10. Appendix

Notations:

$K$ is a finite extension of $Q_p$;

$O_K \subset K$ is the ring of integers;

$m \subset O_K$ is the maximal ideal, $k = O_K/m$, $q = \sharp k$;

$A_K$ is an abelian variety over $K$, $n = \dim A$;

$A_0$ is the Néron model of $A_K$ over $O_K$;

$\omega$ is an invariant differential form of degree $n$ on $A_0$, which does not vanish on the special fiber $A_0 \otimes \text{Spec } k$;

$\mu_\omega$ denotes the corresponding measure on $A_K(K)$.

What follows is a computation of $\mu_\omega(A_K(K))$ in terms of Tate modules $T_l(A_K)$.

The first simple observation (due to A. Weil) is that $\mu_\omega(A_K(K)) = N_k q^{-n}$, where $N_k$ denotes the order of the group $A_0(k)$. Consider the exact
sequence of group schemes over Spec $k$

$$0 \rightarrow A_0^0 \rightarrow A_0 \otimes \text{Spec } k \rightarrow C \rightarrow 0$$

where $A_0^0$ is the connected component of the identity in $A_0 \otimes \text{Spec } k$, and $C$ is a finite smooth group scheme over Spec $k$. Since $H^1(\text{Gal}(\overline{k}/k); A_0^0(\overline{k})) = 0$, it follows that the sequence

$$0 \rightarrow A_0^0(k) \rightarrow A_0(k) \rightarrow C(k) \rightarrow 0$$

is exact. Hence $N_k = \# A_0^0(k) : \# C(k)$.

Let $I \subset \text{Gal}(\overline{K}/K)$ be the inertia subgroup and $J \subset Z[I]$ stand for the augmentation ideal.

Put $a := n - \frac{1}{2} \dim(T_l(A_K) \otimes \mathbb{Q})^J$. We claim that $a$ is an integer. Moreover, one can prove the following.

**Proposition 10.1.** $H_c^i(A_0^0; Q_l) = (A^{2n+2a-i}(T_l(A_K) \otimes \mathbb{Q})^J(-n-a))$ for $i \geq 2a$ and $H_c^i(A_0^0, Q_l) = 0$ otherwise.

Therefore, by the Lefschetz formula

$$\# A_0^0(k) = q^{n+a} \sum_i (-1)^i \text{Tr} \{ \text{Fr}^*; \bigwedge_i T_l(A_K)^I \} \}.$$

The following theorem is essentially due to Grothendieck [4].

**Theorem 3.** Let $C(\overline{k})_l$ denote the $l$-part of $C(\overline{k})$. For any $l \neq p$ there is a canonical isomorphism of $\text{Gal}(\overline{k}/k)$-modules

$$C(\overline{k})_l = \text{coker}(\langle T_l(A_K) \otimes Q_l \rangle^I \rightarrow \langle T_l(A_K) \otimes Q_l/Z_l \rangle^I)$$

The case of $l = p$ is more involved. First we suppose that $A_0$ is semi-abelian.

There is a weight filtration on $T_p(A_K)$:

$$W_{-2}T_p(A_K) \subset W_{-1}T_p(A_K) \subset W_0T_p(A_K) = T_p(A_K),$$

where $W_{-1}T_p(A_K) = \lim \ker(p^n : A_0(O_{\overline{K}}) \rightarrow A_0(O_{\overline{K}}))$, $W_{-2}(T_p(A_K))$ is the orthogonal complement to $W_{-1}T_p(A_K^*), (A_K^* \text{ stands for the dual abelian variety})$. One can prove [4] that the restriction of the representation $\text{Gal}(\overline{K}/K)$ of $T_p(A_K)/W_{-1}$ to $\text{Gal}(\overline{K}/L)$, where $L$ is a finite, unramified extension of $K$, is trivial.

Let $B_{ss}$ denote the Fontaine ring [2]. Consider the space

$$H = (T_p(A_K) \otimes B_{ss})^{\text{Gal}(\overline{K}/K)}.$$

It is a vector space over Fract $W(k)$ equipped with an action of a nilpotent operator (called the monodromy operator) $N : H \rightarrow H$, and Frobenius-linear operator $\phi : H \rightarrow H$, satisfying $N\phi = p\phi N$. The operators $N$ and $\phi$ come from the corresponding operators on $B_{ss}$. The weight filtration defines one on $H$. Moreover, its adjoint factors in weights 0 and $-2$ have canonical sublattices over $W(k)$ of maximal ranks (the lattice in weight 0 is isomorphic to $((T_p(A_K)/W_{-1}) \otimes O_{\overline{K}_{nr}})^{\text{Gal}(\overline{K}_{nr}/K)}$, the definition of the
other lattice involves the dual variety). We denote these lattices by \( L_0 \) and \( L_{-2} \). One can show that \( \phi \) and the monodromy operator \( N \) induce corresponding operators on \( L_i: \phi_i : L_i \to L_i \) \((i = 0; -2)\), \( N : L_0 \to L_{-2} \) with \( N\phi_0 = p\phi_{-2}N \).

**Theorem 4.** a) Suppose that \( A_0 \) is semi-abelian. Then there is an isomorphism

\[
C(k)_p \to (\ker[N : L_0 \otimes \mathbb{Q}_p/\mathbb{Z}_p \to L_{-2} \otimes \mathbb{Q}_p/\mathbb{Z}_p])^{\phi_0}.
\]

b) Let \( L \) be a finite, normal, totally ramified extension of \( K \), such that \( A_K \otimes \text{Spec} L \) admits a semi-abelian Neron model \( A_{0,L} \). Then for any prime \( l \), which does not divide degree of the extension \( L \) over \( K \), there is an isomorphism \( C_l \to C_{L,l} \), where \( C_{L,l} \) stands for the group of connected components of \( A_{0,L} \).

**Remark 10.1.** One can show that for any \( A_K \) of dimension \( n \) there exists a normal extension \( L \) of degree not greater then \( 2GL_{2n}(\mathbb{Z}/12\mathbb{Z}) \) such that \( A_K \otimes \text{Spec} L \) has semi-stable reduction.

**References**