1. Introduction

Realizing a suggestion of Beilinson we construct a DG category of motivic sheaves over a curve gluing the DG categories of Voevodsky motives over all points of the curve. This is the first step in our attempt to develop a theory of motivic sheaves over any base scheme and the formalism of the six operations. 1

Let us explain the idea of our construction. For any scheme $S$, we would like to construct a DG category $M_{\text{eff}}^{gm}(S)$ such that if $S$ is the spectrum of a field $M_{\text{eff}}^{gm}(S)$ coincides with Voevodsky’s category of effective motives [Voe1] 3 and such that, for any closed subscheme $D \hookrightarrow S \twoheadrightarrow S - D = U$ there are functors:

$$M_{\text{eff}}^{gm}(D) \xrightarrow{i^*} M_{\text{eff}}^{gm}(S) \xrightarrow{j_*} M_{\text{eff}}^{gm}(U)$$

which identify $M_{\text{eff}}^{gm}(D)$ with a full subcategory of $M_{\text{eff}}^{gm}(S)$ and $M_{\text{eff}}^{gm}(U)$ with the DG quotient of $M_{\text{eff}}^{gm}(S)$. We refer to this as the localization property of motivic sheaves.

Assume, for the moment, that we know the categories $M_{\text{eff}}^{gm}(D)$ and $M_{\text{eff}}^{gm}(U)$. We shall see in 2.5 in abstract categorical setting that the category of all extensions (??) of $M_{\text{eff}}^{gm}(U)$ by $M_{\text{eff}}^{gm}(D)$ with the property that the subcategory $\text{Ho}(M_{\text{eff}}^{gm}(D)) \xrightarrow{i_*} \text{Ho}(M_{\text{eff}}^{gm}(S))$ is both left and right admissible (we shall call extensions with such a property admissible) is equivalent to the category of quasi-functors $\Phi : M_{\text{eff}}^{gm}(U) \to M_{\text{eff}}^{gm}(D)$. It is important that one should work with DG categories rather than with merely triangulated categories: the category of extensions of triangulated categories does not have a nice description.

Let $C$ be a curve over a field $k$ of characteristic 0 with the generic point $\eta$, and let $a \in C$ be a closed point. Denote by $O_{a,C}$ the local ring of $C$ at $a$. It is clear from the previous paragraph that to glue the category of motivic sheaves over $\text{spec}O_{a,C}$ from the Voevodsky’s categories $M_{\text{eff}}^{gm}(\eta)$, $M_{\text{eff}}^{gm}(k(a))$ all we need is to specify a functor $M_{\text{eff}}^{gm}(\eta) \to M_{\text{eff}}^{gm}(k(a))$. If the motive over $\eta$ is represented by a scheme $X_{\eta}$ over $\eta$ its image under the above functor should be viewed as the motivic punctured tubular neighborhood $PTN_{a}(X_{\eta})$ of $X_{\eta}$ at $a$. The topological picture suggests that if $k(a) = k$ and the scheme $X_{\eta}$ is proper over $\eta$ then the motivic punctured neighborhood $PTN_{a}(X_{\eta})$ should be quasi-isomorphic to the following complex of schemes over $k$:

$$PTN_{a}(X_{\eta}) \simeq \text{cone}(X_{a} \coprod X_{U} \to X)[-1],$$

1See [Voe2], [A], and [Deg] for another approach.
2We refer the reader to the next section for a review of DG categories.
3Voevodsky has defined merely a triangulated category. See [BV] for explanation how to endow it with a DG structure.
where \( X \) is any proper scheme over \( C \) whose fiber over the generic point is \( X_\eta \) and \( U \) denotes the complement to \( a \) in \( C \).

In Section 3 we explain how to construct a functor \( PTN_\eta : M_{gm}^{eff}(\eta) \to M_{gm}^{eff}(k(a)) \) with the above property. This is the key geometric construction of the paper.

If the base scheme \( S \) contains infinitely many closed points (for example, \( S = C \)) the above definition should be modified. The last section contains a precise definition of \( M_{gm}^{eff}(C) \) and some first results about the category. In particular, we prove the Cancellation Theorem for \( M_{gm}^{eff}(C) \) and show (following explanations by Joseph Ayoub) that if \( C \) is a smooth curve the triangulated category of noneffective motivic sheaves \( Ho(M_{gm}(C)) \) (i.e. the category obtained from \( Ho(M_{gm}^{eff}(C)) \) by inverting the Tate motive.) is equivalent to Voevodsky’s category \( DM_{gm}(C) \) from [Voe2].

Some of the most important results about Voevodsky’s category of motives over a field require the ground field to be perfect. This is the true reason for our restriction \( \text{char} \ k = 0 \) : this guarantees that that the residue field \( k(\eta) \) of the generic point is perfect.

We are grateful to Joseph Ayoub for numerous illuminating discussions and for his help with Proposition (??).

2. Some Homological Algebra

In this section we recall some definitions on DG categories referring to [D] and [K] for details. Then we will prove the result cited in the Introduction on admissible extensions of DG categories.

2.1 A Differential Graded (or DG) category is an additive category \( A \) such that for any objects \( X, Y \) the group of morphisms \( Hom_A(X,Y) \) has the structure of a complex of abelian groups (obviously compatible with composition). The homotopy category \( Ho(A) \) of a DG category \( A \) has the same objects as \( A \) and \( Hom_{Ho(A)}(X,Y) = H^0(Hom_A(X,Y)) \).

A DG category is called pretriangulated if it is closed under shifts and if the cone of any closed morphism of degree zero exists, i.e. the following functors are representable:

\[
A \to Hom_A(A,X)[n], \ A \to \text{cone}(Hom_A(A,X)) \to Hom_A(A,Y),
\]

for any closed morphism \( f \in Hom_A(X,Y) \) of degree 0. The homotopy category of a pretriangulated DG category has a canonical structure of a triangulated category. In this paper a DG category means a pretriangulated, homotopically flat over \( \mathbb{Z} \) (i.e. for any objects \( X, Y \) of \( A \) and any acyclic complex \( C \) of abelian groups the complex \( C \otimes Hom_A(X,Y) \) is acyclic) DG category.

2.2. Let \( A, B \) be DG categories. The category \( DGF(A,B) \) of DG functors has a natural DG structure. We define the triangulated category of quasi-functors to be:

\[
\mathcal{T}(A,B) = Ho(DGF(A,B))/\{\text{functors } F \text{ s.t. } Ho(F) : Ho(A) \to Ho(B) \text{ is 0} \}.
\]

(Keller.) Assume that a DG functor \( G : B \to B' \) is a homotopy equivalence i.e. a the functor \( Ho(G) : Ho(B) \to Ho(B') \) is an equivalence of categories. Then \( G_* : \mathcal{T}(A,B) \to \mathcal{T}(A,B') \) is an equivalence as well. According to another result of [K] the category \( \mathcal{T}(A,B) \) is equivalent to the full subcategory of the derived category \( D(A^\circ \otimes B) \) of DG \( A^\circ \otimes B \)-modules, i.e. DG functors \( F : A \otimes B^\circ \to C(\text{Ab}) \) to the category of complexes of abelian groups, whose objects are functors \( F \) with the property that \( F(X) \) belongs to the essential image of the Yoneda embedding \( Ho(B) \to D(B) \), for any object \( X \) of \( A \). In particular any DG functor \( F : A \to B \) such that \( Ho(F) : Ho(A) \to Ho(B) \) has the right (resp. left) adjoint defines a quasi-functor
G \in \mathcal{T}(B, A) \ (\text{unique up to a canonical isomorphism}) \text{ which lifts the adjoint functor } Ho(B) \to Ho(A). \text{ We will refer to } G \text{ as to the right (resp. left) adjoint quasi-functor.}

2.3 \text{ Let } D \text{ be a triangulated category, } I \subset D \text{ be a full, thick (i.e. any direct summand of an object of } I \text{ lies in } I) \text{ triangulated subcategory. } I \text{ is called right admissible if either of the following properties holds:}

i) \text{ The embedding functor } i_* : I \to D \text{ admits the right adjoint } i^! : D \to I. 

ii) \text{ The projection functor } j^* : D \to D/I \text{ admits the right adjoint } j_* : D/I \to D. 

If this happens the projection } j^* : D \to D/I \text{ induces an equivalence } I^\perp \simeq D/I (\text{Recall that the right orthogonal complement } I^\perp \text{ is, by definition, the full subcategory of } D \text{ which consists of objects } X \in D \text{ verifying the following property: } Hom(Y, X) = 0, \text{ for any } Y \text{ in } I.) \text{ Similarly, one defines the notion of left admissible subcategory.}

2.4. \text{ Let } A \text{ be a DG category and let } T \subset A \text{ be a full subcategory. One can prove [D] that there exists a DG category } B \text{ with a DG functor } A \to B \text{ with the following universal property: for any DG category } C \text{ the functor } \mathcal{T}(B, C) \to \mathcal{T}(A, C) \text{ is fully faithful and its essential image consists of functors } F : A \to C \text{ which induce zero functor from } Ho(T) \text{ to } Ho(C). \text{ The category } B \text{ is called the DG quotient of } A. \text{ If } Ho(T) \subset Ho(A) \text{ is right admissible one can take for } B \text{ the full subcategory of } A \text{ whose objects are those of } Ho(T)^\perp.

2.5. \text{ Let } A, C \text{ be DG categories. We define the category } Ext(C, A) \text{ of admissible extensions as follows: objects of } Ext(C, A) \text{ are triples } (B, i_* : A \to B, j^* : B \to C), \text{ where } i_* \text{ is fully faithful, } j^* \text{ verifies the condition from the previous paragraph (i.e. } C \text{ is the DG quotient } B), \text{ and } Ho(A) \subset Ho(B) \text{ is right and left admissible. Morphisms are DG functors } \phi \text{ fitting into a strictly commutative diagram}

\begin{tikzcd}
A \ar[r] \ar[d, equals] & B \ar[r] \ar[d, \phi] & C \\
A \ar[r] & B' \ar[r] & C
\end{tikzcd}

\text{with formally inverted homotopy equivalences.}

There is an equivalence of categories: } Ext(C, A) \simeq \mathcal{T}(C, A).

Proof. \text{ Let } A \xrightarrow{i_0} B \xrightarrow{j_0} C \text{ be an admissible extension. Denote by } i^* : B \to A \text{ (resp. } i^! : B \to A) \text{ the left (resp. right) adjoint to } i_0 \text{ quasi-functor, and by } j_* : C \to B \text{ (resp. } j_! : C \to B) \text{ the right (resp. left) adjoint to } j^*. \text{ The adjoint quasi-functors exist and well defined by 2.2 (and the admissibility assumption). We define the desired quasi-functor to be:}

\begin{equation*}
\Phi = i^! j_! \simeq cone(j_! \to j_*)[-1] \simeq i^* j_*[-1] : C \to A.
\end{equation*}

Conversely, given a quasi-functor } \Phi \text{ we construct a DG category } (A, C, \Phi). \text{ By definition, objects of } (A, C, \Phi) \text{ are triples } (F_A, F_C, \alpha), \text{ where } F_A \text{ is an object of } A, F_C \text{ an object of } C \text{ and } \alpha : \Phi(F_C) \to F_A \text{ is a closed morphism of degree } 0. \text{ Morphisms between such triples are}

\begin{equation*}
Hom((F_A, F_C, \alpha), (F'_A, F'_C, \alpha')) = cone(Hom(F_A, F'_C) \oplus Hom(F_C, F'_A) \to Hom(\Phi(F_C), F'_A)[{-1}].
\end{equation*}

The functors } i_*, } j^* \text{ and the adjoint quasi-functors are given by}

\begin{align*}
i_* &= i : F_A \to (F_A, 0, 0), \text{ and } \\
i^* : (F_A, F_C, \alpha) &\mapsto cone(\alpha) \\
i^! : (F_A, F_C, \alpha) &\mapsto F_A \\
j^* = j : (F_A, F_C, \alpha) &\to F_C, \text{ and } \\
j_* : F_C &\to (\Phi(F_C), F_C, id) \\
j_! : F_C &\to (0, F_C, 0)
\end{align*}
3. Motivic tubular neighborhood

3.1. Let \( C \) be a reduced curve over a field \( k \) of characteristic 0, \( a \in C \) be a closed point, and let \( U := C - a \). Denote by \( \mathcal{O}_{a,C}^h \) the henselization of the local ring \( \mathcal{O}_{a,C} \) and by \( \eta^h \) the complement \( \text{spec} \mathcal{O}_{a,C}^h - a \) (i.e. the finite set of generic points). In this section we construct a functor

\[
\text{PTN}_a : M^\text{eff}_{gm}(\eta^h) \to M^\text{eff}_{gm}(k(a)).
\]

Let \( X \) be a scheme over \( U \). Denote by \( \overline{C}_{k(a)} \) the completion of the curve \( C_{k(a)} := C \times_k \text{spec}(k(a)) \) over \( k(a) \) \(^4\) and by \( a_\Delta \) the diagonal \( k(a) \)-point \( : \text{spec}(k(a)) \to C \times_k \text{spec}(k(a)). \) Let \( \overline{U}_{k(a)} := \overline{C}_{k(a)} - a_\Delta \) be the completion. Let \( X_{k(a)} := X \times_k \text{spec}(k(a)) \) be the scheme over \( k(a) \). Define a presheaf with transfer \( \mathbb{Z}_{tr}^U[X] \) on \( \text{Sm}_{k(a)} \) taking a smooth connected \( k(a) \)-scheme \( Y \) to the free abelian group generated by integral cycles on \( Y \times_k X \) which are

i) quasi-finite and dominant over \( Y \) and

ii) proper over \( Y \times_k(\overline{U}_{k(a)}) \) \(^5\)

There is a natural embedding of presheaves

\[ e : \mathbb{Z}_{tr}^U[X_{k(a)}] \to \mathbb{Z}_{tr}^U[X] \].

By definition, the motivic punctured tubular neighborhood \( \text{PTN}_a(X) \) is the complex \( \text{cone}(c)[-1] \). We can view \( \text{PTN}_a(X) \) as an effective motive over \( k(a) \) via the projection \( D\text{PSh}_{tr}(k(a)) \to \text{DM}^\text{eff}_{gm}(k(a)) \) \(^6\).

Let \( \tilde{X} \) be scheme over \( C \) and let \( \tilde{X}_{k(a)} = \tilde{X} \times_k \text{spec}(k(a)). \) Denote by

\[ \tilde{c} : \mathbb{Z}_{tr}^U[\tilde{X}_{k(a)}] \to \mathbb{Z}_{tr}^U[\tilde{X} \times C U] \]

the morphism which takes a cycle \( Z \subset Y \times_k \tilde{X} \) to \( Z \times C Y \subset Y \times_k X \times C U \). The evident morphism \( \mathbb{Z}_{tr}^U[\tilde{X} \times C \text{spec}(k(a))]/\mathbb{Z}_{tr}^U[\tilde{X} \times k \text{spec}(k(a))] \) yields an identification:

\[
\mathbb{Z}_{tr}^U[\tilde{X} \times C \text{spec}(k(a))] \simeq \ker \tilde{c}
\]

The tubular neighborhood \( \text{T}_a(X) \) is the complex \( \text{cone}(\tilde{c})[-1] \). Thus, by (\(\ref{eq:3.2}\)), we have a canonical morphism:

\[
\mathbb{Z}_tr[\tilde{X} \times C \text{spec}(k(a))] \to \text{T}_a(X).
\]

Observe, also, that for any \( \tilde{X} \), there is a natural morphism

\[ \alpha_a : \text{PTN}_a(\tilde{X} \times C U) \to \text{T}_a(\tilde{X}). \]

---

\(^4\)i.e. a unique proper curve \( \overline{C}_{k(a)} \supset C_{k(a)} \) whose singular locus is contained in \( C_{k(a)} \). Observe that any morphism \( C_{k(a)} \to C_{k(a)} \) between curves over \( \text{spec}(k(a)) \) extends to a morphism between the completions.

\(^5\)We have the morphisms \( Y \times_k X \to Y \times_k U \to Y \times_k(\overline{U}_{k(a)}) \). Hence any cycle on \( Y \times_k X \) can be viewed as a scheme over \( Y \times_k(\overline{U}_{k(a)}) \).

\(^6\)Let us explain the topological picture behind this definition. The cohomology groups of the punctured tubular neighborhood can be computed by the complex \( \text{cone}(\text{RTR}_p \text{R}_p \text{Z}_X) \to \text{RTR}_p \text{R}_p \text{Z}_X) \), where \( p : X \to U \) is the projection and \( \text{Z}_X \) is the constant sheaf on \( X \). One can check that the first complex computes the Betti cohomology of the motive \( \mathbb{Z}_{tr}^U[X] \). For example, if \( C \) is proper over \( k \) and \( X \) is proper over \( U \) this amounts to the well known fact about the presheaf \( \mathbb{Z}_{tr}^U[X] \) of quasi-finite correspondences.
In the rest of this section we explain why $PTN_a(X)$ and $TN_a(X)$ deserve their names. Let $\pi : C' \to C$ be an etale morphism of curves over $k$. Assume that the morphism $C' \times_C \text{spec} \, k(a) \to \text{spec} \, k(a)$ is an isomorphism:

$$
\begin{array}{ccc}
C' \times_C \text{spec} \, k(a) & \xrightarrow{\alpha'} & C' \\
\downarrow & & \downarrow \\
\text{spec} \, k(a) & \xrightarrow{\alpha} & C
\end{array}
$$

Then the canonical morphisms $PTN_a(X \times_C C') \to PTN_a(X)$ and $TN_a(\tilde{X} \times_C C') \to TN_a(\tilde{X})$

are isomorphisms in $DM_{\text{eff}}(k(a))$.

**Proof.** The assertion for $PTN_a(X \times_C C')$ is the special case of the assertion for $TN_a(\tilde{X} \times_C C')$, for $\tilde{X} = X$.

To prove the assertion for $TN_a(\tilde{X} \times_C C')$, it is enough to show for any Nisnevich local $Y$ the exactness of the sequence

$$
0 \to Z_{tr}[\tilde{X}_k(a) \times_C C'](Y) \xrightarrow{(\pi_*, \tilde{\epsilon}_X)} Z_{tr}[\hat{X}_k(a)](Y) \oplus Z_{tr}[\tilde{X} \times_C U'](Y) \xrightarrow{(\tilde{\epsilon}_X, -\pi_*)} Z_{tr}[\tilde{X} \times_C U](Y) \to 0
$$

It follows from (??) that

$$
(3.3) \quad \pi_* : \ker(\tilde{\epsilon}_X) \simeq \ker(\tilde{\epsilon}_X).
$$

This implies *injectivity* of $(\pi_*, \tilde{\epsilon}_X)$.

**Exactness in the middle.** — Let $(Z, W) \in \ker(\tilde{\epsilon}_X, -\pi_*)$. By (??) the problem reduces to showing that $W = \sum n_i W_i$ lies in the image of $\tilde{\epsilon}_X$.

Choose an embedding $\tilde{X}_k(a) \hookrightarrow \overline{\tilde{X}}_k(a)$ into a proper scheme over $\tilde{C}_k(a)$, and let $\overline{W} = \sum n_i W_i$ be the closure of $W$ in $Y \times \overline{\tilde{X}}_k(a) \times \overline{C}_k(a)$. It is enough to show that the support $|\overline{W}|$ of $\overline{W}$ lies in $Y \times \tilde{X}_k(a) \times_C C'$ and that it is finite over $Y$.

Indeed, since $|W|$ is proper over $Y \times U'_{k(a)}$, one has $|\overline{W}| \setminus |W| \subset Y \times \overline{\tilde{X}}_k(a) \times \tilde{C}_k(a) a'_\Delta$, and since $\pi(|W|) = |\overline{\tilde{X}}| \subset |Z| = |\overline{Z}|$, one has $|\overline{W}| \setminus |W| \subset Y \times \tilde{X} \times \tilde{C}_k(a) a'_\Delta$. Hence $|\overline{W}| \subset Y \times \tilde{X}_k(a) \times_C C'$. Moreover, the maps $|\overline{W}| \to |Z| \to Y$ are finite, q. e. d.

**Surjectivity of $\tilde{\epsilon}_X, -\pi_*$.** — Let $Z \in (\tilde{\epsilon}_X, -\pi_*)$; we may assume $Z$ irreducible. If $Z$ is proper over $Y$, $Z \in \text{Im} \tilde{\epsilon}_X$. If not, we are going to show that $Z$ lifts to $Z_{tr}[\tilde{X} \times_C U'](Y)$.

It is enough to show that there exists a lifting $p'$:
Let us, first, show that the closure $\overline{p(Z)}$ is Nisnevich local. Indeed, since $Z$ is proper over $Y \times U_{k(a)}$, $\overline{p(Z)} \subset p(Z) \cup Y \times a_\Delta$. Hence $\overline{p(Z)} \to Y$ is quasi-finite, and since it is also proper, it is finite. Thus, since $Y$ is Nisnevich local, $\overline{p(Z)}$ is Nisnevich local as well. Since $Z$ is not proper over $Y$, $\overline{p(Z)}$ meets $\{\tilde{x}\} \times_k Y$, therefore the closed point of $\overline{p(Z)}$ in $Y \times a_\Delta$. Hence the morphism $Y \times \overline{C}_{k(a)} \to Y \times \overline{C}_{k(a)}$ has a (unique) section over $p(Z)$ and we are done.

\[\begin{array}{c}
Z' \to Y \times \overline{C}_{k(a)} \\
\downarrow {\text{Id}} \\
Z \to Y \times \overline{C}_{k(a)}.
\end{array}\]

3.2. The functor $PTN_a$ extends in the natural way to a functor from the category of bounded complexes of schemes over $U$: $C^b(\mathbb{Z}[\text{Sm}_U]) \to C(\text{PSh}_\text{tr}(k(a)))$.

a) Let $X = W_1 \cup W_2$ be a Zariski covering. Then the complex $PTN_a(Z_{\text{tr}}[W_1 \cap W_2] \to Z_{\text{tr}}[W_1] \oplus Z_{\text{tr}}[W_2] \to Z_{\text{tr}}[X])$ becomes quasi-isomorphic to 0 after the Nisnevich localization.

b) The complex $PTN_a(Z_{\text{tr}}[X \times \mathbb{A}^1] \to Z_{\text{tr}}[X])$ is quasi-isomorphic to 0 after $\mathbb{A}^1$-homotopy localization. Proposition (??) together with the above lemma show that $PTN_a$ extends to a quasi-functor: $PTN_a : M_{gm}^{eff}(\eta^h) \to M_{gm}^{eff}(k(a))$.

Assume that $k$ is a field of characteristic 0.

A) The functor (??) preserves the subcategory of geometric (i.e. perfect) motives:

$PTN_a : M_{gm}^{eff}(\eta^h) \to M_{gm}^{eff}(k(a))$.

B) Let $\tilde{X}$ be a proper scheme over $C$. Then the canonical morphisms of complexes $Z_{\text{tr}}[\tilde{X} \times_C \text{spec } k(a)] \to TN_a(\tilde{X})$

and $\text{cone}(Z_{\text{tr}}[\tilde{X} \times_C \text{spec } k(a)] \oplus Z_{\text{tr}}[\tilde{X} \times_C U] \to Z_{\text{tr}}[\tilde{X}])[\text{spec } k(a)](-1) \to PTN_a(\tilde{X})$ are isomorphisms in $DM_{gm}^{eff}(k(a))$.

C) For any smooth scheme $\tilde{X}$ over $C$, there is a natural isomorphism:

$\text{cone}(PTN_a(\tilde{X} \times_C U) \to TN_a(\tilde{X})) \simeq Z_{\text{tr}}[\tilde{X} \times_C \text{spec } k(a)](1)[2]^7$ in $DM_{gm}^{eff}(k(a))$.

D) For any motives $N$ over $k$ and $M$ over $\eta$ the natural morphism $p_{k(a)/k}^* N \otimes PTN_a(M) \to PTN_a(p_{\eta/k}^* N \otimes M)$ is a quasi-isomorphism. (For a field $K \supset k$, we denote by $p_{K/k}^*$ the obvious functor $M_{gm}^{eff}(K) \to M_{gm}^{eff}(K)$.)

\footnote{The assumption that $\text{char } k = 0$ is not needed for this statement. It is enough to assume that $k$ is perfect.}
If $Z$ is a smooth scheme over $k$ and $h \in Cor_C(Z \times C, Z \times C)$ is a finite correspondence over $C$ the diagram

$$
\begin{array}{ccc}
\mathbb{Z}_tr[Z \times_k \text{spec } k(a)] \otimes PTN_\eta(M) & & \mathbb{Z}_tr[Z \times_k \text{spec } k(a)] \otimes PTN_\eta(M) \\
\downarrow^{h_\# \otimes I_d} & & \downarrow^{h_\# \otimes I_d} \\
PTN_\eta(Z \times \eta) \otimes \eta & & PTN_\eta(Z \times \eta) \otimes \eta
\end{array}
$$

is commutative.

3.3. For any motives $M$ and $N$ over $\eta$ there is a canonical morphism:

$$(3.4) \quad PTN_\eta(M \otimes N) \to PTN_\eta(M) \otimes PTN_\eta(N)$$

Indeed, let $M = (\cdots \to Z_{tr}[X_i] \to Z_{tr}[X_{i+1}] \to \cdots), N = (\cdots \to Z_{tr}[X'_i] \to Z_{tr}[X'_{i+1}] \to \cdots) \in C^b(\mathbb{Z}_tr[Sm_k])$. Replacing $C$ by $C_{k(a)}$ and using Proposition ??, we may assume $k(a) = k$. We shall construct a complex $P = (\cdots \to P_i \to P_{i+1} \to \cdots) \in C(PSh_{tr}(k))$ and morphisms:

$$PTN_\eta(M \otimes N) \to P \leftarrow PTN_\eta(M) \otimes PTN_\eta(N)$$

of complexes such that the second arrow induces an isomorphism in $Ho(M^e_{gm}(k))$.

Denote by $\mathbb{Z}^U_{tr}[X_i \times_k X'_j]$ the presheaf $\mathbb{Z}^U_{tr}[X_i \times_k X'_j]$ where $X_i \times_k X'_j$ is viewed as a scheme over $C$ via the projection to the first factor $X_i \times_k X'_j \to X_i \to C$.

Similarly, we define $\mathbb{Z}^{2U}_{tr}[X_i \times_k X'_j]$. Let $\mathbb{Z}^{U \times U}_{tr}[X_i \times_k X'_j]$ be a presheaf with transfer taking a smooth connected $k$-scheme $Y$ to the free abelian group generated by integral cycles on $Y \times_k X_i \times_k X'_j$ which are quasi-finite, dominant over $Y$, and proper over $Y \times_k U \times_k U$. Our presheaves fit into the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{Z}_{tr}[X_i \times_U X'_j] & & \mathbb{Z}_{tr}[X_i \times_k X'_j] \\
\downarrow & & \downarrow \\
\mathbb{Z}^{U}_{tr}[X_i \times_U X'_j] & & \mathbb{Z}^{1U}_{tr}[X_i \times_k X'_j] \oplus \mathbb{Z}^{2U}_{tr}[X_i \times_k X'_j] \\
\downarrow & & \downarrow \\
0 & & \mathbb{Z}^{U \times U}_{tr}[X_i \times_k X'_j] \oplus \mathbb{Z}^{U \times U}_{tr}[X_i \times_k X'_j]
\end{array}
$$

All the left arrows in the diagram above are isomorphisms $Ho(M^e_{gm}(k))$. We are ready to define $P$. Let $P$ be the double complex:

$$P_\cdot = (Q_1, Q_{1m} = \bigoplus_{i+j=m} \mathbb{Z}_{tr}[X_i \times_k X'_j]) \to (Q_2, Q_{2m} = \bigoplus_{i+j=m} (\mathbb{Z}^{1U}_{tr}[X_i \times_k X'_j] \oplus \mathbb{Z}^{2U}_{tr}[X_i \times_k X'_j]))$$

and let $P$ be the associated single complex.

Assume that $C$ is smooth over $k$ and let $T_{a,C}$ be the tangent space to $C$ at $a$ viewed as a scheme over $k(a)$. For any motive $M$ over $\eta$ there is a canonical morphism:

$$(3.5) \quad PTN_\eta(M) \to \mathbb{Z}_{tr}[T_{a,C} \setminus 0] \otimes PTN_\eta(M)$$

Indeed, choose a section $f$ of the tangent bundle $T_C \to C$ over a neighborhood of $a$ with the following properties: $f(a) = 0$ and $df(a) : T_{a,C} \to T_{a,C} \times T_{a,C}$ is the diagonal morphism. The section $f$ defines a morphism of motives over $\eta$: $f : Z \to \mathbb{Z}_{tr}[(T_C \setminus \text{zero section}) \times_C \eta]$. Thus, by part D of Lemma ??, we get:

$$PTN_\eta(M) \to PTN_\eta(\mathbb{Z}_{tr}[(T_C \setminus \text{zero section}) \times_C \eta] \otimes M) \to \mathbb{Z}_{tr}[T_{a,C} \setminus 0] \otimes PTN_\eta(M).$$

Using the second statement in (??), D we readily see that the composition does not depend on the choice of $f$. 
3.4. Let $X_C \to C$ be a scheme over $C$. Composing (4.2) with the morphism $PTN_a \to TN_a$ we get

$$PTN_a(X_C) \to \mathbb{Z}_{tr}[T_{a, C} \setminus 0] \otimes TN_a(X_C).$$

If $X_C \to C$ is proper $\mathbb{Z}_{tr}[X_a] \cong TN_a(X_C)$, by (4.2). B. If, in addition, $X_C \to C$ is smooth, the canonical morphism $PTN_a(X_C) \to \mathbb{Z}_{tr}[(T_{a, C} \setminus 0) \times X_a]$ is an isomorphism.

4. Motivic sheaves

4.1. First, we consider motivic sheaves over a semilocal base of dimension 1 over a field. Let $C$ be a curve over $k$ and let $S$ be the localization of $C$ at a finite number of closed points $(a_1, a_2, \ldots, a_n)$. Using the general gluing construction from Section 2, we define:

$$M_{gm}^{eff}(S) = \left( \bigoplus_i M_{gm}^{eff}(k(a_i)), M_{gm}^{eff}(\eta), \oplus_i PTN_a \right).$$

A scheme $X \to S$ defines a motivic sheaf $F$ as follows: $F_a = TN_a(X)$, $F_\eta = \mathbb{Z}_{tr}[X_\eta]$ and the morphisms $\alpha_i : PTN_a(F_\eta) \to F_a$ come from the canonical morphism of functors: $PTN_a \to TN_a$.

By part D of Lemma 3.4, for any motive $M$ over $\eta$ the canonical morphism $\mathbb{Z}(1) \otimes PTN_a(M) \to PTN_a(\mathbb{Z}(1) \otimes M)$ is a quasi-isomorphism. Thus we get the Tate twist endofunctor:

$$(4.1) \otimes \mathbb{Z}(1) : M_{gm}^{eff}(S) \to M_{gm}^{eff}(S)$$

The Voevodsky’s Cancellation Theorem [Voe3] applied to the categories of motives over points of $S$ implies the following. The Tate twist endofunctor (4.1) is homotopically fully faithful (i.e. the functor $\otimes \mathbb{Z}(1) : Ho(M_{gm}^{eff}(S)) \to Ho(M_{gm}^{eff}(S))$ is fully faithful). Denote by $M_{gm}(S)$ the category obtained by inverting the Tate twist.

4.2. Assume that $C$ is smooth at the points $a_i$. We have a functor $C^b(\mathbb{Z}_{tr}[Sm_a]) \to M_{gm}^{eff}(S)$ from the category of complexes over the additive category of smooth schemes and finite correspondences. One readily checks that this functor descends to functors from Voevodsky’s motives over $S$ [Voe2]: $DM_{gm}^{eff}(S) \to Ho(M_{gm}^{eff}(S))$, $DM_{gm}(S) \to Ho(M_{gm}(S))$. The following result was explained to us by J. Ayoub. The functor $DM_{gm}(S) \to Ho(M_{gm}(S))$ is an equivalence of categories.

4.3. Let $F = (F_a, F_\eta, \alpha_i)$ be a motivic sheaf over $S$. Define a morphism

$$(4.2) \quad PTN_a(F_\eta) \to \mathbb{Z}_{tr}[T_{a, C} \setminus 0] \otimes F_a$$

to be the composition of the morphism (3.2) and $id \otimes \alpha_i$. The motivic sheaf is called smooth at $a_i$ if (4.2) is a quasi-isomorphism. The notion of smooth motivic sheaf is a motivic analog of the notion of a local system. By 3.4 the motivic sheaf associated to a smooth proper scheme over $S$ is smooth.

4.4. If the base is a global curve $C$ one could try a similar construction, gluing $M_{gm}^{eff}(\eta)$ and $\prod_{a \in C} M_{gm}^{eff}(k(a))$, where the product is taken over all closed points of $C$ (we denote the resulting category by $M_{gm}^{eff}(C)$), but this definition is not reasonable: $Hom(\mathbb{Z}, \mathbb{Z}(1))$ would be quasi-isomorphic to the complex $cone(\eta^*) \to \prod_{a \in C} \mathbb{Z}[-1]$ instead of expected complex $cone(\eta^*) \to \bigoplus_{a \in C} \mathbb{Z}[-1]$. The correct definition is the following.

Consider the functors

$$C^b(\mathbb{Z}_{tr}[Sm_C]) \otimes_{a \in C} TN_a \prod_{a \in C} M_{gm}^{eff}(k(a)) \to \left( \prod / \bigoplus_{a \in C} M_{gm}^{eff}(k(a)) \right).$$
The composition of these functors descends to the category of geometric motives over η:

\[ (4.3) \quad TN : M_{gm}^{eff}(\eta) \to \left( \prod_a / \bigoplus \right)_{a \in C} M_{gm}^{eff}(k(a)). \]

There is the obvious functor

\[ PTN : M_{gm}^{eff}(\eta) \to \left( \prod_a / \bigoplus \right)_{a \in C} M_{gm}^{eff}(k(a)) \]

(this is the composition of \( \prod_a PTN_a \) and the projection \( \prod \to \prod / \bigoplus \)) and the morphism \( \alpha : PTN \to TN \). We define \( M_{gm}^{eff}(C) \) to be the limit of the following diagram of categories:

\[ (4.4) \quad M_{gm, \Pi}^{eff}(C) \rightrightarrows M_{gm, \Pi / \bigoplus}^{eff}(C). \]

Here \( M_{gm, \Pi / \bigoplus}^{eff}(C) \) is the category obtained by gluing \( M_{gm}^{eff}(\eta) \) and \( (\prod / \bigoplus)_{a \in C} M_{gm}^{eff}(k(a)) \).

The first functor in \( (4.4) \) takes \( F = (\prod_a F_a, F_\eta, \prod_a \alpha_a) \) to \( (TN(\eta), F_\eta, \alpha) \). The second one is the obvious functor coming from the projection \( \prod \to \prod / \bigoplus \).

Informally speaking, objects of \( M_{gm}^{eff}(C) \) are allowed to have only finitely many singularities.

**4.5.** Denote by \( p^* : M_{gm}^{eff}(k) \to M_{gm}^{eff}(C) \) the functor which takes a motive \( M \) over \( k \) to the constant sheaf: \( (p^*M)_a = p_{k(a)/k}^*M \). One readily sees that the functor \( p^* \) has the right adjoint quasi-functor \( p_* : M_{gm}^{eff}(C) \to M_{gm}^{eff}(k) \).

**Conjecture.** Let \( k \) be a field of characteristic 0. For any smooth motive \( F \) over \( \mathbb{A}^1_k \) the canonical morphism:

\[ p^* p_* F \to F \]

is a quasi-isomorphism.

For example this conjecture would imply that for any smooth proper scheme \( X \) over \( \mathbb{A}^1_k \) the restriction morphism \( CH^i(X) \to CH^i(X_a) \) is an isomorphism, for any \( a \in \mathbb{A}^1_k(k) \).

**Bibliography.**


[Deg] F. Deglise, private communications.


