Numbers, Functions, and Proofs
Math Boot Camp

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September 17, 2012
Brief Refresher on Logic
- Deductive Reasoning and Logical Connectives

The Set of Real Numbers
- Notation
- Properties
- The Continuum Property

Functions and Graphs
- Function
- Graphs
- Functional Forms

Proofs
- What is a Proof?
- Proof Strategies
Overview of Today’s Class

One of the main characteristics of mathematics is its use of deductive reasoning to find the answers to questions.

- For example, when we solve an equation for $x$ we are using the information given by the equation to deduce what the value of $x$ must be.

In turn, mathematical analysis starts with the study of the set of real numbers and the notion of limit.

- These fundamental concepts cannot be introduced without making reference to quantifiers, functions, etc.

Today: A brief exposition of these basic themes and a discussion of proofs.
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These requirements are satisfied by symbolic or mathematical logic, which gives to every expression an unequivocal meaning and to each symbol an unambiguous interpretation.

A simple way to introduce logical language is given by propositional calculus.

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Here are two propositions:

“3 is a whole number”
“Berk is immortal”

The first one is true, and the second one is false. The expression “x is an even number”, on the other hand, is not a proposition, because we cannot tell with certainty whether it is true or false.
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The first one is true, and the second one is false.

The expression “$x$ is an even number”, on the other hand, is not a proposition, because we cannot tell with certainty whether it is true or false.
Given the two propositions above, we can obtain other propositions. For example:

- “3 is a whole number and Berk is immortal”
- “3 is a whole number or Berk is immortal”
- “If 3 is a whole number, then Berk is immortal”

These new propositions are obtained by compounding the initial ones using the words “and”, “or”, “if-then”, etc.

In logic, this connection is done by defining operations between propositions.
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When we construct a new proposition, we may want to establish whether it is true or false for every truth value of the composing ones.

The truth value of a true proposition is true and the truth value of a false proposition is false.

We usually represent propositions using letters: $p, q, r, s, \ldots$ and we construct tables that give us the necessary information.

These are called truth tables and indicate whether the proposition resulting from an operation is either true or false.
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1 Negation. It is customary to use the symbol “∼” (called a tilde) to express negation. So, ∼p (not p) is a false proposition if p is true, and true if p is false.

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T & F \\
F & T \\
\end{array}
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2 Conjunction. It is customary to use the symbol “∧” to express conjunction (instead of the word “and”). So, the proposition $p \land q$ is true only if $p$ and $q$ are both true.

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Example: $p$=“3 is a whole number” is a true proposition, and $q$=“Berk is immortal”, is false. Thus, its conjunction, “3 is a whole number and Berk is immortal” is a false proposition.
3 Weak Disjunction. It is customary to use the symbol “\( \lor \)” to express disjunction (instead of the word “or”). So, the proposition \( p \lor q \) is false only if \( p \) and \( q \) are both false.

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5 Implication. If p, then q, or \( p \implies q \). A proposition of this kind is false only when p is true and q is false.

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p is called the antecedent, and q is the consequent.

Example: \( p \implies q = "\text{If 3 is an even number, then Berk is immortal}" \), is a true proposition.
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Example: $p \Rightarrow q$="If 3 is an even number, then Berk is immortal", is a true proposition.
As stated before, the expression “\( x \) is an even number” is not a proposition.

However, it becomes a proposition when we replace \( x \) with a number.

We denote such expression as a propositional function of a variable.

A propositional function can be a true proposition for some values of the variable, for all of them or for none of its values.

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Quantifiers

If the propositional function is a true proposition for all the values of the variable, it is customary to indicate this with the “universal quantifier,” symbolized by $\forall$ (which reads “for all”).

- $\forall x : x$ is mortal, means that for any meaningful value of $x$, the proposition that we will obtain will be true.

If the propositional function is a true proposition for at least some of the values of the variable, it is customary to use the “existential quantifier,” symbolized by $\exists$ (which reads “there exists at least one $x$ such that”).

- $\exists x : x$ is a rectangle, reads as “there exists $x$ such that $x$ is a rectangle” and means that there is at least one substitution for $x$ which transforms the propositional function into a true proposition.
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All the propositional functions mentioned thus far have had only affirmative singular propositions as substitution instances. But not all propositions are affirmative.

- To deny a universal proposition we just have to find a case for which it becomes false, that is, we have to find a counterexample.
- For example, the universal proposition $\forall x : x$ is mortal is denied by the existential proposition $\exists x/ x$ is not mortal.

$$\sim [\forall x : p(x)] \iff \exists x/ \sim p(x)$$

Similarly, if we want to deny that a proposition is valid for some value of $x$, we need to show that it is false for all values of $x$:

$$\sim [\exists x/p(x)] \iff \forall x : \sim p(x).$$
All the propositional functions mentioned thus far have had only affirmative singular propositions as substitution instances. But not all propositions are affirmative.

- To deny a universal proposition we just have to find a case for which it becomes false, that is, we have to find a counterexample.
  - For example, the universal proposition \( \forall x : p(x) \) is mortal is denied by the existential proposition \( \exists x / p(x) \) is not mortal.
    \[
    \sim [\forall x : p(x)] \iff \exists x / \sim p(x)
    \]

Similarly, if we want to deny that a proposition is valid for some value of \( x \), we need to show that it is false for all values of \( x \):

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\sim [\exists x / p(x)] \iff \forall x : \sim p(x).
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Quantifiers

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Numbers, Functions and Proofs

1. **Brief Refresher on Logic**
   - Deductive Reasoning and Logical Connectives

2. **The Set of Real Numbers**
   - Notation
   - Properties
   - The Continuum Property

3. **Functions and Graphs**
   - Function
   - Graphs
   - Functional Forms

4. **Proofs**
   - What is a Proof?
   - Proof Strategies
Some terms are used so frequently in mathematics, that we have some special symbols to refer to them. Some of these symbols are:

<table>
<thead>
<tr>
<th>Set</th>
<th>Notation</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural Numbers</td>
<td>N</td>
<td>1, 2, 3, 4, 5,...</td>
</tr>
<tr>
<td>(or whole numbers)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Integers</td>
<td>Z</td>
<td>..., -2, -1, 0, 1, 2, 3,...</td>
</tr>
<tr>
<td>Rational numbers</td>
<td>Q</td>
<td>0, 1, 2, -1, 1/2, 1/3, 2/3,...</td>
</tr>
<tr>
<td>(or fractions)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Real numbers</td>
<td>R</td>
<td>all of the above</td>
</tr>
</tbody>
</table>

Therefore, we have $N \subseteq Z \subseteq Q \subseteq R$. 
Real numbers $\mathbb{R}$ play an important role in mathematics, as they are used to convey numerical information. We can think of the real numbers as being points along a straight line which extends indefinitely in both directions. The line may be then regarded as an ideal ruler with which we may measure the lengths of line segments in Euclidean geometry.
Numbers, Functions and Proofs

- Brief Refresher on Logic
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The following assumptions (and their consequences) are concerned with the properties of the real number system:

Arithmetic. The first assumption is that the real numbers satisfy all the usual laws of addition, subtraction, multiplication and division.

Inequalities. The next assumptions concern inequalities between real numbers and their manipulation.
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- **Arithmetic.** The first assumption is that the real numbers satisfy all the usual laws of addition, subtraction, multiplication and division.
- **Inequalities.** The next assumptions concern inequalities between real numbers and their manipulation.
We assume that, given any two real numbers $a$ and $b$, there are three mutually exclusive possibilities:

(i) $a > b$ (a is greater than $b$)
(ii) $a = b$ (a equals $b$)
(iii) $a < b$ (a is less than $b$).

Observe that $a < b$ means the same thing as $b > a$. We have, for example, the following inequalities:

$1 > 0; 3 > 2; 2 < 3; -1 < 0$. 
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There is often some confusion about the statements

(iv) $a \geq b$ (a is greater than or equal to b)
(v) $a \leq b$ (a is less than or equal to b).

To clear up this confusion, we note that the following are all true statements:

$1 \geq 0$; $3 \geq 2$; $1 \geq 1$; $2 \leq 3$; $-1 \leq 0$; $-3 \leq -3$. 
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$1 \geq 0; \ 3 \geq 2; \ 1 \geq 1; \ 2 \leq 3; \ -1 \leq 0; \ -3 \leq -3$. 
We assume four basic rules for the manipulation of inequalities:

(I) If \( a > b \) and \( b > c \), then \( a > c \).

(II) If \( a > b \) and \( c \) is any real number, then
\[
    a + c > b + c.
\]

(III) If \( a > b \) and \( c > 0 \), then \( ac > bc \) (i.e. inequalities can be multiplied through by a positive factor).

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negative factor reverses the inequality).
Let $n$ be a natural number. You guys should be familiar with the notation $y = x^n$.

For example, $x^2 = (x)(x)$ and $x^3 = (x)(x)(x)$.

Our next assumption about the real number system is the following. Given any $y \geq 0$ there is exactly one value of $x \geq 0$ such that $y = x^n$.

If $y \geq 0$, the value of $x \geq 0$ which satisfies the equation $y = x^n$ is called the $n$th root of $y$ and is denoted by $x = y^{1/n}$.
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If \( y \geq 0 \), the value of \( x \geq 0 \) which satisfies the equation \( y = x^n \) is called the \( n \)-th root of \( y \) and is denoted by \( x = \sqrt[n]{y} \).
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When \( n = 2 \), we also use the notation \( \sqrt{y} = y^{\frac{1}{2}} \):

- Note that with this convention, it is always true that \( \sqrt{y} \geq 0 \).
- If \( y > 0 \), there are, of course, two numbers whose square is \( y \).

The positive one is \( \sqrt{y} \) and the negative one is \( -\sqrt{y} \).

The notation \( \pm \sqrt{y} \) means “\( y \) or \( -y \).”
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  - The notation $\pm \sqrt{y}$ means “$\sqrt{y}$ or $-\sqrt{y}$.”
If \( r = \frac{m}{n} \) is a positive rational number and \( y \geq 0 \), we define

\[
\sqrt[n]{y} = \frac{y^{\frac{1}{n}}}{n}
\]

If \( r \) is a negative rational, then \( -r \) is a positive rational and hence \( y - r \) is defined. If \( y > 0 \) we can therefore define \( y^{r} \) by

\[
y^{r} = y^{\frac{r}{2}}
\]

We also write \( y^{0} = 1 \). With these conventions it follows that, if \( y > 0 \), then \( y^{r} \) is defined for all rational numbers \( r \).
If $r = \frac{m}{n}$ is a positive rational number and $y \geq 0$, we define

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y^r = \sqrt[n]{y}.\]

If $r$ is a negative rational, then $-r$ is a positive rational and hence $y^{-r}$ is defined. If $y > 0$ we can therefore define $y^r$ by

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If $r$ is a negative rational, then $-r$ is a positive rational and hence $y^{-r}$ is defined. If $y > 0$ we can therefore define $y^r$ by

$$y^r = \frac{1}{y^{-r}}.$$  

We also write $y^0 = 1$. With these conventions it follows that, if $y > 0$, then $y^r$ is defined for all rational numbers $r$. 

\[ \text{William Terry} \]  
\[ \text{Numbers, Functions and Proofs} \]
If \( y > 0 \), the equation \( x^2 = y \) has two solutions.

We denote the positive solution by \( \sqrt{y} \).

The negative solution is therefore \( -\sqrt{y} \).

The general quadratic equation has the form

\[ ax^2 + bx + c = 0 \]

where \( a = 0 \).
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### Quadratic Equations

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---

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Numbers, Functions and Proofs
Multiply through by 4a. We obtain

\[4a2x^2 + 4abx + 4ac = 0\]

\[(2ax + b)^2 - b^2 + 4ac = 0\]

\[(2ax + b)^2 = b^2 - 4ac\]

It follows that the quadratic equation has no real solutions if \(b^2 - 4ac < 0\), one real solution if \(b^2 - 4ac = 0\) and two real solutions if \(b^2 - 4ac > 0\).
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If \( b^2 - 4ac \geq 0 \),

\[
2ax + b = \pm \frac{b^2 - 4ac}{2a}
\]

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

The roots of the equation \( ax^2 + bx + c = 0 \) are therefore

\[
\alpha = \frac{-b - \sqrt{b^2 - 4ac}}{2a},
\]

and

\[
\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.
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Suppose that \( x \) is a real number.

Its modulus (or absolute value) \( |x| \) is defined by

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\ 
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\end{cases}
\]

Therefore \(|3| = 3\), \(|-6| = 6\) and \(|0| = 0\).

Obviously, \(|x| \geq 0\) for all values of \( x \).

Note also that \(|x| = x^2\).
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Sebastian M. Saiegh

Numbers, Functions, and Proofs
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William Terry
Numbers, Functions and Proofs
Achilles (Achilles) and the tortoise

The fifth century B.C. Greek philosopher Zeno of Elia invented several famous paradoxes.

The following is one of the most famous: Achilles is to race a tortoise.
Since Achilles runs faster than the tortoise, the tortoise is given a start of \(x_0\) feet.
When Achilles reaches the point where the tortoise started, the tortoise will have advanced a bit, say \(x_1\) feet.
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Achilles soon reaches the tortoise’s new position, but, by then, the tortoise will have advanced a little bit more, say $x_2$ feet.

This argument may be continued indefinitely and so Achilles can never catch the tortoise.
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Achilles (Achilles) and the tortoise

The simplest way to resolve this paradox is to say that Achilles catches the tortoise after he has run a distance of $x$ feet, where $x$ is ‘the smallest real number larger than all of the numbers $x_0, x_0 + x_1, x_0 + x_1 + x_2, \ldots$’

Zeno’s argument then simply reduces to subdividing a line segment of length $x$ into an infinite number of smaller line segments of respective lengths $x_0, x_1, x_2, \ldots$
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Zeno’s argument then simply reduces to subdividing a line segment of length \( x \) into an infinite number of smaller line segments of respective lengths \( x_0, x_1, x_2, \ldots \)
Formulated in this way, the paradox loses its sting.

This solution, though, depends very strongly on the existence of the real number $x$ (i.e. the smallest real number largest than all the numbers $x_0, x_0 + x_1, x_0 + x_1 + x_2, ...$)
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A set $S$ of real numbers is bounded above if there exists a real number $k$ which is greater than or equal to every element of the set.

For some $k$,
$$x \leq k$$
for any $x \in S$.

The number $k$ (if such number exists) is called an upper bound of the set $S$. 

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**Sebastian M. Saiegh**

*Numbers, Functions, and Proofs*
A set $S$ of real numbers is bounded below if there exists a real number $h$ which is less than or equal to every element of the set.

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$$x \geq h$$

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for any $x \in S$.

The number $h$ (if such number exists) is called an lower bound of the set $S$. 
A set $S$ of real numbers is bounded below if there exists a real number $h$ which is less than or equal to every element of the set.

For some $h$,

$x \geq h$

for any $x \in S$.

The number $h$ (if such number exists) is called an lower bound of the set $S$. 
A set which is both bounded above and bounded below is just said to be bounded.

**Definition**

A set $S$ of real numbers is bounded if and only if there exists a real number $k$ such that $|x| \leq k$ for any $x \in S$. 
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The set \( \{x : 1 \leq x < 2\} \) is bounded above. Some upper bounds are 100, 10, 4, and 2. The set is also bounded below. Some lower bounds are -27, 0, and 1.

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The set \( \{x : x > 0\} \) is unbounded above. If \( h > 0 \) is proposed as an upper bound, one has only to point to \( h + 1 \) to obtain an element of the set larger than the supposed upper bound. However, the set \( \{x : x > 0\} \) is bounded below. Some lower bounds are -27, and 0.
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Continuum Property

Definition

Continuum Property. Every non-empty set of real numbers which is bounded above has a smallest upper bound. Every non-empty set of real numbers which is bounded below has a largest lower bound.
An interval $I$ is a set of real numbers with the property that, if $x \in I$ and $y \in I$ and $x \leq z \leq y$, then $z \in I$.

In words, if two numbers belong to $I$, then so does every number between them.
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In words, if two numbers belong to $I$, then so does every number between them.
In describing intervals we use the following notation:

\[(a,b) = \{x : a < x < b\}\]
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These are the bounded intervals (classified by whether or not they have a maximum and whether or not they have a minimum).
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\[(a, \infty) = \{x : x > a\}\]
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Intervals

All intervals (with the exception of the empty set and the set of all real numbers) fall into one of the categories described above.

- We call the intervals \((a, b)\), \((a, \infty)\) and \((\infty, b)\) open intervals.
- We call the intervals \([a, b]\), \([a, \infty)\) and \((\infty, b]\) closed intervals.
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**Numbers, Functions and Proofs**

- **Brief Refresher on Logic**
  - Deductive Reasoning and Logical Connectives

- **The Set of Real Numbers**
  - Notation
  - Properties
  - The Continuum Property

- **Functions and Graphs**
  - Function
  - Graphs
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- **Proofs**
  - What is a Proof?
  - Proof Strategies
Political Science, economics, sociology, as well as biology, chemistry, and physics are frequently concerned with binary relationship between elements of different sets.

For example, we may be interested in the dependence of one quantity upon others. (i.e. a manufacturer might want to know how profit varies with production level).

Among these relationships, we are interested in those where there is a correspondence between every element in one of the sets and one and only one element in the other set.

A relationship of this kind is called a functional relationships or simply a function.
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In traditional calculus a function is defined as a relationship between two variables.

Denote the variables by \( x \) and \( y \).

\[ f : A \to B, \text{ assigns to each } x \in A \text{ a unique element } y \in B. \]

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If every value of \( x \) is associated with exactly one value of \( y \), then \( y \) is said to be a function of \( x \).
The most common way to denote a function is to replace the term y, called the image of the element x, and to write y = f(x) (the symbol f(x) is the value of f at x and it is usually read as “f of x”).

If f is a function from A to B and S ⊂ A, we say that f is defined on the set S.

The largest set on which f is defined is, of course, the set A. We call A the domain of f.

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The set f(A) is called the range of f.
For example, suppose $A = \{1,2,3\}$, and $B = \{2,4\}$, we can define $f$ such that,
Because a number on the left leads to exactly one number on the right, we can say that the numbers on the right (i.e. the images) are a function of those on the left.

The arrows give us a “mapping” of domain to range.
As you can see, more than one number on the left can lead to the same number on the right, but not vice versa.
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As you can see, more than one number on the left can lead to the same number on the right, but not vice versa.
It is customary to use $x$ for what is called the independent variable, and $y$ for what is called the dependent variable because its value depends on the value of $x$.

Values that can be taken by the independent variable are, thus, called the variable’s domain. Values that can be taken by the dependent variable are called the range.

Note also that in our example, we defined the relationship between the elements of the sets $A$ and $B$ using arrows, which represent our “correspondence rules”.

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Alternatively, we may specify our rule using a mathematical equation, such as $y = x^2$.

This equation defines a function from $\mathbb{R}$ to itself.

For each $x \in \mathbb{R}$ there exists a unique $y \in \mathbb{R}$ which satisfies the rule $y = x^2$.

The domain of this function is $\mathbb{R}$. The range of this function is $[0,\infty)$ (recall that the product of two negative numbers is positive).

We can also write this function as $f(x) = x^2$: the dependent variable is the square of $x$. 
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Sebastian M. Saiegh

Numbers, Functions, and Proofs
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Many functions are also given by an algebraic formula.

For example, \( y = f(x) = x^2 - x + 1 \).

In this form, the expression is called an explicit function of \( x \).

The equation may also be expressed as \( x^2 - x - y - 1 = 0 \), and in this case we refer to it as an implicit function of \( x \) because the explicit form is implied by the equation.
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So, if $f(x) = x^2 - x + 1$, what is the value of this function when $x = -1, x = 0, x = \frac{1}{2}$?

We find the value by inserting the designated $x$ value into the formula:

$$f(-1) = (-1)^2 - (-1) + 1 = 1 + 1 + 1 = 3$$

$$f(0) = (0)^2 - (0) + 1 = 1$$

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Explicit and Implicit Functions

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**Evaluating Functions**

Our chief interest will be in rules for evaluating functions defined by formulas.

If the domain is not specified, it will be understood that the domain is the set of all real numbers for which the formula produces a real value, and for which it makes sense.

For example, given the equation \( y = x^3 \), \( x^3 \) is defined for all values of \( x \) except zero; so the range is all real numbers except zero.
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For example, given the equation \( y = x^{1} \cdot \frac{1}{x^{1}} \) is defined for all values of \( x \) except zero; so the range is all real numbers except zero.
Consider now the equation $y^2 = x$.

This equation does not define a function from $\mathbb{R}$ to itself. Take any value for which $x < 0$, and you will find that there is no value of $y$ associated with such value of $x$.

Does this function define a function from $[0, \infty)$ to $\mathbb{R}$?

Again, the answer is no.

This time is certainly true that there is a correspondence between every element in the set $[0, \infty)$ and an element in the range. However, the equation does not assigns to each element in the domain a unique element in the range.
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The equation $y^2 = x$, though, does define a function from $[0, \infty)$ to $[0, \infty)$.

Given any $x \in [0, \infty)$, there is a unique $y \in [0, \infty)$ which satisfies $y^2 = x$.

Recall the discussion about roots: we observe that, for each $x \geq 0$, $f(x) = x^2$. 

William Terry

Numbers, Functions and Proofs
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This picture is called the graph of the function.

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We start by constructing coordinate axes: we construct a pair of mutually perpendicular intersecting lines, one horizontal, the other vertical.
The horizontal line is often called the x-axis.

Values of the independent variable are usually represented by points along this axis.

We call the vertical line the y-axis.

Values of the dependent variable are represented by points along this axis.

The point where these lines meet, called the origin, represents zero.

The scale of the y-axis does not need to be the same as that for the x-axis. In fact, y and x can have different units, such as distance and time.
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We can represent one specific pair of values associated by the function in the following way:

Let \( a \) represent some particular value for the independent variable \( x \), and let \( b \) indicate the corresponding value of \( y = f(x) \). Thus, \( b = f(a) \)

We can now draw a line parallel to the \( y \)-axis at distance \( a \) from the axis, and another line parallel to the \( x \)-axis at distance \( b \).

The point \( P \) at which these two lines intersect is designated by the pair of values \( (a,b) \) for \( x \) and \( y \), respectively.
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The point P at which these two lines intersect is designated by the pair of values (a,b) for x and y, respectively.
The number $a$ is called the x-coordinate of $P$, and the number $b$ is called the y-coordinate of $P$.

In the designation of a typical point by the notation $(a,b)$ we will always designate the x-coordinate first and the y-coordinate second.
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In the designation of a typical point by the notation $(a,b)$ we will always designate the x-coordinate first and the y-coordinate second.
For example, given \( f(x) = x - 1 \), its graph is the set of points \((a,b)\) in the plane such that \( y = x - 1 \).

Notice that the function is defined: each vertical line cuts the graph in one and only one place.
For example, given \( f(x) = x - 1 \), its graph is the set of points \((a,b)\) in the plane such that \( y = x - 1 \).

Notice that the function is defined: each vertical line cuts the graph in one and only one place.
Similarly, the equation $y = x^2$ defines a function from $\mathbb{R}$ to itself.

![Graph of $y = x^2$]

Again, each vertical line cuts the graph in one and only one place.
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Again, each vertical line cuts the graph in one and only one place.
Consider now the correspondence rule

\[ g(x) = \begin{cases} 
2x - 3 & \text{if } x \leq 4 \\
2 & \text{if } x \geq 4.
\end{cases} \]
This correspondence rule does not define a function: the vertical line of the equation $x = 4$ meets the graph in two points: $(4,2)$, and $(4,5)$. In words, the number 4 has two images. But, if we eliminate the second image of 4, for example:

$$h(x) = \begin{cases} 2x - 3 & \text{if } x \leq 4 \\ 2 & \text{if } x > 4. \end{cases}$$

Then, $h$ is a defined function.
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The most direct way to plot the graph of a function $y = f(x)$ is to make a table of reasonably spaced values of $x$ and of the corresponding values of $y = f(x)$.

Then each pair of values can be represented by a point. A graph of the function is obtained by connecting the points with a smooth curve.

A function whose graph is unbroken (i.e., it can be drawn without lifting the pencil from the paper) is said to be continuous, while one whose graph has a gap or a hole is discontinuous.
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Discontinuous Functions

For the functions we will encounter in this class, discontinuities may arise in one of the following two ways:

- A function defined in several pieces will have discontinuities if the graphs of the individual pieces are not connected to each other.
- A function defined as a quotient will have a discontinuity whenever the denominator is zero.
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- A function defined as a quotient will have a discontinuity whenever the denominator is zero.
Discontinuous Functions

For example, the graph of the function \( f(x) = \frac{x^2 + x - 2}{x - 2} \), looks like this:
Plotting Functions: Symmetry

The graphical representation of certain function can be simplified if we take into account its symmetry; the graph for $|x|$ is:

```
  +---+---+---+---+---+
  |   |   |   |   |   |
  |   |   |   |   |   |
  |   |   |   |   |   |
  |   |   |   |   |   |
  +---+---+---+---+---+
      0  1  2  3  4  5
```

William Terry
Numbers, Functions and Proofs
Brief Refresher on Logic
  Deductive Reasoning and Logical Connectives
The Set of Real Numbers
  Notation
  Properties
  The Continuum Property
Functions and Graphs
  Function
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  What is a Proof?
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Common Functions

Constant Function: it assigns a single fixed number $k$ to every value of the independent variable, $x$. Hence, $f(x) = k$.

The graph is just a horizontal line.
Identity Function: A function \( f \) is identical over \( \mathbb{R} \) if and only if the image of every real number is that same real number.

Formally, \( \forall x : f(x) = x \).
Common Functions

Absolute Value Function.

This is no other than the graph for $|x|$.
A function defined by an equation in the form $y = mx + b$, where $m$ and $b$ are constants, is called a linear function because its graph is a straight line.

This is a graph of a typical linear function.
Linear Functions

If a line is not vertical, its direction with respect to the coordinate axes in the plane is described by a number called the slope of the line.

The slope is measured by marking two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on the line and computing the ratio

$$\text{Slope} = \frac{\text{change in y coordinate}}{\text{change in x coordinate}}$$

or

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Sebastian M. Saiegh
Numbers, Functions and Proofs
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$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$
Here is another way to find the slope of a straight line if its equation is given.

If the linear function is in the form $y = mx + b$, then the slope is given by the expression in the previous slide. Substituting for $y$, we have

$$\text{Slope} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{mx_2 - mx_1}{x_2 - x_1} = m(x_2 - x_1) = m.$$ 

So, for example, the slope of $y = 7x - 5$ is simply $m = 7$. 
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So, for example the slope of $y = 7x - 5$ is simply $m = 7$. 

Polynomials

If \( a_0, a_1, a_2, \ldots, a_n \) are all real numbers, then the equation
\[
y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n
\]
defines a function from \( \mathbb{R} \) to itself.

Any value of \( x \) which is substituted on the right hand side
generates a unique corresponding value of \( y \).

If \( a_n = 0 \), we call this function a polynomial of degree \( n \).
A polynomial of degree 0 is called a constant.
If $a_0, a_1, a_2, ..., a_n$ are all real numbers, then the equation

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  Deductive Reasoning and Logical Connectives

The Set of Real Numbers
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Proofs
  What is a Proof?
  Proof Strategies
Now that we have covered the basics on deductive reasoning and the language of mathematics, it is the turn to put this knowledge into practice.

One of the important roles of mathematics in sciences is to deduce complex scientific principles from a collection of generally agreed assumptions. The goal is to arrive at a conclusion from the assumption that some other statements, called premises, are true.

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In mathematics, a statement is a sentence expressed in words (or mathematical symbols) that is either true or false. Examples:

1. $1 + 1 = 2$.
2. $x^2 + 1 = 0$ (x is a positive integer).
3. $3x = 5$ and $y = 1$
4. Given two real numbers $a$ and $b$, if $0 < a < b$, then $a^2 < b^2$.

Note that statement (1) is always true, (2) is always false, and statement (3) is either true or false, depending on the value of a variable. For this reason, (3) is called a conditional statement.
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It is perhaps not as obvious that statement (4) is always true.

It therefore becomes necessary to have some method for proving that such statements are true.

A proof is a logical argument that establishes the truth of a statement beyond any doubt. In other words, a proof is a convincing argument expressed in the language of mathematics.
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Going back to (4), the task is the following: given two statements A and B, each of which may be either true or false, show that the following statement – called an implication – is true:

If A is true, then B is true.

How do we prove it? First, we start with statement A 
(0 < a < b) – called the hypothesis (a statement that it is assumed to be true, and from which some consequence follows)
The consequence, in this case is B (a^2 < b^2). This statement is called the conclusion: (it follows from previously assumed conditions – hypotheses).
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The consequence, in this case is B (a² < b²). This statement is called the conclusion: (it follows from previously assumed conditions – hypotheses).
Recall from the beginning of today’s class, that implication is one of the operations of propositional calculus.

You may also remember that a general feature of statements of the form A implies B is that there is only one case in which the statement is false: when A is true and B is false.

Therefore, we can prove that a statement of the form “If A, then B” is true if it is impossible for A to be true and B to be false at the same time; that is, whenever A is true, B must be true as well.
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Just as there are many ways to express the same idea in language, so there are different ways of proving the same mathematical fact.

A proof should contain enough mathematical details to be convincing to the person(s) to whom it is addressed.

We will now categorize and explain the various proof techniques that are used in all proofs, regardless of the subject matter.
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Direct Proof

A direct proof is based on the assumption that the hypothesis contains enough information to allow the construction of a series of logically connected steps leading to the conclusion.

Note, that from this point of view, a proof of the statement $A$ implies $B$ is not an attempt to verify whether $A$ and $B$ themselves are true but rather to show that $B$ is a logical result of having assumed that $A$ is true.

We can now go back and prove statement (4).
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Direct Proof

Example

Suppose $a$ and $b$ are real numbers. Prove that if $0 < a < b$ then $a^2 < b^2$.

We are given as an ancillary hypothesis the statement that $a$ and $b$ are real numbers. Our problem has the from $A$ implies $B$ where $A$ is the statement $0 < a < b$ and $B$ is the statement $a^2 < b^2$.

According to this proof technique, we should assume that $0 < a < b$ and try to use this assumption to prove that $a^2 < b^2$. 
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According to this proof technique, we should assume that $0 < a < b$ and try to use this assumption to prove that $a^2 < b^2$. 
Comparing the inequalities $a < b$ and $a^2 < b^2$ suggests that multiplying both sides of the given inequality $a < b$ by either $a$ or $b$ might get us closer to proving the statement.

Because we are given that $a$ and $b$ are positive, we won’t need to reverse the direction of the inequality if we do this. Multiplying $a < b$ by $a$ gives us $a^2 < ab$, and multiplying it by $b$ gives us $ab < b^2$. Therefore $a^2 < ab < b^2$, so $a^2 < b^2$, as required. Thus, if $0 < a < b$ then $a^2 < b^2$. 
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Therefore \( a^2 < ab < b^2 \), so \( a^2 < b^2 \), as required.

Thus, if \( 0 < a < b \) then \( a^2 < b^2 \).
Comparing the inequalities $a < b$ and $a^2 < b^2$ suggests that multiplying both sides of the given inequality $a < b$ by either $a$ or $b$ might get us closer to proving the statement.

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Thus, if $0 < a < b$ then $a^2 < b^2$. 
Notice that in the course of proving statement (4) we implicitly used a lot of mathematical knowledge such as for example, the properties of multiplication.

These properties, in turn, depend on statements that have been proved using other statements and so on and so forth. More generally, then, we can characterize the direct way of proving that $A \Rightarrow B$ as the process of finding a sequence of accepted axioms and theorems of the form $A_i \Rightarrow A_{i+1}$ for $i = 1, \ldots, n$ so that $A_0 = A$ and $A_{n+1} = B$. 
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More generally, then, we can characterize the direct way of proving that $A \implies B$ as the process of finding a sequence of accepted axioms and theorems of the form $A_i \implies A_{i+1}$ for $i = 1, \ldots, n$ so that $A_0 = A$ and $A_{n+1} = B$. 
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A theorem is a mathematical statement for which the truth can be established using logical reasoning on the basis of certain assumptions that are explicitly given or implied in the statement (i.e. by constructing a proof).
Theorem

For any \( x, y, z \in \mathbb{R} \), if \( x + z = y + z \) then \( x = y \).

Proof.
1. \( x + z = y + z \) is our hypothesis
2. Using the additive inverse property, we can establish that there exists \((-z)\) such that \( z + (-z) = 0 \).
3. Using again the properties of addition and multiplication, we get \( (x + z) + (-z) = (y + z) + (-z) \).
4. Finally, using the additive associative property, it must be true that \( x + (z + (-z)) = y + (z + (-z)) \).
5. Given step (2), then \( x + 0 = y + 0 \)
6. By the additive identity property, \( x = y \).
As this example show, the theorem says that if certain assumptions
(the hypotheses) are true, then some conclusion must also be true.

Yet, often the hypotheses and conclusion contain free
variables.

An assignment of particular values to these variables is called an
instance of the theorem.

If there is even one instance in which the hypotheses are true
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Theorems

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Indirect Proof

There is a second method that is sometimes used for proving statements of the form $A \Rightarrow B$.

Because any conditional statement $A \Rightarrow B$ is equivalent to its contrapositive $\sim B \Rightarrow \sim A$, this strategy consists in proving $A \Rightarrow B$ by proving $\sim B \Rightarrow \sim A$.

This gives us a different starting point because we will start by assuming that $B$ is false, and we will prove that this implies that $A$ is false, as the contrapositive of the original statement is “If ‘not B,’ then ‘not A.’”
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Indirect Proof

Example

Given the inequality $a > 0$, we are going to prove that $a-1 > 0$.

Proof. We are going to prove this by contradiction. Suppose that $a > 0$, but that $a-1 \leq 0$. It cannot be true that $a-1 = 0$ (since then $0 = 0 \cdot a = 1$). Hence, $a-1 < 0$.

Recall that inequalities can be multiplied through by a positive factor. In this case, we can multiply the previous inequality through by $a$ (since $a > 0$).

Hence, $1 = a-1 \cdot a < 0 \cdot a = 0$. But $1 < 0$ is a contradiction. Therefore the assumption $a-1 \leq 0$ was false.

Hence $a-1 > 0$. 
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Given the inequality $a > 0$, we are going to prove that $a^{-1} > 0$.

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Example
Suppose that, for any \( > 0 \), \( a < b + \). Then, \( a \leq b \).

Proof. Assume that \( a > b \). Then \( a - b > 0 \). But, for any \( a < b + \).

Hence, \( a < b + \) in the particular case when \( a = b - a \). Thus, \( a < b + (a - b) \) and so \( a < a \).

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Example
Suppose that, for any $a > b + \epsilon$. Then, $a \leq b$.

Proof. Assume that $a > b$. Then $a - b > 0$. But, for any $a < b + \epsilon$.

Hence, $a < b + \epsilon$ in the particular case when $a - b = b - a$. Thus, $a < b + (a - b)$ and so $a < a$.

Contradiction! Hence our assumption $a > b$ must be false. Therefore, $a \leq b$. 