MATH 281: Multivariable Calculus

February 10, 2010

Midterm

Name: Weiyong HE

ID: Solution

In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	20 pts	
2	10 pts	
3	8 pts	
4	12 pts	
Total	50 pts	

1. Let

$$\mathbf{r}(t) = \langle 4t, -\cos 3t + 1, -\sin 3t \rangle$$

(a) Find the arclength of the curve between $\langle 0, 0, 0 \rangle$ and $\langle 2\pi, 1, 1 \rangle$; compute the unit tangent vector $\mathbf{T}(t)$.

$0 \leq t \leq \pi/2$ $\qquad t=0, \ \vec{r}(0) = \langle 0, 0, 0 \rangle$

$t = \pi/2, \ \vec{r}(\pi/2) = \langle 2\pi, 1, 1 \rangle$

(1) $\vec{r}\,'(t) = \langle 4, 3\sin 3t, -3\cos 3t \rangle$

$|\vec{r}\,'(t)| = \sqrt{4^2 + 3(\sin 3t)^2 + (-3\cos 3t)^2} = \sqrt{16+9} = 5$

$S = \int_0^{\pi/2} 5\,dt = 5\pi/2$

(2). $T(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left\langle \dfrac{4}{5}, \dfrac{3\sin 3t}{5}, \dfrac{-3\cos 3t}{5} \right\rangle$

$$\mathbf{r}(t) = \langle 4t, -\cos 3t + 1, -\sin 3t\rangle$$

(b) Compute the unit normal vector $\mathbf{N}(t)$ and the unit binormal vector $\mathbf{B}(t)$.

$N(t) = \dfrac{T'(t)}{|T'(t)|}$. $\quad T'(t) = \langle 0, \dfrac{9\cos 3t}{5}, \dfrac{9\sin 3t}{5} \rangle$, $|T'(t)| = \dfrac{9}{5}$

$N(t) = \langle 0, \cos 3t, \sin 3t \rangle$

$B(t) = T \times N = \langle \dfrac{4}{5}, \dfrac{3}{5}\sin 3t, -\dfrac{3\cos 3t}{5} \rangle \times \langle 0, \cos 3t, \sin 3t \rangle$

$= \langle \dfrac{3}{5}, -\dfrac{4}{5}\sin 3t, \dfrac{4}{5}\cos 3t \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{4}{5} & \frac{3}{5}\sin 3t & -\frac{3\cos 3t}{5} \\ 0 & \cos 3t & \sin 3t \end{vmatrix}$$

2. Consider the point $(1, -3, 8)$ and the plane $2x + y + z = 11$.

(a) Parametrize the line passing through the point which is perpendicular to the plane.

$$\vec{n} = \langle 2, 1, 1 \rangle$$

So. the line is parallel to $\vec{n}$

eq.

$$\begin{cases} x = 1 + 2t \\ y = -3 + t \\ z = 8 + t \end{cases}$$

3. (a) For which values of a is the vector $\mathbf{v} = \langle a^2, 3, 1\rangle$ orthogonal to $\mathbf{w} = \langle 1, a, -2\rangle$?

$v \cdot w = 0 \qquad a^2 + 3a - 2 = 0$

$$a = \frac{-3 \pm \sqrt{3^2 - 4(-2)}}{2} = \frac{-3 \pm \sqrt{17}}{2}$$

(b) For which values of a is $\mathbf{v}$ orthogonal to $\mathbf{v} \times \mathbf{w}$?

$\vec{v}$ is always orthogonal to $\vec{v} \times \vec{w}$, for any a.

4. Suppose at time t the location of a bee is given by

$$\mathbf{r}(t) = \langle t\cos t, t\sin t, t\rangle \qquad t \geq 0.$$

(a) Find the velocity and the acceleration of the bee at time $t = 2\pi$.

$v(t) = \vec{r}\,'(t) = \left(t\langle \cos t, \sin t, 1\rangle\right)' = \langle \cos t, \sin t, 1\rangle + t\langle -\sin t, \cos t, 0\rangle$

$a(t) = v'(t) = \langle -\sin t, \cos t, 0\rangle + \langle -\sin t, \cos t, 0\rangle + t\langle -\cos t, -\sin t, 0\rangle$

$= \langle -2\sin t, 2\cos t, 0\rangle + t\langle -\cos t, -\sin t, 0\rangle$

$\underline{t = 2\pi}$, $\quad \vec{v}(2\pi) = \langle 1, 0, 1\rangle + 2\pi\langle 0, 1, 0\rangle$

$= \langle 1, 2\pi, 1\rangle$

$\vec{a}(2\pi) = \langle 0, 2, 0\rangle + 2\pi(-1, 0, 0)$

$= \langle -2\pi, 2, 0\rangle$

(b) Find how fast the bee changes its direction at time $t = 2\pi$ (find the curvature).

$$k = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{|\vec{v} \times \vec{a}|}{|v|^3}$$

One:

$$|v| = \sqrt{2 + t^2}$$

$$\vec{v} = \langle \cos t - t\sin t,\ \cos t + \sin t,\ 1 \rangle$$

$$\vec{a} = \langle -2\sin t - t\cos t,\ 2\cos t - t\sin t,\ 0 \rangle$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t - t\sin t & \cos t + \sin t & 1 \\ -2\sin t - t\cos t & 2\cos t - t\sin t & 0 \end{vmatrix} = \langle t\sin t - 2\cos t,\ -t\cos t - 2\sin t,\ t^2 + 2 \rangle$$

$$|\vec{v} \times \vec{a}| = \sqrt{(t^2+2)^2 + t^2 + 4}$$

$$= \sqrt{t^4 + 4t^2 + 4 + t^2 + 4} = \sqrt{t^4 + 5t^2 + 8}$$

$$\boxed{k = \frac{\sqrt{t^4 + 5t^2 + 8}}{\left(\sqrt{t^2 + 2}\right)^3}}$$ $\boxed{\text{for any } t}$

$$k(2\pi) = \frac{\sqrt{16\pi^4 + 20\pi^2 + 8}}{\left(\sqrt{4\pi^2 + 2}\right)^3}$$

Two:

at $t = 2\pi$,

$$\vec{v} = \langle 1, 2\pi, 1 \rangle$$

$$\vec{a} = \langle -2\pi, 2, 0 \rangle$$

$$\vec{v} \times \vec{a} = \langle -2, -2\pi, 4\pi^2 - 2 \rangle$$

$$|\vec{v} \times \vec{a}| = \sqrt{4 + 4\pi^2 + (4\pi^2 + 2)^2}$$

$$= \sqrt{16\pi^4 + 20\pi^2 + 8}$$

$$|\vec{v}| = \sqrt{4\pi^2 + 2}$$

$$k(2\pi) = \frac{\sqrt{16\pi^4 + 20\pi^2 + 8}}{\left(\sqrt{4\pi^2 + 2}\right)^3}$$