MATH 281: Multivariable Calculus

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Midterm

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ID: Solution for practice.

In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	20 pts	
2	10 pts	
3	8 pts	
4	12 pts	
Total	50 pts	

1. Let

$$\mathbf{r}(t) = \langle 3t, 2\cos 2t, 2\sin 2t \rangle$$

(a) Find the arclength of the curve between $0 \le t \le 3$; compute the unit tangent vector $\mathbf{T}$.

$$\vec{r}\,'(t) = \langle 3, -4\sin 2t, 4\cos 2t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{3^2 + (-4\sin 2t)^2 + (4\cos 2t)^2} = \sqrt{3^2+4^2} = 5$$

$$S = \int_0^3 |\vec{r}\,'(t)|\,dt = 5 \times 3 = 15.$$

$$T(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left\langle \frac{3}{5}, \frac{-4\sin 2t}{5}, \frac{4\cos 2t}{5} \right\rangle$$

$$\mathbf{r}(t) = \langle 3t, 2\cos 2t, 2\sin 2t \rangle$$

(b) Compute the unit normal vector $\mathbf{N}(t)$ and the unit binormal vector $\mathbf{B}(t)$.

$$T'(t) = \left\langle 0, -\frac{8}{5}\cos 2t, -\frac{8}{5}\sin 2t \right\rangle$$

$$|T'(t)| = \frac{8}{5}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \langle 0, -\cos 2t, -\sin 2t \rangle$$

$$B(t) = T(t) \times N(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{3}{5} & \frac{-4\sin 2t}{5} & \frac{4\cos 2t}{5} \\ 0 & -\cos 2t & -\sin 2t \end{vmatrix} = \left\langle \frac{4}{5}, \frac{3}{5}\sin 2t, -\frac{3}{5}\cos 2t \right\rangle$$

2. Consider the point $(2, 3, 1)$ and the plane $2x + y - 3z = 2$.

(a) Find the plane which passes through the given point and parallel to the given plane.

Two parallel planes have ~~the~~ same (parallel) normal vectors.

So $\vec{n} = \langle 2, 1, -3 \rangle$

Suppose the plane equation $2x + y - 3z + d = 0$

plug in $(x_0, y_0, z_0) = (2, 3, 1)$

$2 \cdot 2 + 3 - 3 \cdot 1 + d = 0 \qquad d = -4$

$2x + y - 3z - 4 = 0$

(b) Find the distance between two parallel planes above.

The distance is the distance of any point in one plane to the other plane.

$$d = \frac{|2\cdot 2 + 1\cdot 3 - 3\cdot 1 - 2|}{\sqrt{2^2 + 1^2 + (-3)^2}} = \frac{2}{\sqrt{14}}.$$

3. (a) For which values of a is the vector $\mathbf{v} = \langle 3, 2, a \rangle$ orthogonal to $\mathbf{w} = \langle 2a, 4, a \rangle$?

$\vec{v} \cdot \vec{w} = 0$. $\quad \langle 3, 2, a \rangle \cdot \langle 2a, 4, a \rangle = 0$

$$6a + 8 + a^2 = 0$$

$$(a+2)(a+4) = 0$$

$$a = -2, \quad a = -4.$$

(b) Find a unit vector which is orthogonal to both $\langle 0, 1, 2 \rangle$ and $\langle 1, -2, 3 \rangle$.

$\vec{u} = \langle 0, 1, 2 \rangle \times \langle 1, -2, 3 \rangle$ orthogonal to both.

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & -2 & 3 \end{vmatrix} = \langle 7, 2, -1 \rangle$$

Unit vector, $\quad |\vec{u}| = \sqrt{7^2 + 2^2 + (-1)^2} = \sqrt{54} = 3\sqrt{6}$

$\dfrac{\vec{u}}{|\vec{u}|} = \left\langle \dfrac{7}{3\sqrt{6}}, \dfrac{2}{3\sqrt{6}}, \dfrac{-1}{3\sqrt{6}} \right\rangle$ is a unit vector which is orthogonal to both.

4. (a) Find the velocity and acceleration of the position function

$$\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle.$$

(b) Find the curvature of the curve given by the position function above.

(a) $v(t) = \vec{r}\,'(t) = e^t \langle \cos t, \sin t, 1 \rangle + e^t \langle -\sin t, \cos t, 0 \rangle$

$\vec{a}(t) = \vec{r}\,''(t) = \Big(e^t \langle \cos t, \sin t, 1 \rangle + e^t \langle -\sin t, \cos t, 0 \rangle \Big)$
$\quad + e^t \langle -\sin t, \cos t, 0 \rangle + e^t \langle -\cos t, -\sin t, 0 \rangle \Big)$

$= e^t \langle 0, 0, 1 \rangle + 2e^t \langle -\sin t, \cos t, 0 \rangle$

$= e^t \langle -2\sin t, 2\cos t, 1 \rangle$

(b).

$|\vec{r}\,'(t)| = \sqrt{e^{2t}(\cos^2 t + \sin^2 t + 1) + e^{2t}((-\sin t)^2 + \cos^2 t)} = \sqrt{3}\, e^t$

Note: $\langle \cos t, \sin t, 1 \rangle \perp \langle -\sin t, \cos t, 0 \rangle$

$$\vec{r}\,' \times \vec{r}\,'' = e^{2t} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t - \sin t & \cos t + \sin t & 1 \\ -2\sin t & 2\cos t & 1 \end{vmatrix} = e^{2t} \langle \sin t - \cos t, -\cos t - \sin t, 2 \rangle$$

$|\vec{r}\,' \times \vec{r}\,''| = e^{2t} \sqrt{(\sin t - \cos t)^2 + (\cos t + \sin t)^2 + 2^2} = e^{2t} \sqrt{6}$

$$k = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|r'|^3} = \frac{e^{2t}\sqrt{6}}{(e^t \sqrt{3})^3} = \frac{\sqrt{2}}{3} e^{-t}.$$