be the corresponding complex of sheaves. Since the category of sheaves has enough injective objects (see, e.g. [KS], Th. 9.6.2, 18.1.6) there exists a complex $I = I^0 \to I^1 \to \cdots$ of injective sheaves together with a morphism $\tilde{F}' \to I$ which is an isomorphism in the derived category of sheaves. Let us show that the composition

$$\delta : F' \to \tilde{F}' \to I$$

is an isomorphism in the derived category of presheaves. Indeed, every injective sheaf, viewed as a presheaf, is an object of $\mathcal{R}$. Thus $I$ and $\text{cone}(\delta)$ are in $\mathcal{R}^+$. Assuming that $\text{cone}(\delta) \neq 0$ choose the smallest integer $m$ such that

$$0 \neq H^m(\text{cone}(\delta)) \in PSh.$$~(2.4)

Then, there exist an object $X \otimes X' \in A^o \otimes_k B$ and a nonzero element $a \in H^m(\text{cone}(\delta))(X \otimes X')$. Since the sheafification of $H^m(\text{cone}(\delta))$ is 0 there exists a cover $p : Y \otimes Y' \to X \otimes X'$ such that

$$0 = p^*a \in H^m(\text{cone}(\delta))(Y \otimes Y').$$

Writing $p$ as a composition

$$Y \otimes Y' \xrightarrow{1 \otimes \delta} Y \otimes X' \xrightarrow{f \otimes 1} X \otimes X'$$
we may assume \((f \otimes 1)^*a = 0\) (otherwise, we replace \(X \otimes X'\) by \(Y \otimes X'\)). Let us look at the following fragment of the long exact sequence (2.2) applied to \(F = \text{cone}(\delta)\) and the exact sequence \(0 \rightarrow Z \rightarrow Y \xrightarrow{f} X \rightarrow 0\):

\[
H^{m-1}(\text{cone}(\delta))(Z \otimes X') \rightarrow H^m(\text{cone}(\delta))(X \otimes X') \rightarrow H^m(\text{cone}(\delta))(Y \otimes X').
\]

Since, by our assumption, \(H^{m-1}(\text{cone}(\delta)) = 0\), it follows that \((f \otimes 1)^*\) is injective and, hence, \(a = 0\). This contradiction proves that \(\text{cone}(\delta) = 0\) i.e., \(\delta\) is a quasi-isomorphism. Thus, to complete the proof of (2.4) it suffices to show that

\[
\text{Hom}_{D(PSh)}(G, I) = 0,
\]

for every \(G \in \mathcal{I}_{lac}\) and every bounded from below complex of injective sheaves \(I\). Indeed, every morphism \(h : G' \rightarrow I\) in the derived category is represented by a diagram in \(C(PSh(A^0 \otimes_k B))\)

\[
G' \leftarrow G^' \xrightarrow{K'} I',
\]

where the first arrow is a quasi-isomorphism (and, in particular, \(G^' \in \mathcal{I}_{lac}\)). If \(h'\) is homotopic to 0 then \(h = 0\) in the derived category. Thus, it is enough to show that

\[
\text{Hom}_{K(PSh)}(G^', I) = 0,
\]

where \(K(PSh)\) denotes the homotopy category of complexes. We have

\[
\text{Hom}_{K(PSh)}(G^', I) \xrightarrow{\sim} \text{Hom}_{K(Sh)}(\tilde{G}'', I) \xrightarrow{\sim} \text{Hom}_{D(Sh)}(\tilde{G}'', I).
\]

The first arrow is an isomorphism because all terms of the complex \(I'\) are sheaves; the second arrow is an isomorphism by ([KS], Lemma 13.2.4). Finally, the group \(\text{Hom}_{D(Sh)}(\tilde{G}'', I)\) is trivial because the sheafification \(\tilde{G}''\) is 0 in \(D(Sh)\).

To finish the proof of the lemma, we observe that, for every triangulated category \(\mathcal{C}\) and its full triangulated subcategory \(\mathcal{I}\), the composition

\[
\mathcal{I}^+ \rightarrow \mathcal{C} \rightarrow \mathcal{C}/\mathcal{I}
\]

is a fully faithful embedding: for every \(X, Y \in \mathcal{C}\)

\[
\text{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) := \text{colim}_{f \in X' \rightarrow X} \text{Hom}_{\mathcal{C}}(X', Y),
\]

where the colimit is taken over the filtrant category of pairs \((X' \in \mathcal{C}, f : X' \rightarrow X)\) such that \(\text{cone} f \in \mathcal{I}\). If \(Y \in \mathcal{I}^+\), then

\[
\text{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(X', Y),
\]

and, hence,

\[
\text{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y).
\]

Applying this remark to \(\mathcal{C} = D(PSh), \mathcal{I} = \mathcal{I}_{lac}\) and using (2.4) we conclude that the functor \(\mathcal{R}^+ \xrightarrow{\sim} D(Sh)\) is fully faithful and, hence, so is the composition \(\mathcal{D}^+ \xrightarrow{\sim} \mathcal{R}^+ \xrightarrow{\sim} D(Sh)\). The essential image the functor \(\mathcal{R}^+ \xrightarrow{\sim} D(Sh)\) coincides with \(D^+(Sh)\) because every complex of injective sheaves viewed as a complex of presheaves is an object of \(\mathcal{R}^+\).

**Remark 2.2.** Applying Lemma 2.1 to \(k = \mathbb{Z}\) and \(A\) being the category of free abelian groups of finite rank we obtain the following statement: for every small abelian category \(B\)

\[
\mathcal{D}^+(D^b(A)(B)) \xrightarrow{\sim} D^+(PSh(B)) = D^+(Ind(B)),
\]
Let us show that complex of projective objects. Thus it is enough to show that every projective object of $B$ in $D$ is a fully faithful embedding.

that commutes with arbitrary direct sums ([BV], §2.5) $\mathbb{D}(D^b_{dg}(B)) \to D(Ind(B))$ is not an equivalence of categories in general. In fact, the functor (2.5) factors as

\[ \mathbb{D}(D^b_{dg}(B)) \to HoC(Ind(B))/HoC^b_{ac}(B) \xrightarrow{p} D(Ind(B)), \]

where $HoC^b_{ac}(B)$ is the smallest triangulated subcategory of the homotopy category of acyclic complexes $HoC_{ac}(Ind(B))$ that contains finite acyclic complexes $HoC^b_{ac}(B)$ and closed under arbitrary direct sums; the functor $p$ is the projection

\[ HoC(Ind(B))/HoC^b_{ac}(B) \to HoC(Ind(B))/HoC_{ac}(Ind(B)). \]

The equivalence $\phi$ can be constructed as follows. Let $C^b_{ac}(B)$ be the full subcategory of the DG category $C(Ind(B))$ whose objects are those of $HoC^b_{ac}(B)$. The DG quasi-functor $D^b_{dg}(B) \to C(Ind(B))/C^b_{ac}(B)$ extends uniquely to a quasi-functor

\[ \phi_{dg} : D^b_{dg}(B) \to C(Ind(B))/C^b_{ac}(B) \]

that commutes with arbitrary direct sums ([BV], §1.6.1). Define

\[ \phi := Ho\phi_{dg}. \]

Let us show that $\phi$ is an equivalence of categories. The subcategory $HoC^b_{ac}(B) \subset HoC(Ind(B))$ is generated by compact objects (e.g., objects of $HoC^b_{ac}(B)$); it follows that the projection $HoC(Ind(B)) \to HoC(Ind(B))/HoC^b_{ac}(B)$ carries compact objects of $HoC(Ind(B))$ to compact objects of the quotient category ([BV], §1.4.2). In particular, in the following commutative diagram

\[
\begin{array}{ccc}
D^b_{dg}(B) & = & D^b_{dg}(B) \\
\downarrow i & & \downarrow j \\
\mathbb{D}(D^b_{dg}(B)) & \xrightarrow{\phi} & HoC(Ind(B))/HoC^b_{ac}(B)
\end{array}
\]

the image of $j$ consists of compact objects. The same is true for the image of $i$ ([BV], §1.7). The functors $i,j$ are fully faithful and their images generate the categories $\mathbb{D}(D^b_{dg}(B)), HoC(Ind(B))/HoC^b_{ac}(B)$ respectively. It follows that $\phi$ is an equivalence of categories.

In general, (e.g., if $B$ is the category of finitely generated modules over a finite group) the projection $p$ is not conservative. However, if the category $B$ has finite homological dimension the objects of $D^b_{dg}(B)$ are compact in $D^b_{dg}(Ind(B))$ \(^6\) and the above argument proves that (2.5) is an equivalence of categories.

**Corollary 2.3.** The composition

\[ \begin{array}{ccc}
S : T^+ & \xrightarrow{\alpha} & \mathbb{D}(D^b_{dg}(A)^{\circ} \otimes_k D^b_{dg}(B)) \\
& & \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh)
\end{array} \]

is a fully faithful embedding.

\(^6\) Indeed, under our finiteness assumption every complex in $D^b_{dg}(B)$ is quasi-isomorphic to a finite complex of projective objects. Thus it is enough to show that every projective object of $B$ is compact in $D(Ind(B))$. This is clear because every such object is projective and compact in $Ind(B)$.
Consider the Yoneda embedding

\[ s : \text{Fun}(\mathcal{A}, \mathcal{B}) \to \text{PSh} \]

that takes a functor \( F \in \text{Fun}(\mathcal{A}, \mathcal{B}) \) to the presheaf

\[ s(F)(X \times X') = \text{Hom}_\mathcal{B}(X', F(X)) \]

If \( F \) is left exact then \( s(F) \) is actually a sheaf.

Let \( \mathcal{F} \in \mathcal{T} \) be a DG quasi-functor satisfying property (P). It follows from the definition of \( \mathcal{T}^+ \) given at the beginning of this section that \( \mathcal{F} \in \mathcal{T}^+ \). We shall prove that \( S(\mathcal{F}) \xrightarrow{\sim} s(H^0\mathcal{F}) \). Having in mind applications to Theorem 2 we will actually show a slightly more general statement. Namely, let us extend the functor (2.7) to a larger category:

\[ S' : \mathcal{T}(D^b_{dg}(A), D^+_{dg}(B)) \to \mathbb{D}(D^b_{dg}(A) \otimes_k D^+_{dg}(B)) \xrightarrow{\beta'} D(\text{PSh}) \xrightarrow{\gamma} D(\text{Sh}). \]

**Lemma 2.4.** Let \( \mathcal{F} \in \mathcal{T}(D^b_{dg}(A), D^+_{dg}(B)) \) be a DG quasi-functor such that \( H^i\mathcal{F} \) is zero for \( i < 0 \) and effaceable for \( i > 0 \). Set \( s(F) = s(H^0\mathcal{F}) \subset \text{Sh} \subset D(\text{Sh}) \).

Then the complex \( S'(\mathcal{F}) \in D(\text{Sh}) \) is canonically quasi-isomorphic to \( s(\mathcal{F}) \).

**Proof.** By definition, the cohomology presheaves of the complex \( \beta'\alpha'(\mathcal{F}) \in D(\text{PSh}) \) are given by the formula

\[ H^i(\beta'\alpha'(\mathcal{F}))(X \otimes X') = \text{Hom}_{D^+_{dg}(B)}(X', \text{Ho}(\mathcal{F})(X)[i]). \]

Since the negative cohomology of the complex \( \text{Ho}(\mathcal{F})(X) \in D^+(B) \) vanishes the same is true for \( \beta'\alpha' \mathcal{F} \) and, thus, we have

\[ H^0(\beta'\alpha'(\mathcal{F}))(X \otimes X') = \text{Hom}_{D^+_{dg}(B)}(X', H^0\mathcal{F}(X)) = s(F). \]

It remains to prove that for every \( i > 0 \) the sheafification of the presheaf \( H^i(\beta'\alpha'(\mathcal{F})) \) equals zero. Given an integer \( j \) define presheaves \( G^{i,j} \) to be

\[ G^{i,j}(X \otimes X') = \text{Hom}_{D^+_{dg}(B)}(X', \tau_{\leq j}(\text{Ho}(\mathcal{F})(X))[i]). \]

We shall show by induction on \( j \) that for every \( i > 0 \) and every \( j \) the sheafification of \( G^{i,j} \) is 0. This would complete the proof since \( G^{i,j} \) is isomorphic to \( H^i(\beta'\alpha'(\mathcal{F}))(X \otimes X') \) for \( j \geq i \). For every \( i > 0 \) and every element \( v \) of the group

\[ G^{i,0}(X \otimes X') = \text{Ext}_B^{i}(X', H^0\mathcal{F}(X)) \]

there exists an epimorphism \( Y' \to X' \) such that \( v \) is annihilated by the map

\[ \text{Ext}_B^{i}(X', H^0\mathcal{F}(X)) \to \text{Ext}_B^{i}(Y', H^0\mathcal{F}(X)) \]

([KS], Exercise 13.17). This proves that the sheafification of \( G^{i,0} \) is 0. For the induction step, consider the distinguished triangle

\[ \tau_{\leq j}(\text{Ho}(\mathcal{F})(X)) \to \tau_{\leq j+1}(\text{Ho}(\mathcal{F})(X)) \to H^{j+1}\mathcal{F}(X)[-j-1] \]

and the corresponding long exact sequence

\[ \cdots \to G^{i,j}(X \otimes X') \to G^{i,j+1}(X \otimes X') \to \text{Hom}_{D^+_{dg}(B)}(X', H^{j+1}\mathcal{F}(X)[-j-1+i]) \to \cdots. \]

It follows that \( G^{i,j+1} \) fits in a long exact sequence

\[ \cdots \to G^{i,j} \to G^{i,j+1} \to \text{Ext}_B^{-j-1}(\cdot, H^{j+1}\mathcal{F}(\cdot)) \to \cdots. \]

---

7The vanishing of \( H^i\mathcal{F} \) implies that \( F \) is left exact and, hence, \( s(F) \) is a sheaf.
The sheafification of $G^{i,j}$ is 0 by the induction assumption, the sheafification of $Ext^j_{R^c}(\cdot, H^{j+1}F(\cdot))$ is 0 because the functor $H^{j+1}F$ is effaceable. Hence, the sheafification of $G^{i,j+1}$ is 0 as well. □

Now we are ready to prove the second part of the theorem. Given quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ satisfying property (P) we have by Lemmas 2.1, 2.4

\[(2.8) \quad \text{Hom}_T(\mathcal{F}, \mathcal{G}[i]) \sim Hom_{D(Sh)}(S(\mathcal{F}), S(\mathcal{G})[i]) \sim Ext^i_{Sh}(s(H^0\mathcal{F}), s(H^0\mathcal{G})).\]

In particular, $\text{Hom}_T(\mathcal{F}, \mathcal{G}[i])$ is isomorphic to $\text{Hom}_{Fun(A,B)}(H^0\mathcal{F}, H^0\mathcal{G})$ for $i = 0$ (since the functor $s : Fun(A,B) \to PSh$ is fully faithful) and to 0 for $i < 0$.

To prove the first part of the theorem we need to recall some facts about DG categories and derived functors. Let $f : C_1 \to C_2$ be a DG functor between small DG categories. Then the restriction functor $f_* : D(C_2) \to D(C_1)$ admits a left and a right adjoint functors (the derived induction and co-induction functors)

\[(2.9) \quad f^*, f^! : D(C_1) \to D(C_2)\]

([Dri], §14.12). In particular, we have the canonical morphisms

\[(2.10) \quad Id \to f_* f^*, \quad f_* f^! \to Id\]

\[Id \to f^! f_* \quad f^* f_* \to Id.\]

It also follows from the adjunction property that $f^*$ commutes with arbitrary direct sums and that $f^!$ commutes with arbitrary direct products. If the the functor $Ho(f) : Ho(C_1) \to Ho(C_2)$ is fully faithful so is $f_*$ and the first two morphisms in (2.10) are isomorphisms.

Recall the definition of the derived DG quasi-functor $RF$ of a left exact functor $F : A \to B$ from ([Dri], §16). Consider the functor

\[T(A, D^b_{dg}(B)) \hookrightarrow D(C^b_{dg}(A)^\circ \otimes_k D^b_{dg}(B)) \xrightarrow{j^*} D(D^b_{dg}(A)^\circ \otimes_k D^b_{dg}(B))\]

induced by the projection

\[f : C^b_{dg}(A)^\circ \otimes_k D^b_{dg}(B) \to D^b_{dg}(A)^\circ \otimes_k D^b_{dg}(B).\]

Given a $k$-linear functor $F \in Fun(A,B) \to T(A, D^b_{dg}(B))$ we define the "derived functor"

\[(2.11) \quad "RF" = f^*(F) \in D(D^b_{dg}(A)^{op} \otimes_k D^b_{dg}(B)).\]

The right derived DG quasi-functor $RF : D^b_{dg}(A) \to D^b_{dg}(B)$, if it exists, is an object of $T(D^b_{dg}(A), D^b_{dg}(B))$ whose image in $D(D^b_{dg}(A)^{op} \otimes_k D^b_{dg}(B))$ is $"RF"$.

**Lemma 2.5.** Assume that $F$ is left exact. Then $"RF" \in D^+ \subset D(D^b_{dg}(A)^{op} \otimes_k D^b_{dg}(B))$ and the functor $S : D^+ \hookrightarrow D(Sh)$ takes $"RF"$ to $s(F)$.

**Proof.** Let $\beta : D(D^b_{dg}(A)^{op} \otimes_k D^b_{dg}(B)) \to D(PSh)$ be the restriction functor, and let $\gamma : D(PSh) \to D(Sh)$ be the sheafification functor. As explained in ([Dri], §5) the presheaves $H^i(\beta("RF"))$ can be computed as follows:

\[(2.12) \quad H^i(\beta("RF"))(X \otimes X') = colim_Q Hom_{D^b_{dg}(B)}(X', F(Y)[i]),\]

where the colimit is taken over the filtrant category $Q$ of pairs $(Y' \in HoC^b_{dg}(A), f \in Hom_{HoC^b_{dg}(A)}(X, Y'))$ such that cone$(f)$ is acyclic. As the subcategory $Q' \subset Q$ consisting of pairs $(Y', f)$ with $Y' = 0$ for $j < 0$ is cofinal in $Q$, the category $Q$ in the
equation (2.12) can be replaced by $Q’$. This proves that $\text{``RF''} \in \mathbb{D}^+$. Let us show that $\gamma \circ \beta(\text{``RF''}) \simeq s(F)$. We have

$$H^0(\beta(\text{``RF''}))(X \otimes X’) = \text{colim}_{Q’} \text{Hom}_{\mathbb{D}^+(B)}(X’, F(Y’)) \simeq \text{colim}_{Q’} \text{Hom}_{\mathbb{D}^+(B)}(X’, F(X)) = s(F)(X \otimes X’).$$

It remains to prove that, for every $i > 0$, the sheafification of $H^i(\beta(\text{``RF''}))$ is 0. Let $s$ be the section of $H^i(\beta(\text{``RF''}))(X \otimes X’)$ represented by an element

$$\tilde{s} \in \text{Hom}_{\mathbb{D}^+(B)}(X’, F(Y’)[i]),$$

where $X \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots$ is an object of $Q’$. Looking at the diagram

$$
\begin{array}{c}
X \xrightarrow{f} Y^0 \rightarrow Y^1 \rightarrow \cdots \\
\downarrow \text{\text{Id}} \downarrow \text{Id} \downarrow \\
Y^0 \xrightarrow{\text{Id}} Y^0 \rightarrow 0 \rightarrow \cdots
\end{array}
$$

we see that the pullback $(f \otimes \text{Id})^*s \in H^i(\beta(\text{``RF''}))(Y^0 \otimes X’)$ is represented by an element of the group $\text{Hom}_{\mathbb{D}^+(B)}(X’, F(Y’)[i]) = \text{Ext}^i_{B}(X’, F(Y’))$. For any positive $i$ every element of this group is annihilated by the map $\text{Ext}^i_{B}(X’, F(Y’)) \rightarrow \text{Ext}^i_{B}(Y’, F(Y’))$ for some epimorphism $Y’ \rightarrow X’$.

Let us prove the first part of the theorem. Let $F \in \mathcal{T} \subset \mathbb{D}(D_{dg}^b(A)^0 \otimes_k D_{dg}^b(B))$ be a DG quasi-functor satisfying property (P) together with an isomorphism $F \simeq H^0F$. We need to construct an isomorphism $F \simeq \text{``RF''}$. By Lemmas 2.4, 2.5 $F$, $\text{``RF''}$ are objects of $\mathbb{D}^+$. By Lemma 2.1 the functor $S : \mathbb{D}^+ \rightarrow D(Sh)$ is fully faithful. Thus, constructing an isomorphism $F \simeq \text{``RF''}$ is equivalent to producing an isomorphism $S(F) \simeq S(\text{``RF''})$ in $D(Sh)$ which was done in Lemmas 2.4, 2.5. Theorem 1 is proved.

**Proof of theorem 2.** Let $T^+ \subset \mathcal{T} := \mathcal{T}(D_{dg}^b(A), D_{dg}^+(B))$ be the full triangulated subcategory whose objects are quasi-functors $F$ such that, for some integer $n$, we have

$$\text{Ho}(F)(D_{\geq 0}(A)) \subset D_{\geq n}(B).$$

We shall prove that the composition

$$T^+ \hookrightarrow \mathbb{D}(D_{dg}^b(A)^0 \otimes_k D_{dg}^+(B)) \xrightarrow{\text{Res}} \mathbb{D}(D_{dg}^b(A)^0 \otimes_k D_{dg}^+(B)) \rightarrow D(Sh)$$

is a fully faithful embedding. Here $\text{Res}$ denotes the restriction functor induced by the embedding

$$D_{dg}^b(A)^0 \otimes_k D_{dg}^+(B) \rightarrow D_{dg}^+(A)^0 \otimes_k D_{dg}^+(B).$$

To show this we need to introduce a bit of notation. If $C$ is an abelian category closed under countable direct sums and

$$X^0 \xrightarrow{\phi_0} X^1 \xrightarrow{\phi_1} X^2 \xrightarrow{\phi_2} \cdots$$

is a diagram of complexes $X^i \in C(C)$, we set

$$\text{hocolim} \ X^i = \text{cone} \left( \bigoplus_{i} X^i \xrightarrow{\oplus \phi_i} \bigoplus_{i} X^i \right) \in C(C),$$

where $\nu_{X^i} := \text{Id}_{X^i} - \phi_i : X^i \rightarrow \bigoplus_{i} X^i$. There is a canonical morphism

$$\text{hocolim} \ X^i \rightarrow \text{colim} \ X^i,$$
which is a quasi-isomorphism if countable direct limits in \( \mathcal{C} \) are exact. If this is the case, every morphism \( X \rightarrow X' \) of diagrams that is a term-wise quasi-isomorphism induces a quasi-isomorphism of the homotopy colimits \(^8\). Dually, for a category \( \mathcal{C} \) closed under countable products and a diagram

\[
\cdots \rightarrow X_2 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0,
\]

we set

\[
\text{holim} X_i = \text{cone} (\prod_i X_i \xrightarrow{\nu} \prod_i X_i)[-1],
\]

where \( \nu_i := p_i - \phi_i p_{i+1} : \prod X_i \rightarrow X_i \) and \( p_i : \prod X_i \rightarrow X_i \) are the projections.

Let \( \mathbb{D}' \subset \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})) \) be the full subcategory whose objects are the covariant DG functors \( M : D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})^o \rightarrow C(\text{Mod}(k)) \) such that, for every \( X \in D_{dg}^+(\mathcal{A}) \) and \( X' \in D_{dg}^+(\mathcal{B}) \), the canonical morphism

\[
(2.14) \quad M(X \otimes X') \rightarrow \text{holim} M(X \otimes \tau_{<1} X'),
\]

is a quasi-isomorphism, and, for every \( X \in D_{dg}^+(\mathcal{A}) \) and every bounded \( X' \in D_{dg}^+(\mathcal{B}) \), the canonical morphism

\[
(2.15) \quad \text{hocolim} M(\tau_{<1} X \otimes X') \rightarrow M(X \otimes X'),
\]

is a quasi-isomorphism.

**Remark 2.6.** Since countable direct limits are exact in \( \mathcal{B} \), the morphism \( \text{hocolim} \tau_{<1} X' \rightarrow X' \) is a quasi-isomorphism. Thus, property (2.14) is implied by the following: for every integer \( n \) and a countable collection \( X^n \in D_{dg}^{\geq n}(\mathcal{B}) \), the morphism

\[
(2.16) \quad M(X \otimes \oplus_i X^n) \rightarrow \prod_i M(X \otimes X^n)
\]

is a quasi-isomorphism.

**Remark 2.7.** Since directed limits are exact in \( \text{Mod}(k) \) property (2.15) is equivalent to the following: for every \( X \in D_{dg}^+(\mathcal{A}) \) and \( X' \in \mathcal{B} \), we have

\[
(2.17) \quad \text{colim} H^0(\text{hocolim} \tau_{<1} X \otimes X')) \xrightarrow{\sim} H^0(\text{holim} M(X \otimes X')).
\]

**Lemma 2.8.** The restriction functor

\[
\mathbb{D}' \xrightarrow{\text{Res}} \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))
\]

is an equivalence of categories.

**Proof.** We shall first consider the restriction

\[
f_* : \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})) \rightarrow \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))
\]

and prove that \( f' \) and \( f_* \) define mutually inverse equivalences of categories

\[
(2.18) \quad f'(\mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))) \subset \mathbb{D}',
\]

where \( \mathbb{D}' \) is the full subcategory of \( \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})) \) whose objects are DG functors \( M \) satisfying the property (2.14). Let us check that

\[
(2.19) \quad f'(\mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))) \subset \mathbb{D}'.
\]

---

\(^8\)For the last property, it suffices to assume that countable direct sums are exact in \( \mathcal{C} \).
For every DG functor $f : C_1 \to C_2$ between DG categories over a field, the functor
$f^\dagger : \mathbb{D}(C_1) \to \mathbb{D}(C_2)$ admits the following concrete description: if $M : C_1 \to C(\text{Mod}(k))$
is a contravariant DG functor and $X$ is an object of $C_2$, we have

$$f^\dagger(M)(X) = \text{Hom}_{\mathbb{D}(C_1)}(f^\dagger(M), X, M).$$

(2.19)

Here $\mathbb{D}(C_1)$ denotes the DG derived category of right $C_1$-modules, $f^\dagger(M)$ the derived restriction functor, and $\text{Hom}_{C_2}(\cdot, X)$ is the image of $X$ under the Yoneda embedding $C_2 \to \mathbb{D}(C_2)$.

We shall prove that

$$\text{hocolim} \text{Hom}(\cdot, X \otimes_{\tau_{<i}} X') \to f_* \text{Hom}(\cdot, X \otimes X')$$

is an isomorphism in $\mathbb{D}(D^+_d(A)^o \otimes_k D^b_d(B))$. Together with (2.19) it will imply

(2.18). By definition of the tensor product of DG categories, for every $Y \otimes Y' \in D^+_d(A)^o \otimes_k D^b_d(B)$,

$$\text{Hom}(Y \otimes Y', X \otimes X') = \text{Hom}(Y, X) \otimes_k \text{Hom}(Y', X').$$

Hence, it is enough to check that the morphism

$$\text{hocolim} \text{Hom}_{D^+_d(B)}(Y', \tau_{<i} Y) \to \text{Hom}_{D^+_d(B)}(Y', Y)$$

is a quasi-isomorphism, for every $Y' \in D^b_d(B)$. Using the exactness of direct limits in $\text{Mod}(k)$ the last assertion is reduced to the formula

$$\text{colim} \text{Hom}_{D^+_d(B)^o}(Y', \tau_{<i} Y) \simeq \text{Hom}_{D^+_d(B)^o}(Y', Y),$$

which holds because the group $\text{Hom}_{D^+_d(B)^o}(Y', \tau_{>i} Y)$ is trivial for large $i$. This proves the

(2.18).

Since the functor $\text{Ho}(f)$ is fully faithful, we have

$$f_* f^\dagger \simeq \text{Id}.$$
proved above (with $\mathcal{A}$ replaced by $\mathcal{B}$). Since $g^*$ commutes with arbitrary direct sums and since $\mathbb{D}(D^b_{dg}(\mathcal{A})^\circ \otimes_k D^b_{dg}(\mathcal{B}))$ is the smallest triangulated subcategory that contains representable functors and closed under direct sums, $g^*(M)$ is an object of $\mathbb{D}(D^+_d(A)^\circ \otimes_k D^b_{dg}(\mathcal{B}))$ for every $M$. By (2.15) the functor $g_*$ is conservative when restricted to $\mathbb{D}^f$ and the adjoint functor $g^*$ is fully faithful (because $Ho(g)$ is fully faithful). Hence, we have

$$Id \sim g_* g^*, \quad (g^* g_*)|_{\mathbb{D}^f} \sim Id.$$  

Combining equations (2.17) and (2.20) we see that the functors $Res$ and $f^! g^*$ define mutually inverse equivalences between the category $\mathbb{D}^f$ and the category $\mathbb{D}(D^b_{dg}(\mathcal{A})^\circ \otimes_k D^b_{dg}(\mathcal{B}))$.

Consider the composition

(2.22) \quad $\mathbb{D}^f \xrightarrow{Res} \mathbb{D}(D^b_{dg}(\mathcal{A})^\circ \otimes_k D^b_{dg}(\mathcal{B})) \xrightarrow{\beta} D(PSh) \to D(Sh).$

Combining Lemmas 2.1 and 2.8 we get the following.

Corollary 2.9. Let $\mathbb{D}^f \subset \mathbb{D}^f$ be the full subcategory whose objects are DG modules $\mathcal{M}$ such that $\beta \circ Res(\mathcal{M})$ is bounded from below. Then (2.22) induces an equivalence of categories

$$S: \mathbb{D}^f \sim D^+(Sh).$$

Lemma 2.10. The functor $\mathcal{T} \hookrightarrow \mathbb{D}(D^b_{dg}(\mathcal{A})^\circ \otimes_k D^b_{dg}(\mathcal{B}))$ carries $\mathcal{T}^+$ into $\mathbb{D}^f$.

Proof. Let us show that every $\mathcal{F} \in \mathcal{T}$ satisfies property (2.6). By definition of $\mathcal{T}$, for every $X \in D^+_{dg}(\mathcal{A})$, there exists $Y \in D^+_{dg}(\mathcal{B})$ and an isomorphism

$$\mathcal{F}(X \otimes ?) \simeq Ho_{D^+_d(\mathcal{B})}(?, Y)$$

in the derived category of right $D^+_d(\mathcal{B})$-modules. Property (2.6) follows because the morphism

$$Ho_{D^+_d(\mathcal{B})}(?, X^\otimes Y) \to \prod_i Ho_{D^+_d(\mathcal{B})}(X^\otimes Y)_i$$

is a quasi-isomorphism.

Let us show that every $\mathcal{F} \in \mathcal{T}^+$ satisfies the property (2.7). Denote by $Ho(\mathcal{F}) : D^+_{dg}(\mathcal{A}) \to D^+(\mathcal{B})$ the triangulated functor associated with $\mathcal{F}$. By definition of $Ho(\mathcal{F})$ there is a functorial isomorphism

(2.23) \quad $Ho(\mathcal{F}(X \otimes X')) \simeq Ho_{D^+(\mathcal{B})}(X', Ho(\mathcal{F}(X)))$

In order to check (2.7) we will prove a stronger statement: for every $X' \in \mathcal{B}$ the morphism

(2.24) \quad $Ho_{D^+(\mathcal{B})}(X', Ho(\mathcal{F}(\tau_{< n} X)) \to Ho_{D^+(\mathcal{B})}(X', Ho(\mathcal{F}(X)))$

is an isomorphism for sufficiently large $n$. By definition of $\mathcal{T}^+$ we can find an integer $N$ such that the functor $Ho\mathcal{F}$ carries every object of $D^>_{> N}(\mathcal{A})$ to an object $D^>_{> N}(\mathcal{B})$. In particular, for every $n > N$, the complex $Ho\mathcal{F}(\text{cone}(\tau_{<n} X \to X))$ has trivial cohomology in non-positive degrees. Hence, we have

$$Ho_{D^+(\mathcal{B})}(X', Ho(\mathcal{F}(\tau_{< n} X \to X))) = 0.$$
Combining Lemma 2.10 and Corollary 2.9 we get a fully faithful embedding

\[(2.25) \quad S : T^+ \hookrightarrow D(Sh).\]

By Lemma 2.4 $S$ carries every quasi-functor $F$ satisfying $(P')$ to $s(H^0F) \in Sh$. This proves the second part of Theorem 2. For the first part, let $F \in Fun(A, B)$ be a $k$-linear functor, and let

\[(2.26) \quad RF = \mathbb{D}(D_{dg}^+(A)^o \otimes_k D_{dg}^+(B))\]

be the "derived functor" (see (2.11)). To complete the proof of Theorem it suffices to show the following.

**Lemma 2.11.** Assume that $F$ is left exact. Then $RF$ is an object of $\mathbb{D}^{f+}$ and $S(RF)$ is isomorphic to $s(F)$.

**Proof.** Let us show that $RF$ satisfies property (2.14). According Remark 2.6 it will suffice to show that, for every integer $n$, $Y^i \in D_{dg}^{\geq n}(B)$ and $X \in HoC^{+}(A)$

\[H^n(RF^n(X \otimes Y^i)) \cong \prod_i H^n(RF^n(X \otimes X^n)).\]

We have ([Dri], §5)

\[(2.27) \quad H^n(RF^n(X \otimes X')) \cong colim_{Q_X} Hom_{D^+(B)}(X', F(Y)),\]

where $Q_X$ is the filtrant category of pairs

\[(Y \in HoC_{dg}^+(A), f \in Hom_{HoC_{dg}^+(A)}(X, Y))\]

such that $cone(f)$ is acyclic. If $X$ is in $HoC^{\geq n}(A)$ the subcategory $Q'_X \subset Q_X$ formed by pairs $(Y, f)$ with $Y \in HoC^{\geq n}(A)$ is cofinal in $Q_X$ and, hence, $Q_X$ in equation (2.27) can be replaced by $Q'_X$. Thus, it is enough to prove that the category $Q_X$ has the following property: for every countable collection $w_i = (Y_i, f_i) \in Q'_X$, $(i = 1, 2, \cdots)$, there exists $v \in Q_X$ such that, for every $i$, the set $Mor_{Q_X}(w_i, v)$ is not empty. In fact, the object

\[v = (cone(\bigoplus_i X \xrightarrow{\phi} \bigoplus_i Y_i), g),\]

where $\phi_j : X \to \bigoplus_i Y_i$ equals $f_j - f_{j-1}$ and $g$ is induced by the morphisms $X \xrightarrow{f_j} Y_i$ \xrightarrow{\phi} $Y_i$, does the job.

Let us show that $RF$ satisfies property (2.15). As we explained in Remark 2.7 it suffices to show that

\[colim H^n(RF^n(\tau_{<i}X \otimes X'))) \cong H^n(RF^n(X \otimes X'))\]

for every $X' \in B$. In fact, formula (2.27) with $Q_{\tau_{>i}X}$ replaced by $Q_{\tau_{>i}X}$ shows that $H^n(RF^n(\tau_{\geq i}X \otimes X'))$ is trivial for $i > 0$. Hence, the morphism $H^n(RF^n(\tau_{<i}X \otimes X')) \to H^n(RF^n(X \otimes X'))$ is an isomorphism for $i > 1$. This proves that $RF$ belongs to $\mathbb{D}^{f+}$.

For the second claim, observe that the restriction $Res(RF') \in \mathbb{D}(D_{dg}^k(A)^o \otimes_k D_{dg}^k(B))$ is the bounded "derived functor" (2.11). Thus, we are done by Lemma 2.5.

**Proof of theorem 3.** Apply Corollary 2.3 and equation (2.25).
References


