MATH 636, PROBLEMS FOR THE MIDTERM

1. For a finitely generated abelian group $G$ denote by $T(G)$ the subgroup of $G$ consisting of all elements of finite order and by $F(G)$ the quotient $G/T(G)$. Let $X$ be a space such that the groups $H_q(X)$ are finitely generated for all $q$. Prove that $H^q(X;\mathbb{Z})$ are also finitely generated and

$$H^q(X;\mathbb{Z}) \xrightarrow{\sim} F(H_q(X)) \oplus T(H_{q-1}(X)).$$

Remark: the above isomorphism is not canonical.

2. Is it true that for any space $X$ and a field $F$ one has

$$H^q(X, F) \xrightarrow{\sim} \text{Hom}(H_q(X), F)?$$

3. Define the Alexander-Whitehead map

$$C^*(X, A) \otimes C^*(X, A) \to C^*(X \times X, A)$$

and then explain how it gives rise to the external product

$$H^p(X, A) \otimes H^q(X, A) \to H^{p+q}(X \otimes X, A).$$

4. State the Poincaré Duality Theorem. Use Poincaré duality construct a ring isomorphism

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}[x]/x^{n+1}.$$

5. Use the previous problem to show that $\mathbb{C}P^n$ ($\infty > n > 0$) does not have a structure of an $H$-space.

6. Compute the ring structure on $H^*(\mathbb{R}P^n;\mathbb{Z}/2^k)$.

7. Let $M$ be a compact, connected $n$-manifold, where $n \geq 2$. Assume there is a map $f : S^n \to M$ that is injective on $H_n(-)$. Prove that $H_i(M;\mathbb{Q}) = 0$ for $0 < i < n$.

8. Prove that $\mathbb{R}P^3$ is not homotopy equivalent to $S^3 \vee \mathbb{R}P^2$.

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Date: May 5, 2015.