Midterm Review

1. Every equations of n-th order $x^{(n)} = g(t, x, x', \ldots, x^{(n-1)})$ can be transformed into the form $y' = f(t, y)$ by setting $y_1 = x, y_2 = x', \ldots, y_n = x^{(n-1)}$, so are the corresponding initial values.

2. **Structure of Solutions of Linear System Equations**
   (a) Homogeneous equation $\mathbf{y}' = A(t)\mathbf{y}$, $\mathbf{y}(t_0) = \mathbf{y}_0$: $\mathbf{y}(t) = \Phi(t)\mathbf{y}_0$, where $\Phi(t)$ is a fundamental matrix of solutions, $\Phi(t_0) = I$.
   (b) Non-homogeneous equation $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{g}(t)$: $\mathbf{y}(t) = \Phi(t)\mathbf{y}_0 + \mathbf{y}_p(t)$, where $\mathbf{y}_p$ is any particular solution for the non-homogeneous equation. The method of variation of parameters gives $\mathbf{y}(t) = \Phi(t)\mathbf{y}_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)\mathbf{g}(s)ds$, where $\Phi(t_0) = I$.
   (c) Homogeneous equation with constant coefficients $\mathbf{y}' = A\mathbf{y}$: fundamental solutions are given in terms of eigenvalues and (generalized) eigenvectors of A.

3. **Existence and Uniqueness of the Solutions**
   (a) Equivalent integral equation. Picard successive approximation.
   (b) Let $\mathbf{f} : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $\mathbf{y}_0 \in \mathbb{R}^n$. If $\mathbf{f}$ and $\partial_{x_i}\mathbf{f}$ $(1 \le i \le n)$ are continuous in the domain $R = \{(t, \mathbf{y}) : |t - t_0| \le a,\ \|\mathbf{y} - \mathbf{y}_0\| \le b\}$, then the

   $$\text{IVP}: \qquad \mathbf{y}' = \mathbf{f}(\mathrm{t}, \mathbf{y}), \qquad \mathbf{f}(\mathrm{t}_0) = \mathbf{y}_0$$

   has a unique solution $\mathbf{y}(t)$ on the interval $I = \{t : |t - t_0| \le \alpha\}$ where $\alpha = \min\{a, b/M\}$.
   (c) The condition $\partial_{x_i}\mathbf{f}$ is continuous can be replaced by the Lipschitz condtion

   $$\|\mathbf{f}(t, \mathbf{y}_1) - \mathbf{f}(t, \mathbf{y}_2)\| \le M\|\mathbf{y}_1 - \mathbf{y}_2\|, \qquad \text{for all} \quad (t, \mathbf{y}_1), (t, \mathbf{y}_2) \in R.$$

   (d) If $\mathbf{f}$ is continuous and $\mathbf{f}$ satisfies Lipschitz condition on the strip $R_a = \{(t, \mathbf{y}) : |t - t_0| \le a\}$, then the IVP has a unique solution on the entire interval $I = \{t : |t - t_0| \le a\}$.
   (e) If $A(t)$ and $\mathbf{g}(t)$ are continuous on the interval $I = \{t : |t - t_0| < a\}$, then the IVP of the linear system

   $$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0$$

   has a unique solution on I.

4. **Autonomous Systems $\mathbf{y}' = \mathbf{f}(\mathbf{y})$**
   (a) Concepts: critical points, isolated critical point (equilibrium), trajectory, stable, asymptotically stable, unstable, phase space.
   (b) Linear system with constant coefficients: $\mathbf{y}' = A\mathbf{y}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The behavior of the critical point is determined by the eigenvalues of A (node, improper node, spiral points, center, saddle points). Phase portrait.
   (c) Nonlinear system: For $y_1' = f_1(y_1, y_2), y_2' = f_2(y_1, y_2)$, determine equilibrum solutions and characterize them by considering approximating linear equations.