GEOMETRIC CONSTRUCTION OF THE CANONICAL COORDINATES

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Abstract.

1. Introduction

Let $R$ be a commutative ring flat over $\mathbb{Z}$ and $\mathcal{X}$ a strictly proper semi-stable scheme over $\mathcal{Y} = \text{spec } R[[t]]$. Assume that $X = \mathcal{X} \times_{\mathcal{Y}} Y$ is Calabi-Yau scheme over $Y = \text{spec } R((t))$ of dimension $d > 1$. I also assume that $t = 0$ is the maximal degeneracy point. By definition, this means that

$$H^d_{\text{Zar}}(X_0, \mathbb{Q}) = \mathbb{Q},$$

where $X_0 := \mathcal{X} \times_{\mathcal{Y}} \text{spec } R$ is the special fiber.

(Note that for any smooth $D$, $H^i_{\text{Zar}}(D, \mathbb{Z}) = 0$, unless $i = 0$. It follows $H^i_{\text{Zar}}(X_0, \mathbb{Z})$ is cohomology of the dual polytope of $X_0$: the simplicial set whose vertexes are irreducible components of $X_0$, edges are components of pairwise intersections and so on. For any semistable $\mathcal{X}$ over $\mathbb{C}[[t]]$, $H^i_{\text{Zar}}(X_0, \mathbb{Q})$ is isomorphic to the $W_0$ term of the limiting Hodge structure. It follows that our definition of maximal degeneration agrees with the usual over $\mathbb{C}$.)

Define a sheaf of groups on $X_0$ to be

$$M^{gr} = \lim_n j_\ast \mathcal{O}_X^\ast / \mathcal{O}_{\mathcal{X},n}^\ast,$$

where $j$ stands for the embedding $X \hookrightarrow \mathcal{X}$ and $\mathcal{O}_{\mathcal{X},n}^\ast$ for the subsheaf of $\mathcal{O}_X^\ast$ consisting of functions which are equal to 1 modulo $t^{n+1}$. The function $t$ determines a morphism of sheaves $\mathbb{Z} \hookrightarrow M^{gr}$; I will denote by $M^{gr}_{red}$ the cokernel of this morphism.

We have exact sequences of sheaves on $X_0$

$$0 \to \mathcal{O}_X^\ast \to M^{gr} \to L \to 0$$

$$0 \to \mathcal{O}_X^\ast \to M^{gr}_{red} \to L_{red} \to 0$$

$$0 \to \mathbb{Z} \to L \to L_{red} \to 0$$

where $\mathcal{O}_X^\ast$ the sheaf of invertible functions on the formal completion $\hat{X}$ of $\mathcal{X}$ along $X_0$ and $\hat{L}$ is the direct sum $\bigoplus D_j$ of constant sheaves supported on the irreducible components of $X_0$. Since $H^i_{\text{Zar}}(X_0, L) = 0$ for $i > 0$ we have from the third sequence that

$$H^{d-1}_{\text{Zar}}(X_0, L_{red}) \sim H^{d}_{\text{Zar}}(X_0, \mathbb{Z}),$$

and thus, by our maximal degeneracy assumption

$$H^{d-1}_{\text{Zar}}(X_0, L_{red}) \otimes \mathbb{Q} \sim \mathbb{Q}.$$
Now look on cohomology sequence associated with the second exact sequence
\[ H^{d-1}_{Zar}(X_0, L_{red}) \to H^d_{Zar}(X_0, O^*_X) \to H^d_{Zar}(X_0, M_{red}^{gr}). \]
We also consider the morphism
\[ \gamma : H^d_{Zar}(X_0, \mathbb{Z}) \otimes (R[[t]])^* \to H^d_{Zar}(X_0, O^*_X) \]
induced by the cup product.

**Lemma 1.1.** \( \gamma \otimes Q \) is injective and \( \text{Im} (\alpha \otimes Q) \subset \text{Im} (\gamma \otimes Q) \).

We get from the lemma a canonical morphism
\[ Q = H^{d-1}_{Zar}(X_0, L_{red}) \otimes Q \to H^d_{Zar}(X_0, Q) \otimes (R[[t]])^* = (R[[t]])^* \otimes Q. \]
The series
\[ c(1) \in (R[[t]])^* \otimes Q \]
viewed as an element \( O^*_Y \otimes Q \) does depend on the choice of the coordinate \( t \) on \( Y \), however
\[ tc(1) \in O^*_Y \otimes Q \]
is well defined. This is the canonical coordinate.