1. Regard \( \mathbb{R}^5 \) as a subspace of \( \mathbb{R}^6 \) in the usual way. Prove that there is no way to continuously assign, to every 1-dimensional subspace \( \ell \) in \( \mathbb{R}^6 \), a corresponding 1-dimensional subspace \( F(\ell) \) of \( \mathbb{R}^5 \) if we require the property that for \( \ell \subseteq \mathbb{R}^5 \) one has \( F(\ell) = \ell \).

2. Explain why the pair \((D^n, S^{n-1})\) has the homotopy extension property. If \( X \) is obtained from \( A \) by attaching an \( n \)-cell, explain why \((X, A)\) also has the homotopy extension property.

3. Let \( p: E \to B \) be a fiber bundle with fiber \( F \). Define the connecting homomorphism \( \partial: \pi_k(B) \to \pi_{k-1}(F) \) and prove that

\[ \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \]

is exact in the middle spot.

4. Compute \( \pi_i(\mathbb{R}P^7 \vee S^4) \) for \( i \leq 4 \).

5. Let \( M \) be a compact, connected \( n \)-manifold, where \( n \geq 2 \). Assume there is a map \( f: S^n \to M \) \((n = \text{dim } M)\) that is injective on \( H_n(-) \). Prove that \( H_i(M; \mathbb{Q}) = 0 \) for \( 0 < i < n \).

6. Let \( K \) be the Klein bottle, and recall that \( \pi_1(K) \cong \langle a, b \mid aba = b \rangle \).

   (a) How many 3-fold path-connected covering spaces of \( K \) are there, up to isomorphism? Explain.
   (b) Is the torus a 3-fold cover of the Klein bottle? Explain.
   (c) Is the Klein bottle a 3-fold cover of \( S^2? \) Explain.

7. Let \( a_1, \ldots, a_k \in \mathbb{C} \) and \( b_1, \ldots, b_r \in \mathbb{C} \) satisfy \( |a_i| < 1, |b_i| < 1 \) for all \( i \). Let

\[ f(z) = (\text{Re}(z)^4 + 5) \cdot (z - a_1) \cdots (z - a_k) \cdot (\bar{z} - b_1) \cdots (\bar{z} - b_r) \]

Define \( F: S^1 \to S^1 \) by \( F(z) = f(z)/||f(z)||. \) Show that the degree of \( F \) is \( k - r \).

8. Consider the map \( f: S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1 \) given by \( f(x, y, z) = (xy, xz, z) \).

   (a) Calculate \( f_*: H_1(S^1 \times S^1 \times S^1) \to H_1(S^1 \times S^1 \times S^1) \) and \( f^*: H^1(S^1 \times S^1 \times S^1) \to H^1(S^1 \times S^1 \times S^1) \) as maps \( \mathbb{Z}^3 \to \mathbb{Z}^3 \).
   (b) Let \( X \) be the result of gluing two copies of \( D^2 \times S^1 \times S^1 \) along their boundaries, twisting via the map \( f \). That is,

\[ X = [(D^2 \times S^1 \times S^1) \amalg (D^2 \times S^1 \times S^1)]/ \sim \]

where \( x \sim f(x) \) for \( x \in S^1 \times S^1 \times S^1 \). Compute \( H_i(X) \) for all \( i \). [Hint: It might be easier to compute \( H^2(X) \) before \( H_2(X) \).]

9. Let \( M \) be a compact, connected, orientable 6-manifold. Assume that \( H_3(M) = \mathbb{Z}^2 \oplus \mathbb{Z}/2, \chi(M) = 10, \) and that the universal cover of \( M \) is a 7-fold cover. Determine \( H_i(M) \) for all \( i \).

10. Let \( n \geq 1. \)

   (a) If \( M \) and \( N \) are \( n \)-manifolds, prove that \( \chi(M \# N) = \chi(M) + \chi(N) - (1 + (-1)^n). \)
   (b) Prove that no matter what \( n \) is, there is a compact, connected, orientable \( n \)-manifold with Euler characteristic zero.
   (c) Prove that every integer is equal to the Euler characteristic of some compact, connected, orientable 4-manifold.
1. Let $X$ be the 2-manifold formed from two copies of $\mathbb{RP}^2$ according to the following picture:

Calculate $H_\ast(X)$ and determine the homeomorphism type of $X$ in terms of the standard classification of surfaces.

2. Subdivide the following picture to give a $\Delta$-complex structure to $X = \mathbb{RP}^2 \# T$:

Consider the basis $\{\alpha, \gamma, \delta\}$ for $H_1(X; \mathbb{Z}/2)$ and let $\{\hat{\alpha}, \hat{\gamma}, \hat{\delta}\}$ be the dual basis under the Kronecker pairing.

(a) Using your $\Delta$-complex structure, give explicit 1-cochains that represent $\hat{\alpha}$ and $\hat{\gamma}$.

(b) Compute $\hat{\alpha} \cup \hat{\alpha}$ and $\hat{\alpha} \cup \hat{\gamma}$, and decide whether each of these is zero in cohomology.

(c) Determine the basis of $H_1(X; \mathbb{Z}/2)$ that is Poincaré Dual dual to the basis $\{\hat{\alpha}, \hat{\gamma}, \hat{\delta}\}$.

3. Let $M$ be a 5-manifold with $H_1(M)$ finite of odd order. Prove that every map $M \to \mathbb{RP}^6$ is null homotopic. (You may assume that $M$ is a CW-complex).

4. Let $p: E \to B$ be a fiber bundle, where $B$ is path-connected, and let $G = \pi_1(B, b)$. Explain how to get a left action of $G$ on $p^{-1}(b)$, together with a bijection $\pi_0(p^{-1}(b))/G \to \pi_0(E)$. (You do not need to prove that your action is indeed an action, but you should prove that your bijection is indeed a bijection).

5. Let $X$ be a CW-complex, with $X_n$ its $n$-skeleton. Prove that $\pi_i(X_n) \to \pi_i(X)$ is surjective for $i \leq n$ and injective for $i \leq n - 1$. Is this also true for $H_i(X_n) \to H_i(X)$? Explain.

6. (a) Calculate the degree of the map $\mathbb{CP}^1 \to \mathbb{CP}^1$ given by $[z_0 : z_1] \to [z_0^2 : z_1^2]$.

(b) Define $f: \mathbb{CP}^3 \to \mathbb{CP}^3$ by $[z_0 : z_1 : z_2 : z_3] \to [z_0^2 : z_1^2 : z_2^2 : z_3^2]$. Calculate the induced map $f_*: H^6(\mathbb{CP}^3) \to H^6(\mathbb{CP}^3)$.

7. Prove that there does not exist a continuous way to select a nonzero vector on every line in $\mathbb{R}^3$.

8. Fix a point $x \in S^3$ and let $\mu: S^3 \times S^3 \to S^3$ be a continuous multiplication on $S^3$ having $x$ as a two-sided identity element. If $f, g: S^3 \to S^3$ then prove that the degree of the composite

$$S^3 \xrightarrow{\Delta} S^3 \times S^3 \xrightarrow{f \times g} S^3 \times S^3 \xrightarrow{\mu} S^3$$

is equal to $\deg(f) + \deg(g)$, where $\Delta(u) = (u, u)$. [Hint: Compute the induced map $H^3(S^3) \to H^3(S^3)$.]
9. Compute the cohomology groups (with \( \mathbb{Z} \) coefficients) of \((\mathbb{R}P^6/\mathbb{R}P^2) \times \mathbb{R}P^5\).

10. Let \( X \) and \( Y \) be two pointed spaces, and write \( \pi_1: X \vee Y \to X \) and \( \pi_2: X \vee Y \to Y \) for the two projections. If \( \alpha \in H^i(X) \) and \( \beta \in H^j(Y) \) for \( i > 0 \) and \( j > 0 \), prove that \((\pi_1)^*(\alpha) \cup (\pi_2)^*(\beta) = 0\).
LETOVERS

1. Prove that $\Sigma T$ is homotopy equivalent to $S^2 \vee S^2 \vee S^3$ (hint: analyze the suspension of the attaching map of the 2-cell for $T$).

2. Let $(X, B)$ be a relative CW-complex where all cells have dimension at least $n + 2$. Prove that any map $B \to K(A, n)$ extends to a map $X \to K(A, n)$.

3. (a) Prove that every map $f: \mathbb{C}P^2 \to \mathbb{C}P^2$ has a fixed point.
   (b) Prove that any covering space $p: \mathbb{C}P^2 \to X$ must be a homeomorphism.

4. Let $X$ be a topological space with $H_0 = \mathbb{Z}$, $H_1(X) = \mathbb{Z}/12$, $H_2(X) = \mathbb{Z}$, $H_3(X) = 0$, $H_4(X) = \mathbb{Z}$, and $H_i(X) = 0$ for $i \geq 5$.
   (a) Prove that $X$ is not homotopy equivalent to a compact, orientable manifold.
   (b) Could $X$ be homotopy equivalent to a compact, non-orientable manifold? Explain.

5. Explain how to construct an Eilenberg-MacLane space of type $K(\mathbb{Z}/2, 5)$, and prove that your construction really is this type of space.

6. Let $X$ be a space with finitely-generated homology groups. Suppose $f: X \to \mathbb{C}P^n$ induces a nonzero map $H_{2n}(X) \to H_{2n}(\mathbb{C}P^n)$. Prove that $H^2(X; \mathbb{Z})$ and $H_2(X; \mathbb{Z})$ are both nonzero.

7. Recall that the space $U_n$ of complex, unitary matrices is a path-connected manifold.
   (a) For any $A \in U_n$, prove that $R_A: U_n \to U_n$ given by $B \mapsto B \cdot A$ is homotopic to the identity.
   (b) Show that $\chi(U_n) = 0$ (Hint: Lefschetz Fixed Point Theorem).

8. Let $X$ be the 4-manifold $\mathbb{R}P^4 \# \mathbb{C}P^2$. Calculate $\pi_1(X)$ and the cohomology ring $H^*(X)$.

9. (Note: The $\pi_1$ calculation might be too hard.) Let $X = (S^2 \times S^1)/ \sim$ where $(x, y) \sim (-x, -y)$ for $x \in S^2$, $y \in S^1$.
   (a) Calculate $\pi_i(X)$ for $1 \leq i \leq 3$.
   (b) Show that $p: X \to \mathbb{R}P^2$ given by $p([(x, y)]) = [x]$ is a fiber bundle with fiber $S^1$.
   (c) Is $X$ homeomorphic to $S^1 \times \mathbb{R}P^2$? Explain.