Some applications of Goodwillie’s Theorem

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Abstract

Our main result is Theorem 1, which provides a description of the cokernel of the restriction map $K_i(\tilde{X}) \to K_i(X)$ from $K$-group of an infinitesimal deformation $\tilde{X}$ of a smooth projective variety $X$. The key ingredient in the proof is an appropriate generalization ("Deligne crystalline cohomology") of Deligne cohomology groups to the case of singular schemes. We define the Chern characters with values in these groups, and then derive from results of Goodwillie that the Chern character from the relative $K$-groups $K^*_m(\tilde{X}, (\epsilon))$ to the relative Deligne crystalline cohomology is an isomorphism. This allows us to reduce the original problem about the $K$-groups to a question in Hodge Theory.

1. Fix a field $k$ of characteristic 0. Let $p : X \to S_n$ be a smooth, proper scheme over $S_n = \text{spec } k[\epsilon]/\epsilon^{n+1}$ and let $X_0$ be the special fiber. We are looking on the kernel and the cokernel of the map:

$$K_0(X) \to K_0(X_0).$$

2. The pushforward $R^mp_{\text{cris}}O_X$ is a vector bundle with an integrable connection over $S_\infty$. Its restriction $H^m_{DR}(X/S_n)$ to $S_n$ is equipped with the Hodge filtration. Denote by

$$F^i_XH^m_{DR}(X_0) \subset F^iH^m_{DR}(X_0) \subset H^m_{DR}(X_0)$$

the subspace of elements of $F^iH^m_{DR}(X_0)$ whose parallel transport belongs to the $i$-th term of the Hodge filtration over $S_n$.

3. It can be shown that $F^i_XH^m_{DR}(X_0)$ coincides with the crystalline Hodge filtration on the absolute cohomology groups $H^m_{cris}(X, O_X) = H^m_{DR}(X_0)$ i.e.

$$F^i_XH^m_{DR}(X_0) = \text{Im}(H^m_{cris}(X, J_X^i) \to H^m_{cris}(X, O_X)),$$

where $J_X$ is the subsheaf of the structure sheaf $O_X$, that takes an object $(U \subset X, U \to T)$ to the ideal $\ker(O(T) \to O(U))$.

4. Take any $E \in K_0(X_0)$ and consider the Chern character

$$ch(E) \in \bigoplus_p F^pH^{2p}_{DR}(X_0).$$

It is easy to see that if $E$ lies in the image of the restriction map (1) then

$$ch(E) \in \bigoplus_p F^p_XH^{2p}_{DR}(X_0).$$

Theorem 1. Assume $k = \mathbb{Q}$. Then

any class $E \in K_0(X_0)$ satisfying the property (2) lies in the image of (1)

the map $K_0(X) \to K_0(X_0)$ is surjective, for any $m > 0$.

5. Let us explain how to generalize the first part of the theorem to the case of any field $k$ of characteristic 0. There is the absolute Chern character:

$$ch : K_0(X_0) \to \bigoplus_p H^p(X_0, \Omega^p_{X/k}).$$

Let $\delta \in H^1(X, T_{X/k})$ be the class corresponding to the deformation $X$ over $S_1$. 

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Theorem 1'. Assume that \( n = 1 \). Then the sequence

\[
K_0(X) \rightarrow K_0(X_0) \xrightarrow{\delta \times \text{ch}} \bigoplus_p H^{p+1}(X_0, \Omega^p_{X_0}/\mathbb{Q})
\]

is exact.

We shall explain the idea of the proof of Theorem 1.

6. Deligne crystalline cohomology. Let \( Y \) be a scheme over \( \mathbb{C} \). Denote by \( \text{Cris}_{an}(Y/\mathbb{C}) \) the analytic crystalline site; objects of \( \text{Cris}_{an}(Y/\mathbb{C}) \) are pairs \((U, T)\), where \( U \) is an open subspace of \( Y_{an} \) and \( T \hookrightarrow U \) is a nilpotent thickinning. Denote by \( \pi_{an} : (Y/\mathbb{C})_{\text{cris}} \rightarrow Y(\mathbb{C}) \) the natural map of topoi. There is an obvious morphism \( Z \rightarrow R\pi_{an}\mathcal{O}_{Y,an} \). Put

\[
A_{Y,an}(i) := \text{cone}(\mathbb{Z} \rightarrow R\pi_{an}\mathcal{O}_{Y,an}/J^i_{Y,an})[-1]
\]

(see 3 for the definition of \( J_{Y,an} \)). We define the Deligne crystalline cohomology of \( Y \) to be

\[
H^r_{\text{Dcris,an}}(Y, \mathbb{Z}(i)) := H^r(Y(\mathbb{C}), A_{Y,an}(i)).
\]

Explicitly, if \( Y \hookrightarrow P \) is an embedding into a smooth scheme \( P \), the complex \( A_{Y,an}(i) \) is canonically isomorphic to

\[
\mathbb{Z} \rightarrow \mathcal{O}_{P,an}/I^1 \rightarrow \cdots \rightarrow \Omega^i_{P,an}/I^i \rightarrow \Omega^{i-1}_{P,an}/I,
\]

\( \hat{P} \) denotes the formal completion of \( P \) along \( Y \) and \( I \) denotes the sheaf of ideals of \( Y \hookrightarrow P \). In particular, if \( Y \) is smooth over \( \mathbb{C} \) we recover the usual Deligne complex \([B]\).

For a pair \( Z \hookrightarrow Y \), one defines naturally the relative Deligne crystalline cohomology \( H^r_{\text{Dcris,an}}(Y, Z, \mathbb{Z}(i)) \).

Let \( X \rightarrow S_1 = \text{spec} \mathbb{C}[\epsilon]/\epsilon^2 \) be a smooth scheme and let \( X_0 \) be the special fiber. The cohomology sheaves of the relative complex

\[
A_{X,X_0,an}(i) := \text{cone}(R\pi_{an}\mathcal{O}_{X,an}/J^i_{X,an} \rightarrow R\pi_{an}\mathcal{O}_{X,an}/J^i_{X,an})[-2]
\]

are canonically isomorphic to \( H^r(A_{X,X_0,an}(i)) = \Omega^{r-1}_{X,an} \), for \( r \leq i \), and they are 0 otherwise. Using the Hodge Theorem, one can show that, if \( X \) is projective over \( S_1 \), the spectral sequence convergent to \( H^r_{\text{Dcris,an}}(X, X_0, \mathbb{Z}(i)) \) associated to the canonical filtration on \( A_{X,X_0}(i) \) degenerates at the first term. Thus the group \( H^r_{\text{Dcris,an}}(X, X_0, \mathbb{Z}(i)) \) has a canonical filtration with the successive quotients isomorphic to \( H^i(X_0, \Omega^m_{X_0,an}) \), where \( i + m = r - 1 \), \( m \leq i \).

7. Let \( k \) be a subfield of \( \mathbb{C} \). Given a \( k \)-scheme \( Z \) and an infinitesimal thickinning \( Z \hookrightarrow Y \) over \( k \) we put

\[
A_{Y,Z,\text{alg}}(i) := \text{cone}(R\pi_{an}\mathcal{O}_Y/J^i_Y \rightarrow R\pi_{an}\mathcal{O}_Z/J^i_Z)[-2]
\]

Note that if \( Y \) is proper over \( k \) the canonical morphism

\[
H^r_{\text{Dcris,alg}}(Y, Z, \mathbb{Z}(i)) \otimes_k \mathbb{C} := H^r(Y, A_{Y,Z,\text{alg}}(i)) \otimes_k \mathbb{C} \rightarrow H^r_{\text{Dcris,an}}(Y, Z, \mathbb{Z}(i))
\]

is an isomorphism.

8. Chern characters. One can show that Deligne crystalline cohomology groups satisfy the axioms of Beilinson \([B]\). Thus, for any scheme \( Y \) over \( k \subset C \) the general construction from \textit{loc. cit.} and \([S]\) gives rise to the Chern character morphism:

\[
\text{ch} : K_m(Y) \rightarrow \bigoplus_r H^{2r-m}_{\text{Dcris,alg}}(Y, \mathbb{Z}(r)) \otimes \mathbb{Q} =: \bigoplus_r H^{2r-m}_{\text{Dcris,an}}(Y, \mathbb{Q}(r))
\]

If \( Z \hookrightarrow Y \) is an infinitesimal thickinning over \( k \) the relative Chern character \( \text{ch} : K_m(Y, Z) \rightarrow \bigoplus_r H^{2r-m}_{\text{Dcris,alg}}(Y, Z, \mathbb{Z}(r)) \otimes \mathbb{Q} \) factors canonically through

\[
\text{ch} : K_m(Y, Z) \rightarrow \bigoplus_r H^{2r-m}_{\text{Dcris,alg}}(Y, Z, \mathbb{Z}(r))
\]
Theorem 3. Let $X$ be a smooth scheme over $S_n$, and let $X_0 = X \times_{S_n} S_0$. Then the morphism (3):

$$
ch : K_m(X, X_0) \to \bigoplus_r H^{2r-m}_{Dcris,alg}(X, X_0, \mathbb{Z}(r))
$$

is an isomorphism.

The proof of this result consists of two steps. First, using [S] one extends (3) to a homotopy morphism from the K-theory $\Omega$-spectrum to the Eilenberg-MacLane spectrum corresponding to the complex of abelian groups $R\Gamma(X, A_X, X_0(r))$. The two spectra have the Mayer-Vietoris property with respect to open coverings of $X$. This reduces the proof of the theorem to the case of an affine $X_0$ and the constant deformation $X$. In the affine case Goodwillie and Hesselholt, [G], [H], constructed an isomorphism

$$
K_m(X, X_0) \simeq \bigoplus_{r \geq 1} (\Omega^{m+1-2r})^n
$$

To complete the proof it is enough to check that the map (4) from loc. cit. coincides with ours.

9. Let us explain how Theorem 3 implies Theorem 1. Consider the commutative diagram:

$$
\begin{array}{ccccccc}
\cdots \to K_m(X) & \to K_m(X_0) & \to K_{m-1}(X, X_0) & \to \cdots \\
\downarrow & \downarrow & \downarrow & \\
\cdots \to \bigoplus_r H^{2r-m}_{Dcris,an}(X, \mathbb{Q}(r)) & \to \bigoplus_r H^{2r-m}_{Dcris,an}(X_0, \mathbb{Q}(r)) & \to \bigoplus_r H^{2r-m-1}_{Dcris,an}(X, X_0, \mathbb{Q}(r)) & \to \cdots
\end{array}
$$

Theorem 3 together with the remark in 7 imply that the right vertical morphism is injective. Thus Theorem 1 would follow if we show that the morphism $H^*_{cris}(X, \mathcal{O}_X) \to H^*_{cris}(X_0, \mathcal{O}_{X_0})$ is surjective. Indeed, the Hodge theorem implies that the surjectivity of the composition:

$$
H^*_{cris}(X, \mathcal{O}_X) \simeq H^*_{cris}(X_0, \mathcal{O}_{X_0}) \to H^*_{cris}(X_0, \mathcal{O}_{X_0}/J_{X_0}),
$$

and we are done.

References.


[S] V. Schechtman, On the delooping of Chern character and Adams operations. Lecture Notes in Mathematics (v. 1289)