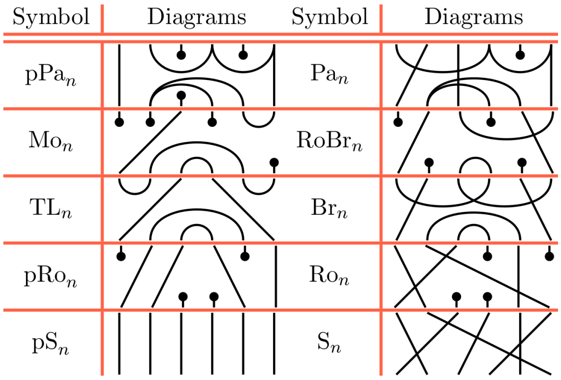
Exercises 2

1. Diagram algebras and H-reduction

Recall the arguably most important diagram monoids:

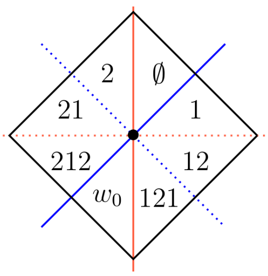


Fix some field $\mathbb{K}$. In all cases, the respective algebras are obtained by evaluating floating components to a fixed $\delta \in \mathbb{K}$. (If that doesn't make sense to you, then I have messed up: my bad...)

a) Classify the simple modules for your favorite(s) of these diagram algebras.
b) (') If you know the quantum versions of these algebras, such as the BMW algebra, then try those as well.

2. Finite fun with dihedral groups – classical

Let $\emptyset$ denote the unit and let $D_n = \langle 1, 2 | 1^2 = 2^2 = (12)^n = \emptyset \rangle$ be the dihedral group of the n gon.

$D_4 \leftrightsquigarrow$ 

a) Use *e.g.* the Magma online calculator (see below) to guess the classification of simple D_n-modules over $\mathbb{C}$. You can use the code

```
n:=5;
CharacterTable(DihedralGroup(n))
```

$n = 4$:

```
-----------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  2  2  4
-----------------------
p  =  2   1  1  1  1  2
-----------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1  1 -1
X.3   +   1  1  1 -1 -1
X.4   +   1  1 -1 -1  1
X.5   +   2 -2  0  0  0
```

, $n = 5$:

```
------------------------
Class |   1  2    3    4
Size  |   1  5    2    2
Order |   1  2    5    5
------------------------
p  =  2   1  1    4    3
p  =  5   1  2    1    1
------------------------
X.1   +   1  1    1    1
X.2   +   1 -1    1    1
X.3   +   2  0   Z1 Z1#2
X.4   +   2  0 Z1#2   Z1
```

and vary n.

b) Show that your guessed classification is true.

c) (*) What happens for general fields?

3. Infinite fun with dihedral groups – à la KL

Retain the notation from Exercise 2. For a field $\mathbb{K}$ consider the group algebra $S = \mathbb{K}[D_\infty]$ of the infinite dihedral group $D_\infty = \langle 1,2 | 1^2 = 2^2 = \emptyset \rangle$. Every element of D_∞ has a unique reduced expression. We write $k,1,2$ and $k,2,1$ for the reduced expressions $\dots 12$ and $\dots 21$ in k symbols.

The algebra S has a KL basis $\{b_w | w \in D_\infty\}$ (whose precise definition does not matter) with identity $b_\emptyset$. Set $b_{0,a,b} = 0$. The nonidentity multiplication rules are given by the Clebsch–Gordan formula:

$$b_{k,1,2} b_{j,1,2} = \begin{cases} 2b_{|k-j|+1,1,2} + \dots + 2b_{|k+j|-1,1,2} & \text{j,1,2=2\dots 12,} \\ b_{|k-j|,1,2} + 2b_{|k-j|+2,1,2} + \dots + 2b_{|k+j|-2,1,2} + b_{|k+j|,1,2} & \text{j,1,2=1\dots 12.} \end{cases}$$

There are also similar formulas with $b_{j,2,1}$ on the right or $b_{k,1,2}$ on the left.

For example:

$$b_{1212} b_{21212} = 2b_{12} + 2b_{1212} + 2b_{121212} + 2b_{12121212},$$
$$b_{1212} b_{121212} = b_{12} + 2b_{1212} + 2b_{121212} + 2b_{12121212} + b_{1212121212}.$$

a) Compute the cell structure of S with respect to the KL basis $\{b_w | w \in D_\infty\}$ for $\mathrm{char}(\mathbb{K}) \neq 2$. Skip the identification of the nontrivial $S_{\mathcal{H}}$ for now.
b) (') Compare the nontrivial $S_{\mathcal{H}}$ of S to the Grothendieck algebra of complex finite dimensional $\mathrm{SO}_3(\mathbb{C})$-representations.
c) What happens in characteristic two?

4. Finite fun with dihedral groups – à la KL

Retain the notation from Exercise 3. Let $S = D_n = \langle 1,2 | 1^2 = 2^2 = (12)^n = \emptyset \rangle$ be the dihedral group of the n gon. The longest element is $w_0 = n,1,2 = n,2,1$.

With respect to the KL basis and its multiplication rules, the only change compare to D_∞ is that expressions of the form (here $d > 0$)

$$b_{n-d,1,2} + b_{n+d,1,2} \longmapsto 2b_{w_0}, \quad b_{n-d,2,1} + b_{n+d,2,1} \longmapsto 2b_{w_0}.$$

are replaced as indicated. This is the truncated Clebsch–Gordan formula.

For example, for $n = 6$ one gets:

$$b_{1212} b_{21212} = 2b_{12} + \boxed{2b_{1212}} + \underline{2b_{121212}} + \boxed{2b_{12121212}} = 2b_{12} + 6b_{121212},$$

$$b_{1212} b_{121212} = \boxed{b_{12}} + \boxed{2b_{1212}} + \underline{2b_{121212}} + \boxed{2b_{12121212}} + \boxed{b_{1212121212}} = 8b_{121212}.$$

a) Compute the cell structure of S with respect to the KL basis $\{b_w | w \in D_n\}$ for $\mathbb{K} = \mathbb{C}$ and odd n. Skip the identification of the nontrivial $S_{\mathcal{H}}$ for now.
b) (') In Exercise 3 we have seen that the representation theory of the infinite dihedral group for the middle cell is controlled by $\mathrm{SO}_3(\mathbb{C})$. Show that the same is true in finite type when working with an appropriate semisimplification of $\mathrm{SO}_3(\mathbb{C})$-representations.
c) (*) What are the nontrivial $S_{\mathcal{H}}$ explicitly?
d) What is the difference between odd and even n?
e) (*) What happens over general fields?

- There might be typos on the exercise sheets, my bad. Be prepared.
- Star exercises are a bit trickier; prime exercises use notions I haven't explained.