FINAL EXAM—MATH 251—PRACTICE

Time: 3:15pm—5:15pm, Dec 7th

Name (print): Weiyong HE

Student ID No.:

Signature: :

Grade: Solution Manual.

Problem	Points	Grades
No. 1	32	
No. 2	10	
No. 3	10	
No. 4	10	
No. 5	12	
No. 6	12	
No. 7	14	

Instructions: To receive full credits, all answers must be supported with clear and correct derivations. **No Credit** will be given for the answer without the detailed correct work.

Date: Dec 7th, 2009.

1. Short answer problems.
(1).

$$\lim_{s\to\infty} \frac{\sqrt{-s^2+s^4}}{3s^2-s-2}$$

Quotiant by highest power on bottom.

$$\frac{\sqrt{-s^2+s^4}}{3s^2-s-2} = \frac{\sqrt{-s^2+s^4}/s^2}{(3s^2-s-2)/s^2} = \frac{\sqrt{-\frac{1}{s^2}+1}}{3-\frac{1}{s}-\frac{2}{s^2}}$$

$$\lim_{s\to+\infty} \frac{\sqrt{-s^2+s^4}}{3s^2-s-2} = \lim_{s\to+\infty} \frac{\sqrt{-\frac{1}{s^2}+1}}{3-\frac{1}{s}-\frac{2}{s^2}} = \frac{1}{3}$$

5'

(2). Find the limit of

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$$

With algebra:

$$\tan x - \sin x = \frac{\sin x}{\cos x} - \sin x = \frac{\sin x\,(1-\cos x)}{\cos x}$$

Then

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\frac{\sin x\,(1-\cos x)}{\cos x}}{x^3} = \lim_{x\to 0} \frac{1}{\cos x}\,\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2}$$

$$\lim_{x\to 0}\frac{1}{\cos x} = 1 \qquad \lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2} \overset{L-R}{=} \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}.$$

So,

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}$$

5'

You can also use L'Hospital Rule directly to get the same answer. But the computation is slightly more involved!

(3). Find the derivative of $y = x^{\sin x}$.

$$\ln y = \ln x^{\sin x} = \sin x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x \ln x) = \cos x \ln x + \frac{\sin x}{x}$$

$$\frac{d}{dx}\ln y = \frac{1}{y}\cdot\frac{dy}{dx}$$

5

So, $\frac{dy}{dx} = y\cdot\left(\cos x \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

(4). Find the equation of the tangent line at $(1, 2)$ to the curve $\ln(x)+y^2 = e^{2x-y}+3$.

Take derivative w.r.t. x on both sides, (Remember, y is a function of x).

$$\frac{d}{dx}\left(\ln(x)+y^2\right) = \frac{d}{dx}\left(e^{2x-y}+3\right)$$

$$\frac{1}{x} + 2y\frac{dy}{dx} = e^{2x-y}\left(2-\frac{dy}{dx}\right)$$

5

when $x=1$, $y=2$, (check, $(1,2)$ is on the curve $\ln x + y^2 = e^{2x-y}+3$.)

$$1 + 4\frac{dy}{dx} = e^{2-2}\left(2-\frac{dy}{dx}\right) \quad \text{at } (1,2)$$

$$\Rightarrow 5\frac{dy}{dx} = 1 \qquad \frac{dy}{dx} = \frac{1}{5}$$

Tangent line: $y-2 = \frac{1}{5}(x-1)$.

(5). Let $h(x) = f(g(x))$. If $g(1) = 2, g'(1) = -1$, $f(2) = 2$, $f'(2) = -3$. Find $h(1)$ and $h'(1)$. Then use linear approximation to estimate $h(0.98)$.

$h(1) = f(g(1)) = f(2) = 2$ (1)'

$h'(x) = f'(g(x)) \cdot g'(x)$ (chain rule). So $h'(1) = f'(g(1))\, g'(1)$
$= f'(2)\, g'(1) = 3$ (3)'

$h(0.98) \approx h(1) + h'(1)(0.98 - 1)$ (linear. approx.)

$= 2 + 3(-0.02) = 1.94$ (2)'

(6). Suppose $f(x)$ is a continuous function, and suppose

$$\lim_{x\to 1} \frac{f(x) - 2}{x - 1} = 3.$$

Find $f(1), f'(1)$.

Since $f(x)$ is continuous,

$$f(1) - 2 = \lim_{x\to 1} (f(x) - 2) = \lim_{x\to 1}\left[\left(\frac{f(x)-2}{x-1}\right)\cdot(x-1)\right]$$

$$= \lim_{x\to 1} \frac{f(x)-2}{x-2} \cdot \lim_{x\to 1}(x-1) = 0$$

So: $f(1) = 2$. (3)'

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{f(x)-2}{x-1} = 3.$$ (3)'

2. Use definition to find the derivative of

$$f(x) = \sqrt{1+x^2}.$$

No points will be given without using definition of derivatives.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+(x+h)^2} - \sqrt{1+x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{1+(x+h)^2} - \sqrt{1+x^2}\right)}{h} \frac{\left(\sqrt{1+(x+h)^2} + \sqrt{1+x^2}\right)}{\sqrt{1+(x+h)^2} + \sqrt{1+x^2}}$$

$$= \lim_{h \to 0} \frac{1+(x+h)^2 - (1+x^2)}{h} \cdot \frac{1}{\sqrt{1+(x+h)^2} + \sqrt{1+x^2}}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} \cdot \frac{1}{\sqrt{1+(x+h)^2} + \sqrt{1+x^2}}$$

$$= \lim_{h \to 0} (2x+h) \cdot \frac{1}{\sqrt{1+(x+h)^2} + \sqrt{1+x^2}}$$

10

$$= \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

3. Some bones are exhumed from an ancient burial ground. If the ratio of Carbon 14 in the bones is 65% of the ratio in the bones of a living human, then how old are the bones? (The half-life of Carbon 14 is 5730 years and it begins to decay immediately after a person dies).

(1): The model for Carbon decay is

$$P(t) = P_0 \cdot e^{-kt} \quad (\text{for some } k > 0. \quad 2$$

The half life is 5730, it means

$P(5730) = \frac{1}{2} P_0$. So: $\frac{1}{2} P_0 = P_0 e^{-k \cdot 5730}$ 2

$\frac{1}{2} = e^{-k \cdot 5730}$ $\quad \ln \frac{1}{2} = -5730k, \quad k = \frac{\ln 2}{5730}$. 1

(2) Suppose the bones are T years old.

Then $P(T) = P_0 e^{-kT} = 65\% \cdot P_0$ 2

So: $\ln(e^{-kT}) = \ln(0.65)$

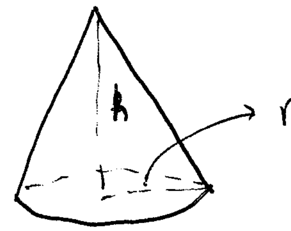
$-kT = \ln(0.65)$ 2

$$T = \frac{\ln(0.65)}{-k} = -\frac{\ln(0.65)}{\ln(2)} \cdot 5730$$

≈ 3561 years. 1

4. Gravel is dumped from a conveyor belt at a rate of $200 ft^3/min$. It forms a pile in the shape of a right circular cone whose base radius and height are always the same. How fast is the height of the pile increasing when the pile is $20 ft$ high?

$\underline{Vol = \frac{\pi}{3} h r^2}$ 2



Since $r = h$, $\underline{so}$. $V = \frac{\pi}{3} h^3$ 1 (1)

$\underline{Now}$ $\underline{\frac{dV}{dt} = 200\ ft^3/min.}$ 1

Note that both h, and V are functions of t.

Taking derivative w.r.t t on both sides of (1).

$$\boxed{\frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{3} h^3\right) = \frac{\pi}{3} \cdot 3h^2 \cdot \frac{dh}{dt}}$$ (chain rule) 4

$\underline{So}$: $\frac{dh}{dt} = \frac{dV}{dt} \frac{1}{\pi h^2}$ 2

When $h = 20$, $\left(\frac{dV}{dt} = 200\right)$, $\frac{dh}{dt} = \frac{200}{\pi \cdot (20)^2} = \frac{1}{2\pi}\ ft/min.$

5. How many roots does $f(x) = x^4 + 4x - 6$ have? Justify your answer clearly. Then use Newton's method to get the second approximation of the negative root; you can start with $x_0 = -2$.

(1): $f'(x) = 4x^3 + 4 = 4(x+1)(x^2-x+1)$ (2)

$f'(x) = 0, \Rightarrow x = -1$, or $x^2 - x + 1 = 0$

But there is no solution for $x^2 - x + 1 = 0$

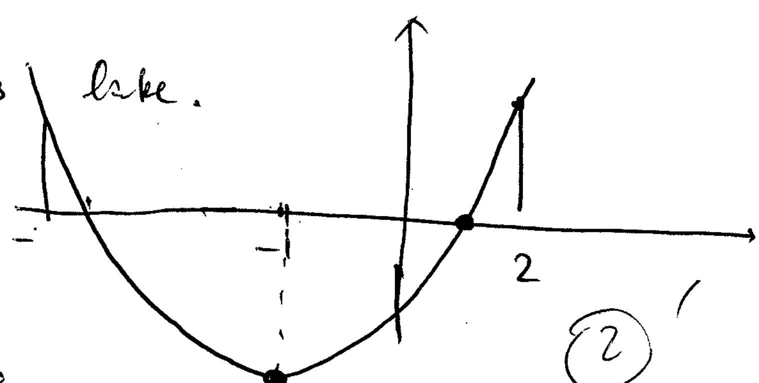
So: $x = -1$. (2)

when $x = -1$, $f(-1) = (-1)^4 + 4(-1) - 6 = -9$.

and $x < -1$, $f'(x) < 0$ (test say for $x = -2$)

$x > -1$ $f'(x) > 0$ (test say for $x = 0$) (2)

So: roughly, $f(x)$ looks like.



in particular,

when $x = 2$,

$f(2) = 2^4 + 8 - 6 = 18 > 0$, $f(0) = -6 < 0$ (2)

$f(-2) = (-2)^4 + 4\cdot(-2) - 6 = 2 > 0$

This implies that there are two roots of $f(x)$, one is in $(-2, -1)$ the other is in $(0, 2)$.

(2) N-method. (2) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -2 + \frac{1}{18} \approx -1.944$

(2) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx -1.924$

6. A manufacture has been selling 1700 television sets a week at \$540 each. A market survey indicates that for each 24 rebate offered to a buyer, the number of sets sold will increase by 240 per week.

a) Find the demand function $p = p(x)$, where x is the number of the television sets sold per week.

It's clear $\frac{dp}{dx} = \frac{-24}{240} = -\frac{1}{10}$ 2

$x = 1700, \quad p = 540$

So: $p = -\frac{1}{10}(x - 1700) + 540 = \underline{-\frac{1}{10}x + 710.}$ 2

b) How large rebate should the company offer to a buyer, in order to maximize its revenue?

$R(x) = p(x) \cdot x = x\left(-\frac{1}{10}x + 710\right) = -\frac{1}{10}x^2 + 710x$ 2

$R'(x) = -\frac{1}{5}x + 710 = 0 \quad x = 3550.$

$p = \frac{3550}{-10} + 710 = 710 - 355 = 355$

The rebate should be. $540 - 355 = 185$ \$. 2

c) If the weekly cost function is $153000 + 180x$, how should it set the size of the rebate to maximize its profit?

Profit $P(x) = R(x) - C(x)$

$= -\frac{1}{10}x^2 + 710x - (153{,}000 + 180x)$

$= -\frac{1}{10}x^2 + 530x - 153{,}000$ 2

$P'(x) = -\frac{1}{5}x + 530 = 0 \quad x = 2650.$

$p(2650) = -\frac{2650}{10} + 710 = 710 - 265 = 445.$ 2

Rebate: $540 - 445 = 95$ \$.

7. Let $f(x) = \cancel{\frac{\ln(x+1)}{x}}$. $\frac{\ln x}{x}$

(a). Find the domain of $f(x)$ and asymptotes of $f(x)$, if there is any.

domain: $x > 0$. 1'

$x \to +\infty$ $\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{\ln x}{x} \overset{L.R}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$ 1'

$y = 0$ is H. A.

$x \to 0$, $\lim_{x \to 0} \frac{\ln x}{x} = \lim_{x \to 0} (\ln x)\left(\frac{1}{x}\right) = -\infty$. 1'

$x = 0$ is V. A.

(b). Find the first and section derivatives of $f(x)$.

$$f'(x) = \left(\frac{\ln x}{x}\right)' = \frac{(\ln x)' \cdot x - \ln x \cdot x'}{x^2} = \frac{1 - \ln x}{x^2}$$ 1'

$$f''(x) = \left(\frac{1 - \ln x}{x^2}\right)' = \frac{x^2 (1 - \ln x)' - (1 - \ln x) \cdot (x^2)'}{(x^2)^2}$$

$$= \frac{-x - 2x(1 - \ln x)}{x^4} = \frac{\cancel{x}(2\ln x - 3)}{x^3}$$ 2'

(c). Find the critical points, local maximum, and local minimum of $f(x)$, and the intervals where $f(x)$ is decreasing and increasing.

$f'(x)=0$, $1-\ln x=0$, $x=e$. 1'

	$f'(x)$	
$x<e$	$+$	$x=1$
$x>e$	$-$	$x=e^2$

So: $x=e$, $f(x)$ is local maximum.

$f(x)$ is increasing on $(0, e)$

decreasing on $(e, +\infty)$. 1'

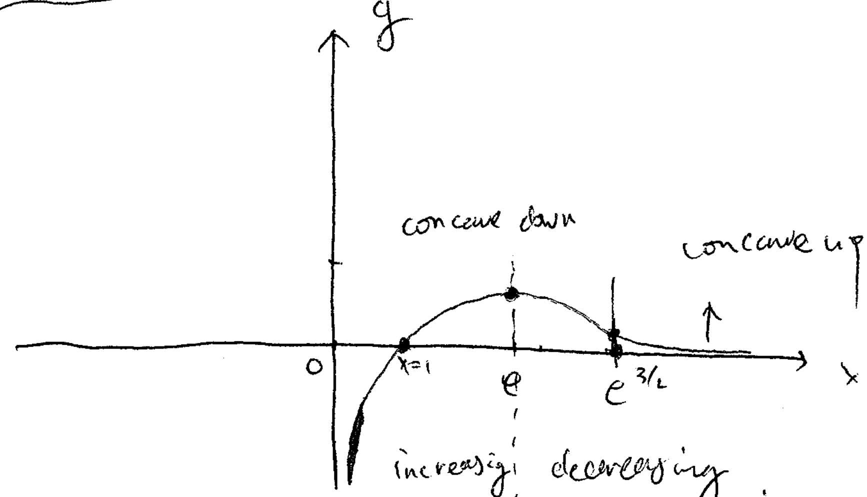
(d). Find the intervals where $f(x)$ is concave up and down, inflection points. Then use the information $(a)-(d)$ to sketch the graph of $y=f(x)$

$f''(x)=0$, $2\ln x-3=0$ $\quad x=e^{3/2} = e\sqrt{e}$ 1'

	$f''(x)$	
$0<x<e^{3/2}$	$-$	$x=1$
$e^{3/2}<x$	$+$	$x=e^2$

So $x=e^{3/2}$ is an inflection point. 1'

$f(x)$ is concave up $x\in(e^{3/2}, +\infty)$, concave down $x\in(0, e^{3/2})$



4'