MATH 281: Multivariable Calculus

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Final exam—practice

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In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations. This is a practice exam; the **actual** exam contains **only eight** problems which resemble the problems in the practice exam.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	15 pts	
2	12 pts	
3	10 pts	
4	12 pts	
5	12 pts	
6	12 pts	
7	12 pts	
8	15 pts	
9	15 pts	
Total	115 pts	

1. Consider the curve $\mathbf{r}(t) = (e^t \cos t, e^t \sin t, e^t)$. Compute at $t = \pi$:

(a) The unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$ and the unit binormal vector $\mathbf{B}$.

(b) The curvature k.

(a). $\vec{\gamma}'(t) = \left(e^t (\cos t, \sin t, 1)\right)' \underset{\text{Product R.}}{=} e^t \langle \cos t, \sin t, 1\rangle + e^t \langle -\sin t, \cos t, 0\rangle$

Since $\langle \cos t, \sin t, 1\rangle \cdot \langle -\sin t, \cos t, 0\rangle = 0$ $\quad$ $(|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$ when $\vec{u}\cdot\vec{v}=0)$

$|\vec{\gamma}'(t)| = e^t \sqrt{|\langle \cos t, \sin t, 1\rangle|^2 + |\langle -\sin t, \cos t, 0\rangle|^2}$

$= e^t \cdot \sqrt{3}$

$T(t) = \dfrac{\vec{\gamma}'(t)}{|\vec{\gamma}'(t)|} = \dfrac{\langle \cos t, \sin t, 1\rangle}{\sqrt{3}} + \dfrac{\langle -\sin t, \cos t, 0\rangle}{\sqrt{3}}$, $\quad T'(t) = \dfrac{\langle -\sin t, \cos t, 0\rangle}{\sqrt{3}} + \dfrac{\langle -\cos t, -\sin t, 0\rangle}{\sqrt{3}}$

$|T'(t)| = \frac{1}{\sqrt{3}} \sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \sqrt{\frac{2}{3}}$, $\quad N(t) = \dfrac{T'(t)}{|T'(t)|} = \left\langle \dfrac{-\cos t - \sin t}{\sqrt{2}}, \dfrac{\cos t - \sin t}{\sqrt{2}}, 0\right\rangle$.

at $t = \pi$, $\quad T(\pi) = \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$, $\quad N(\pi) = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\rangle$

$$B(\pi) = T \times N = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{vmatrix} = \frac{1}{\sqrt{6}} \langle -1, 1, 2\rangle.$$

(b). $k = \dfrac{|\vec{\gamma}'(t) \times \vec{\gamma}''(t)|}{|\vec{\gamma}'(t)|^3}$ $\quad$ at $t = \pi$, $\quad |\vec{\gamma}'(t)| = \sqrt{3} e^{\pi}$. $\quad \vec{\gamma}'(t) = e^{\pi} \langle -1, -1, 1\rangle$

$\vec{\gamma}''(t) \underset{\text{Product Rule}}{=} e^t \langle \cos t, \sin t, 1\rangle + 2 e^t \langle -\sin t, \cos t, 0\rangle + e^t \langle -\cos t, -\sin t, 0\rangle$ $\quad$ (at $t = \pi$).

$= e^t \langle -1, 0, 1\rangle + 2e^t \langle 0, -1, 0\rangle + e^t \langle 1, 0, 0\rangle = e^{\pi} \langle 0, -2, 1\rangle$

$$k = \frac{e^{2\pi} |\langle -1, -1, 1\rangle \times \langle 0, -2, 1\rangle|}{(\sqrt{3} e^{\pi})^3} = \frac{\sqrt{6} e^{2\pi}}{e^{3\pi} 3\sqrt{3}} = \frac{\sqrt{2}}{3} e^{-\pi}.$$

2. (a) Find equations of the tangent plane and normal line to the surface $z + 2 = xe^y \cos(z)$ at the point $(2, 0, 0)$.

(b) If a plane which passes through the point $(-1, 2, 3)$ is parallel to the given plane above, find the distance of the two planes.

(a). $F(x,y,z) = z - xe^y\cos(z) = 2$

$\nabla F = \langle -e^y\cos(z), -xe^y\cos(z), 1 + xe^y\sin(z) \rangle$.

at $(2,0,0)$ $\nabla F = \langle -1, -2, 1 \rangle$

tangent plane : $\langle -1, -2, 1 \rangle \cdot \langle x-2, y-0, z-0 \rangle = 0$

$-(x-2) - 2y + z = 0, \quad x + 2y - z = 2.$

(b) $d = \dfrac{|x+2y-z-2|}{\sqrt{1^2+2^2+1^2}} = \dfrac{|-1+4-3-2|}{\sqrt{6}} = \dfrac{2}{\sqrt{6}}$

3. Let $z = f(x, y)$ defined by

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}, (x,y) \neq (0,0),$$
$$f(0,0) = 0.$$

Is $f(x, y)$ a continuous function? Why? **Hint:** find if the following limit exists

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

Step 1:

along: $y = x$, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^3}{x^2 + x^4} = \lim_{x\to 0} \frac{x}{1+x^2} = 0$

along $y = x^2$ $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4 + x^4} = \frac{1}{2}$

so: $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

Step 2: so. $f(x,y)$ is not continuous at $(0,0)$.

4. The volume of a torus with inner radius r and outer radius R is given by

$$V = \frac{\pi^2}{4}(R-r)^2(R+r).$$

A donut, represented by a torus with $R = 7$ and $r = 3$ is dipped completely in chocolate. If the thickness of the chocolate is 0.2 (that is $dR = 0.2$ and $dr = 0.2$), use differentials to estimate how much chocolate (the volume) was used.

<u>step 1</u>: $dV = \left(\frac{\pi^2}{4}(R-r)^2(R+r)\right)_R dR + \left(\frac{\pi^2}{4}(R-r)^2(R+r)\right)_r dr$

$= \left(2\frac{\pi^2}{4}(R-r)(R+r) + \frac{\pi^2}{4}(R-r)^2\right) dR + \left(\frac{\pi^2}{4}(R-r)^2 - \frac{\pi^2}{4}2(R-r)(R+r)\right) dr$

$(df = f_x dx + f_y dy, \quad dV = V_R dR + V_r dr)$

<u>Step 2</u>: Realize that when $dR = 0.2$, $dr = 0.2$ (the thickness of chocolate.)

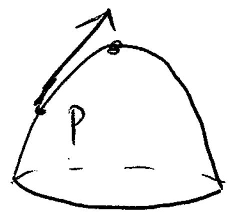
<u>then</u> dV is an approximation of chocolate used to cover a donut.

$dV = \left(\frac{\pi^2}{2}(7-3)(7+3) + \frac{\pi^2}{4}(7-3)^2\right)0.2 + \left[\frac{\pi^2}{4}(7-3)^2 - \frac{\pi^2}{2}(7-3)(7+3)\right]0.2$

$= \frac{\pi^2}{2} 16 \cdot 0.2 = 1.6\pi^2$

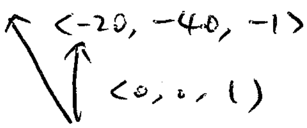
<u>So</u>: $\Delta V \approx dV = 1.6\pi^2$

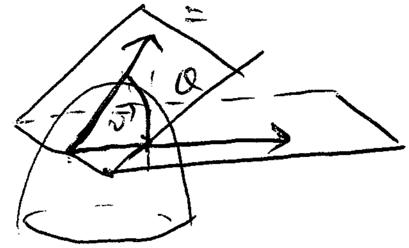
5. Suppose you are climbing a hill whose shape is given by $z = 500 - x^2 - 2y^2$ and you are at the point $(10, 10, 200)$. In which direction should you proceed initially in order to reach the top of the hill in the shortest path? If you climb in that direction, at what angle above horizontal will you be climbing initially?



(1) To reach the top in the shortest path, we always climb along the direction of the gradient vector field, in which direction the height (the function value) changes fastest.

So: $\nabla z = \langle -2x, -4y \rangle = \langle -20, -40 \rangle = \vec{v}$





(2). along this direction, if we climb the hill, there is an angle above horizontal θ, which is the angle between the tangent plane of $z = 500 - x^2 - 2y^2$ at $(10, 10, 200)$ and horizontal plane. the normal direction of the tangent plane

is: $\langle f_x, f_y, -1 \rangle = \langle -20, -40, -1 \rangle$.

~~cos θ =~~

$$\cos\theta = \frac{|\langle -20, -40, -1 \rangle \cdot \langle 0, 0, 1 \rangle|}{\sqrt{(-20)^2 + (40)^2 + (-1)^2}} = \frac{1}{\sqrt{2001}}.$$

6. Find and classify the critical points of the function $z = x^3y + 6x^2 - 8y$.

$$\begin{cases} z_x = 3x^2y + 12x = 0 & (1) \\ z_y = x^3 - 8 = 0 & (2) \end{cases}$$

From (2) we get $x = 2$, hence $y = -2$

$z_{xx} = 6xy + 12$ $\qquad$ $z_{xx} = -12$

$z_{yy} = 0$ $\qquad$ at $(2, -2)$ $\qquad$ $z_{yy} = 0$

$z_{xy} = 3x^2$ $\qquad$ $z_{xy} = 12$

$$D = \begin{vmatrix} -12 & 12 \\ 12 & 0 \end{vmatrix} = -144 < 0$$

So $(x=2, y=-2)$ is a saddle point.

7. (a) Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, where $z = \sqrt{x^2 + 2y^2}, x = s + 2t, y = st^{-1}$.

(b) The radius of a right circular cone is increasing at a rate of 4 inches per minute and its height is decreasing at a rate of 2 inches per minute. At what rate is the volume of the cone changing when the radius is 10 inches and the height is 20 inches?

(a)

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+2y^2}}, \quad \frac{\partial z}{\partial y} = \frac{4y}{2\sqrt{x^2+2y^2}} = \frac{2y}{\sqrt{x^2+2y^2}}$$

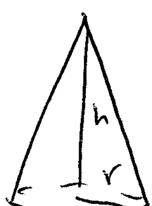
$$\frac{\partial x}{\partial s} = 1 \quad \frac{\partial x}{\partial t} = 2, \quad \frac{\partial y}{\partial s} = t^{-1} \quad \frac{\partial y}{\partial t} = -st^{-2}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \frac{x}{\sqrt{x^2+2y^2}} + \frac{2y}{\sqrt{x^2+2y^2}} \cdot t^{-1}$$

$$= \frac{s+t}{\sqrt{(s+t)^2+2s^2t^{-2}}} + \frac{2st^{-2}}{\sqrt{(s+t)^2+2s^2t^{-2}}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \frac{2(s+t)}{\sqrt{(s+t)^2+2s^2t^{-2}}} + \frac{2s^2t^{-3}}{\sqrt{(s+t)^2+2s^2t^{-2}}}$$

(b)



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= \frac{2}{3}\pi r h \cdot \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

$$r = 10, \ h = 20, \ \frac{dr}{dt} = 4, \ \frac{dh}{dt} = -2$$

$$\frac{dV}{dt} = \frac{2}{3}\pi \cdot 200 \cdot 4 - \frac{\pi}{3} 100 \cdot 2 = \frac{1400}{3}\pi.$$

8. Find the global maximum and the global minimum of the function $f(x,y) = 2x^3+y^4$ subject to the constraint $x^2+y^2 \le 16$.

typo

8. (1) $f(x,y)=2x^3+y^4$

$f_x = 6x^2$, $f_y=4y^3$ $\qquad f_x=f_y=0 \Rightarrow (x,y)=(0,0)$

the only critical pt. $\qquad f(0,0)=0$.

By Extremum Value Thm: max & min occur at critical pt or on the boundary.

On the boundary: $x^2+y^2=16$. max/min $f(x,y)=2x^3+y^4$.

Lagrange multiplier: $F(x,y,\lambda) = (2x^3+y^4) - \lambda(x^2+y^2-16)$

$F_x=0$, $F_y=0$, $\Rightarrow$

$$\begin{cases} 6x^2-2\lambda x=0 & (1) \\ 4y^3-2\lambda y=0 & (2) \\ x^2+y^2=16. & (3) \end{cases}$$

(1) either $x=0$ or $\lambda=3x$

if $x=0$, $y^2=16$ (by (3)), $f(x,y)=16^2=256$.

(2) either $y=0$ or $\lambda=2y^2$

if $y=0$, $x^2=16$ (by (3)), $x=\pm 4$, $f(4,0)=2\cdot 4^3=8\times 16=128$

$f(-4,0)=2(-4)^3=-128$

(3) if $\lambda=3x=2y^2$. (On the back)

then $y^2=\frac{3}{2}x$

$$x^2 + \frac{3}{2}x = 16.$$

$$2x^2 + 3x - 32 = 0$$

$$x = \frac{-3 \pm \sqrt{9 + 4\times2\times32}}{4}$$

$$= \frac{-3 \pm \sqrt{265}}{4}$$

when $x = \frac{-3-\sqrt{265}}{4} \simeq -4.82$ is not allowed $(x^2+y^2 \le 16)$

$$x = \frac{-3+\sqrt{265}}{4} = 3.32.$$

$$y^2 = \frac{3}{2}x = \frac{3\times3.32}{2} = 4.98$$

$$y^4 \simeq 24.8$$

$$2x^3 = 109.8$$

$$f(x,y) = 109.8 + 24.8 = 134.6$$

max: $x = 0, \quad y^2 = 16 \quad (y = \pm 4) \quad f(0, \pm 4) = 256$

min: $x = -4, \quad y = 0 \quad f(-4, 0) = -128.$

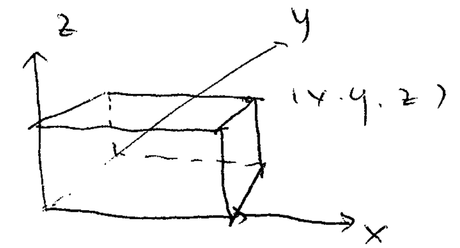
9. Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid

$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1.$$

Hint: Week 8 homework, no. 4.

Solution: By symmetry, we can focus on the first octant.

If we have a point (x, y, z) on the ellipsoid, the rectangular box formed by coordinates planes has volume xyz.



By symmetry, we have 8 (small) rectangular boxes

and so: $V = 8xyz$. Subject to constraint $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1$

To max V, we need to maximize $f(x,y,z) = xyz$

$$F(x,y,z,\lambda) = xyz - \lambda\left(\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}-1\right)$$

$$\begin{cases} F_x = yz - \frac{2\lambda x}{4} = 0 \\ F_y = xz - \frac{2y}{9} = 0 \\ F_z = yx - \frac{2z}{25} = 0 \end{cases}$$

We can assume $x, y, z > 0$

So. $\lambda = \frac{4yz}{2x} = \frac{9xz}{2y} = \frac{25xy}{2z}$

We get: $\frac{y^2}{9} = \frac{x^2}{4} = \frac{z^2}{25}$.

So: $\frac{x^2}{4} = \frac{y^2}{9} = \frac{z^2}{25} = \frac{1}{3}$ $\qquad x = \frac{2}{\sqrt{3}},\ y = \sqrt{3},\ z = \frac{5}{\sqrt{3}}$

$$V = 8xyz = 8\cdot\frac{2}{\sqrt{3}}\cdot\sqrt{3}\cdot\frac{5}{\sqrt{3}} = \frac{80}{\sqrt{3}}$$